Title: Tensor renormalization group in bosonic field theory

Speakers: Esperanza Lopez

Series: Tensor Network

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Abstract: We compute the partition function of a massive free boson in a square lattice using a tensor network algorithm. We introduce a singular value decomposition (SVD) of continuous matrices that leads to very accurate numerical results. It is shown the emergence of a CDL fixed point structure. In the massless limit, we reproduce the results of conformal field theory including a precise value of the central charge.

TENSOR RENORMALIZATION GROUP FOR BOSONIC FIELDS

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with Manuel Campos and German Sierra, arXiv:1902.02362 tensor network techniques are very successful

variational ansatz for ground states and low lying states MPS, PEPS, MERA

real space renormalization group TRG, TNR

main ingredient ENTANGLEMENT PROPERTIES

tensor network techniques are very successful

variational ansatz for ground states and low lying states MPS, PEPS, MERA

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main ingredient ENTANGLEMENT PROPERTIES

APPLY TENSOR NETWORK TECHNIQUES TO QUANTUM FIELD THEORIES

MPS, PEPS, TRG for lattice gauge field theories

continuous tensor networks: cMPS, cPEPS, cMERA different approach

ADAPTED TRG PROTOCOL TO EVALUATE QFT PATH INTEGRALS

revisit QFT with the focus on entanglement RG flow

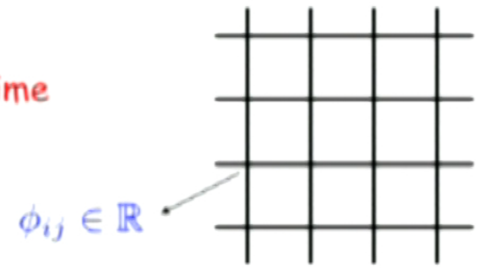
free boson in 2d

$$L = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

starting point for perturbation theory analysis

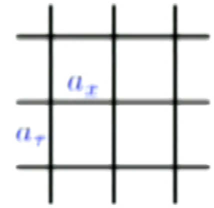
computationally efficient: discretize space-time

lattice variables: continuous fields

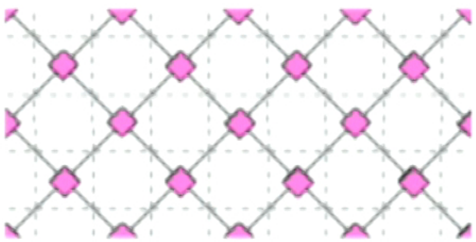


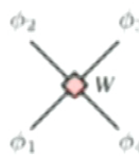
euclidean partition function

$$Z = \int \prod_{ij} d\phi_{ij} e^{-\frac{1}{2}\sum_{ij} a_\tau a_x \left(\frac{(\phi_{i+1j}-\phi_{ij})^2}{a_x^2} + \frac{(\phi_{ij+1}-\phi_{ij})^2}{a_\tau^2} + m^2\phi_{ij}^2 \right)}$$



change to a vertex model

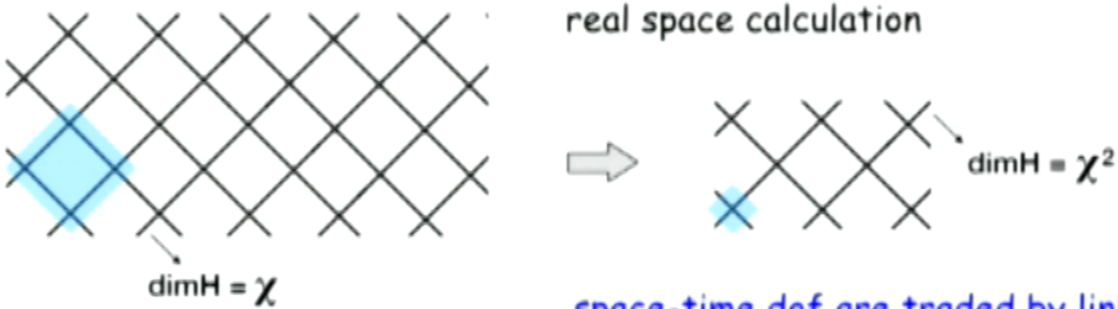




$a_\tau = a_x = 1$

$$W(\phi_i) = e^{-\frac{1}{2}\sum_{i=1}^{4}[(\phi_i - \phi_{i+1})^2 + \frac{m^2}{2}\phi_i^2]}$$

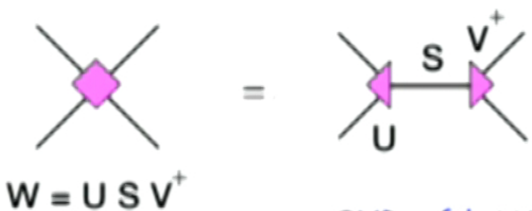
real space calculation

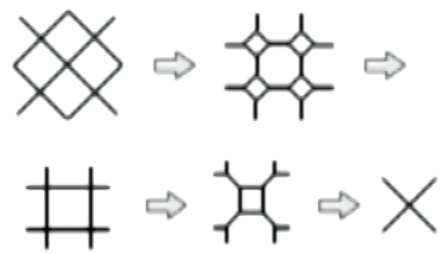


space-time dof are traded by link variables

link variables grows exponentially, truncation is necessary

tensor renormalization group (TRG)



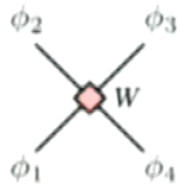


SVD of lattice weights

truncate small singular values

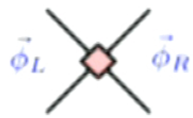
(Levin and Nave, 2007)

initial lattice



$$W(\phi_i) = e^{-\frac{1}{2}\sum_{i=1}^{4}[(\phi_i - \phi_{i+1})^2 + \frac{m^2}{2}\phi_i^2]}$$

generalization: several fields per lattice link $\vec{\phi}_i = \{\phi_i^{(1)}, \phi_i^{(2)}, \ldots, \phi_i^{(\chi)}\}$



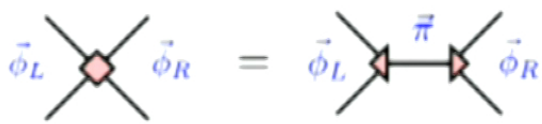
$$W(\vec{\phi}_i) = \rho\, e^{-\frac{1}{2}\vec{\phi}_L^T A_L \vec{\phi}_L - \frac{1}{2}\vec{\phi}_R^T A_R \vec{\phi}_R + \vec{\phi}_L^T B \vec{\phi}_R}$$

gaussian W

$A_{L,R}$, B: real matrices $\vec{\phi}_L = \{\vec{\phi}_1, \vec{\phi}_2\}$

ρ: normalization constant $\vec{\phi}_R = \{\vec{\phi}_3, \vec{\phi}_4\}$

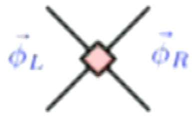
search for a SVD-type decomposition



$\vec{\pi}$: fields

W always gaussian

WORK ON THE QUADRATIC FORMS OF THE GAUSSIAN EXPONENT



$$W(\vec{\phi}_i) = \rho\, e^{-\frac{1}{2}\vec{\phi}_L^T A_L \vec{\phi}_L - \frac{1}{2}\vec{\phi}_R^T A_R \vec{\phi}_R + \vec{\phi}_L^T B \vec{\phi}_R}$$

B connects L and R fields $\quad B = U\, D\, V^{+}$

diagonal matrix with positive entries

introduce $\tilde{\chi}$ new fields π

$\tilde{\chi}$ = dim D = rank B

$$W(\vec{\phi}_L, \vec{\phi}_R) = G_L(\vec{\phi}_L)\, \widehat{W}(\vec{\phi}_L, \vec{\phi}_R)\, G_R(\vec{\phi}_R)$$

$$G_L = e^{-\frac{1}{2}\vec{\phi}_L^T (A_L - U D U^T)\vec{\phi}_L}$$

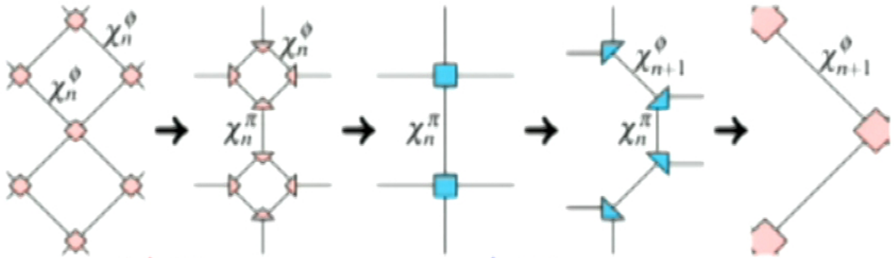
deviates from SVD

$$\widehat{W} = \int d\pi\; e^{i\vec{\phi}_L^T U \vec{\pi}}\, S(\vec{\pi})\, e^{-i\vec{\pi}^T V^T \vec{\phi}_R}$$

$$S = \frac{1}{\sqrt{(2\pi)^{\tilde{\chi}} \det D}}\, e^{-\frac{1}{2}\vec{\pi}^T D^{-1} \vec{\pi}}$$

SVD of $\widehat{W}$: S singular value matrix

GAUSSIAN SVD



bond dimension χ_n^φ

fields per link

each TRG iteration changes 45° the lattice orientation

RG cycle: two TRG iterations

B_n^φ is a $2\chi_n^\varphi \times 2\chi_n^\varphi$ matrix but $\chi_{n+1}^\varphi = 2\chi_n^\varphi$

$\chi_1^\phi = 1$

$\chi_1^\pi = \mathrm{rank}\, B_1^\phi = 2$

$B_1^\phi = \mathbf{1}_2$

ϕ lattice ⇨ π lattice		χ doubles
π lattice ⇨ ϕ lattice		χ remains constant

TRUNCATION   requirement: acts on the number of fields

$$W(\vec{\phi}_L, \vec{\phi}_R) = G_L(\vec{\phi}_L)\, \widehat{W}(\vec{\phi}_L, \vec{\phi}_R)\, G_R(\vec{\phi}_R)$$

$$G_L = e^{-\frac{1}{2}\vec{\phi}_L^T (A_L - UDU^T)\vec{\phi}_L}$$

$$\widehat{W} = \int d\pi\, e^{i\vec{\phi}_L^T U \vec{\pi}}\, S(\vec{\pi})\, e^{-i\vec{\pi}^T V^T \vec{\phi}_R}$$

$$S = \frac{1}{\sqrt{(2\pi)^{\tilde{\chi}} \det D}} e^{-\frac{1}{2}\vec{\pi}^T D^{-1} \vec{\pi}}$$

$$B = UDV^+ \quad \Rightarrow \quad B = U_1 D_1 V_1^+ + U_2 D_2 V_2^+$$

largest singular values, $\vec{\pi}_1$ smallest singular values, $\vec{\pi}_2$

eliminate dependence on $\vec{\pi}_2$

$$\frac{1}{\sqrt{2\pi\alpha}} e^{-\frac{1}{2\alpha}x^2} \sim \delta(x) \quad \Rightarrow \quad S(\vec{\pi}) \sim S(\vec{\pi}_1)\delta(\vec{\pi}_2)$$

not unique choice: G_{LR} not modified difference between exact and truncated W

$$\Delta W = G_L\,\widehat{W}\Delta\widehat{W}\,G_R \quad \Rightarrow \quad \Delta\widehat{W} = 1 - e^{\frac{1}{2}(\phi_L^T U_2 - \phi_R^T V_2)\,D_2\,(U_2^T \phi_L - V_2^T \phi_R)}$$

can be arbitrarily large when $|\phi_{L,R}| \to \infty$

<u>example</u>

$$f(x,y) = e^{-\alpha(x^2+y^2)+2\beta xy} \quad = e^{-(\alpha-\beta)x^2} \int dp\, e^{2ip(x-y)}\, \frac{e^{-p^2/\beta}}{\sqrt{\pi\beta}} \quad e^{-(\alpha-\beta)y^2}$$

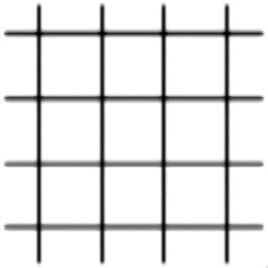
GL $\widehat{W}$ GR

$\Downarrow$

$$\Delta\hat{f} = 1 - e^{\beta(x-y)^2}$$

GLR regulate the large field limit

error $\mathcal{O}(\beta/\alpha)$

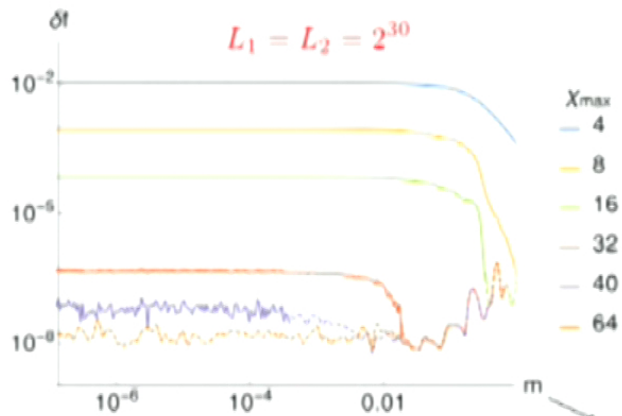


$L_1 \times L_2$ lattice

exact partition function

$$Z^{\text{exact}}_{L_1 L_2} = \left(\frac{\pi}{2}\right)^{\frac{L_1 L_2}{2}} \prod_{n_1, n_2} \left(\sin^2 \frac{\pi n_1}{L_1} + \sin^2 \frac{\pi n_2}{L_2} + \frac{m^2}{4}\right)^{-\frac{1}{2}}$$

numerical results



scalar mass

$f = -\ln Z / L_1 L_2$

$\delta f = (f_{TRG} - f_{ex})/f_{ex}$

maximal bond dimension: $\chi_{\max}$

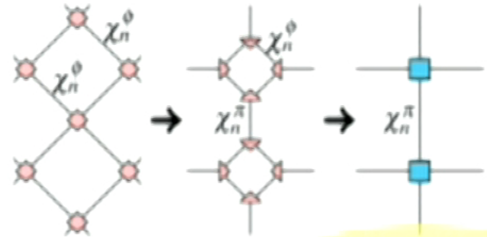
$\chi_{\max} = 32$ ⇨ $\delta f < 10^{-6}$

$\chi_{\max} = 64$ ⇨ average precision 10^{-8}

DETAILS OF TRUNCATION

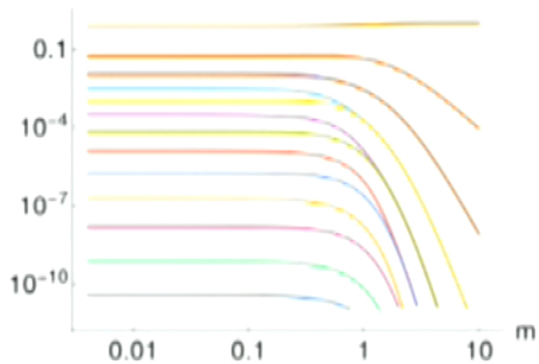
truncation is introduced in

ϕ lattice ⇨ π lattice



$$\chi_n^\pi = 2\chi_n^\phi$$

<u>example</u>: singular values of B_4^ϕ ⇨ 3 RG cycles, $\chi_4^\phi = 8$



strongly decreasing

even for very small masses

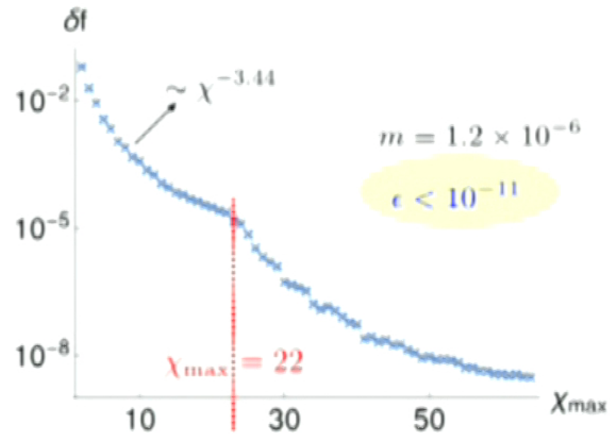
general pattern

⇨ efficient truncation

to reduce numerical noise

discard singular values smaller than ε

relevant for large $\chi_{\max}$



truncation for small bond dimension

$$\{\chi_1^\phi, \chi_1^\pi, \chi_2^\phi, \chi_2^\pi, \dots\} = \{1, 2, 2, 4, 4, 8, 8, 16, 16, 20, 20, 20, 20, \dots\}$$

truncation triggered by ε ⇨ involves ϕ and π lattices

$$\{\chi_1^\phi, \chi_1^\pi, \chi_2^\phi, \chi_2^\pi, \dots\} = \{1, 2, 2, 4, 4, 8, 8, 16, 16, 22, 30, 35, 41, 46, 54, 60, 64, 64, 64, \dots\}$$

precision improves at lower computational cost

MASSLESS LIMIT

free boson central charge: **c = 1**

$$Z_{L_1 L_2} \simeq \frac{2}{m}\left(\frac{\pi}{2}\right)^{\frac{1}{2}L_1 L_2} \prod_{(n_1,n_2)\neq(L_1,L_2)} \left(\sin^2\frac{\pi n_1}{L_1} + \sin^2\frac{\pi n_2}{L_2}\right)^{-1/2}$$

$L_1, L_2 \gg 1$

L_2/L_1 fixed

perform n_2 sum using $\prod_{n=1}^{L}(x^2 + \sin^2\frac{\pi n}{L}) = (2^{1-L}\sinh(L\,\mathrm{arcsinh}\,(x))^2$

$$Z_{L_1 L_2} \simeq \frac{(2\pi)^{\frac{1}{2}L_1 L_2}}{m L_2} \prod_{n_1=1}^{L_1-1} b(n_1, L_1)^{L_2} \times \prod_{n_1=1}^{L_1-1}\left[1 - b(n_1, L_1)^{2L_2}\right]^{-1}$$

$q = e^{-2\pi L_2/L_1}$

$b(n_1, L_1) = -\sin\frac{\pi n_1}{L_1} + \sqrt{1 + \sin^2\frac{\pi n_1}{L_1}}$

$$\simeq e^{-\frac{2G}{\pi}L_1 L_2} q^{-\frac{1}{12}}$$

$$\simeq \prod_{n=1}^{\infty}(1-q^n)^{-2}$$

G: *Catalan constant*

$$Z_{\rm CFT}(\tau) = \frac{1}{(\mathrm{Im}\tau)^{1/2}|\eta(q)|^2} \qquad q = e^{2\pi i\tau}, \quad \eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

$$Z^{\text{exact}}_{L_1 L_2} \simeq \frac{e^{-f_\infty L_1 L_2}}{m(L_1 L_2)^{1/2}} \; Z_{\text{CFT}}(\tau)$$

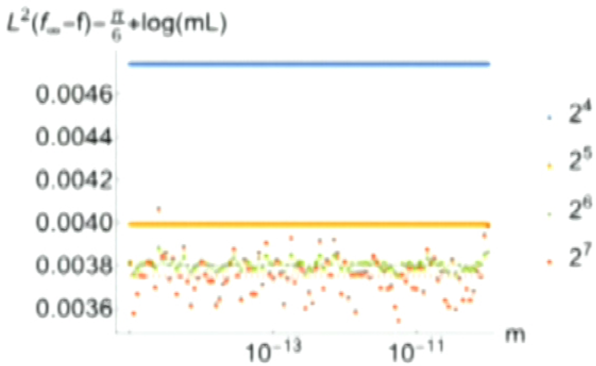
$$f_\infty = \frac{2G}{\pi} - \frac{\ln(2\pi)}{2} \qquad \tau = iL_2/L_1$$

leading finite size effects: Z_{CFT}

$$\frac{\pi}{6} c_{\text{th}} = L^2(f_\infty - f) + \ln(mL) + 2\sum_{n=1}^{\infty} \ln(1 - e^{-2\pi n})$$

$L_1 = L_2 = L$

$$\frac{\pi}{6} + 2\log\frac{\Gamma(1/4)}{2\pi^{3/4}} = -0.00375\ldots$$



medium size lattices

$$c_{\text{gTRG}} - c_{\text{th}} = O(10^{-5}) \quad L = 2^6$$
$$= O(10^{-6}) \quad L = 2^7$$

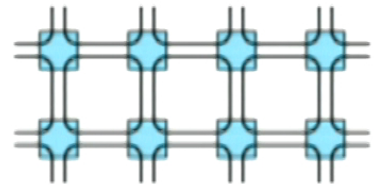
RENORMALIZATION GROUP

mass becomes O(1) in lattice units after

$$n(m) \sim -\frac{\log m}{\log 2} \qquad \text{RG cycles}$$

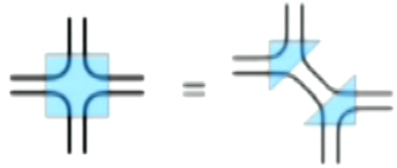
$n \gtrsim n(m)$: correlations confined to a single plaquette

⇨ corner double line structure (CDL)



TRG is not able to eliminate ultralocal entanglement

CDL is a IR fixed point of TRG

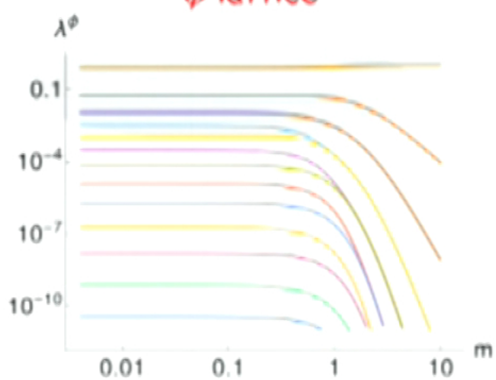


$\mathcal{H}_{\text{link}} = \widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$

no interaction

⇨ singular values of B should come in pairs

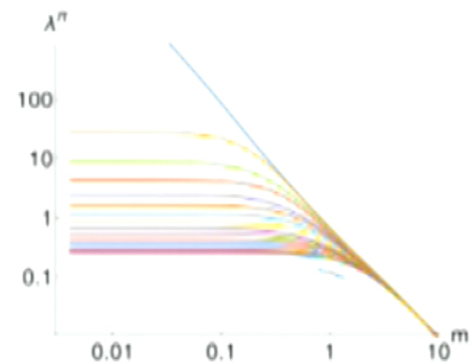
ϕ lattice



3 RG cycles

no truncation

π lattice



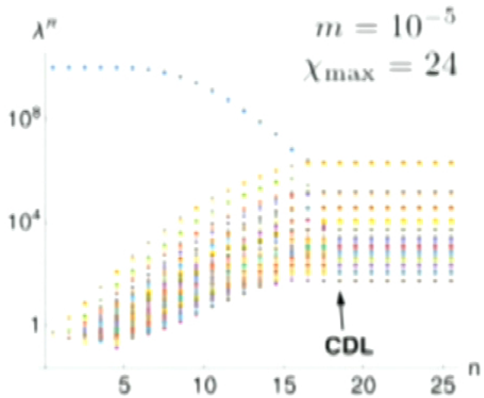
strongly decreasing

tend to pair up after few RG cycles

$\lambda_1^{\pi} \simeq \dfrac{1}{m^2}$ $\quad \lambda_i^{\pi} > \mathcal{O}(10^{-1})$

do not pair

RG flow of the singular values of B^{π}

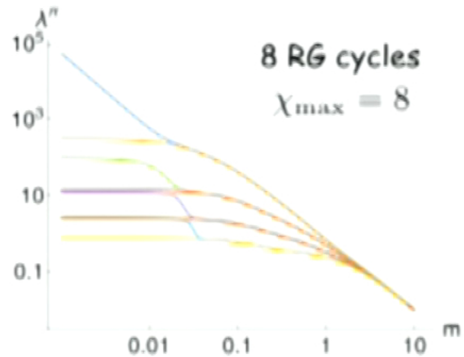


gap between λ_1^{π} and λ_2^{π} closes up along RG

the smaller m, the more RG cycles are needed

$n_{\mathrm{CDL}} \simeq 19$ consistent with

$n(m = 10^{-5}) \simeq 16 - 17$

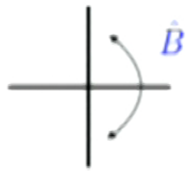


singular values pair up for m>0.03

in lattice units $0.03 \times 2^{8} \simeq 8$

CDL not before scales larger than $\xi = 1/m$

pairing criterium not enough for CDL

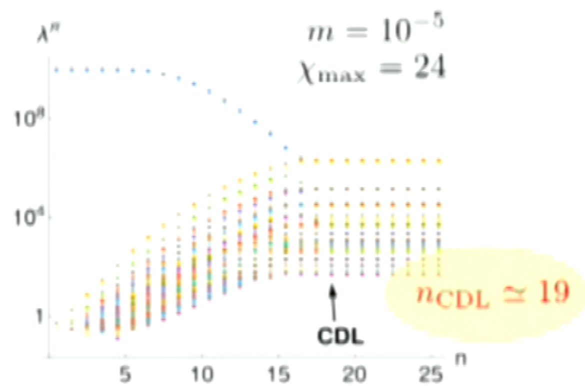


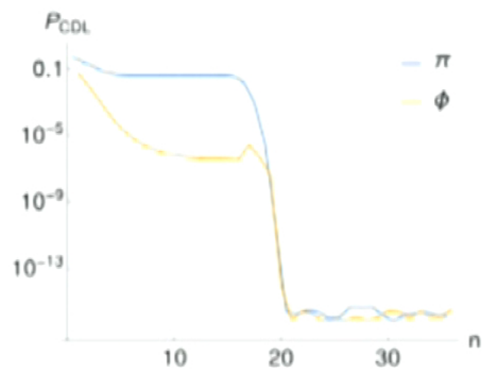
submatrix of B connecting fields on opposite links ⇨

$$P_{\rm CDL} = \frac{1}{\chi}\frac{||\hat{B}||}{\lambda_1}$$

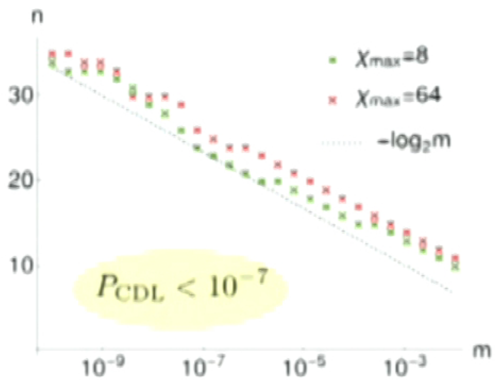
vanishing of $P_{\rm CDL}$ on 2 successive gTRG iterations

⇨ complete CDL structure





RG cycles necessary to reach a CDL IR fixed point



similar results for large and small $\chi_{\rm max}$

consistent with the scaling argument

some long distance info kept for $\chi_{\rm max}$

$$W(\vec{\phi}_L, \vec{\phi}_R) = G_L(\vec{\phi}_L)\, \widehat{W}(\vec{\phi}_L, \vec{\phi}_R)\, G_R(\vec{\phi}_R)$$

$$\widehat{W} = \int d\pi\, e^{i\vec{\phi}_L^T U \vec{\pi}}\, S(\vec{\pi})\, e^{-i\vec{\pi}^T V^T \vec{\phi}_R}$$

$$S = \frac{1}{\sqrt{(2\pi)^{\chi} \det D}}\, e^{-\frac{1}{2}\vec{\pi}^T D^{-1} \vec{\pi}}$$

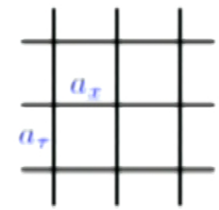
singular values for gTRG

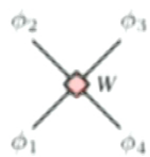
arbitrarily small singular values always kept

$$e^{-\frac{1}{2\lambda}\pi^2}, \qquad |\pi| \to \infty$$

INTEGRABILITY OF THE BOSONIC LATTICE

$$Z = \int \prod_{ij} d\phi_{ij} e^{-\frac{1}{2}\sum_{ij} a_\tau a_x \left(\frac{(\phi_{i+1j}-\phi_{ij})^2}{a_x^2} + \frac{(\phi_{ij+1}-\phi_{ij})^2}{a_\tau^2} + m^2\phi_{ij}^2 \right)}$$

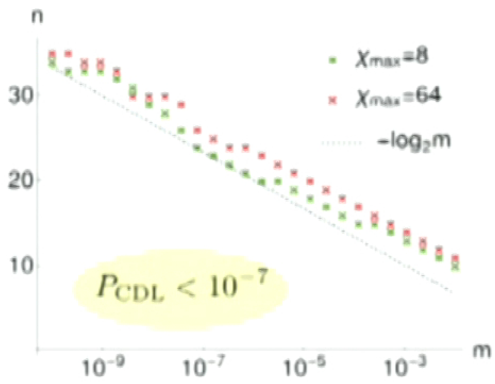




$$W(\phi_i) = e^{-\frac{1}{2}\sum_{i=1}^{4}[(\phi_i-\phi_{i+1})^2+\frac{m^2}{2}\phi_i^2]}$$

$$a_\tau = a_x = 1$$

RG cycles necessary to reach a CDL IR fixed point



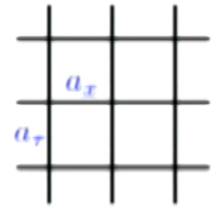
similar results for large and small $\chi_{\rm max}$

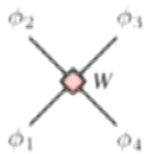
consistent with the scaling argument

some long distance info kept for $\chi_{\rm max}$

INTEGRABILITY OF THE BOSONIC LATTICE

$$Z = \int \prod_{ij} d\phi_{ij} e^{-\frac{1}{2}\sum_{ij} a_\tau a_x \left(\frac{(\phi_{i+1j}-\phi_{ij})^2}{a_x^2} + \frac{(\phi_{ij+1}-\phi_{ij})^2}{a_\tau^2} + m^2\phi_{ij}^2 \right)}$$



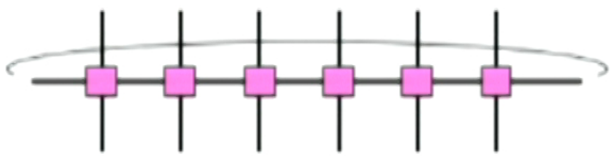


$$W(\phi_i) = e^{-\frac{1}{2}\sum_{i=1}^{4}[(\phi_i-\phi_{i+1})^2 + \frac{m^2}{2}\phi_i^2]}$$

$a_\tau = a_x = 1$

$a_\tau / a_x = c$ espectral parameter

$$W(\phi_i) = e^{-\frac{1}{2}\sum_{i=1}^{4}[c\,(\phi_1-\phi_2)^2 + \frac{1}{c}(\phi_2-\phi_3)^2 + c\,(\phi_3-\phi_4)^2 + \frac{1}{c}(\phi_4-\phi_1)^2 + \frac{m(c)^2}{2}\phi_i^2]}$$



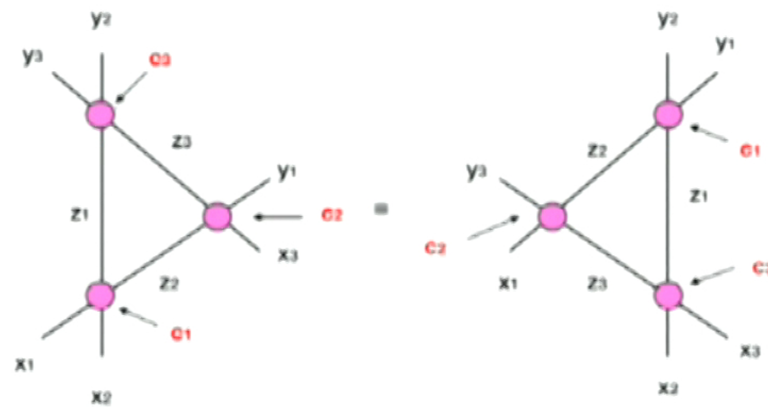
T(c) : transfer matrix

when $c \to 0$

provided $\lim_{c \to 0} m(c) = 0$

Yang-baxter equation

$$[T(c), T(\bar{c})] = 0$$



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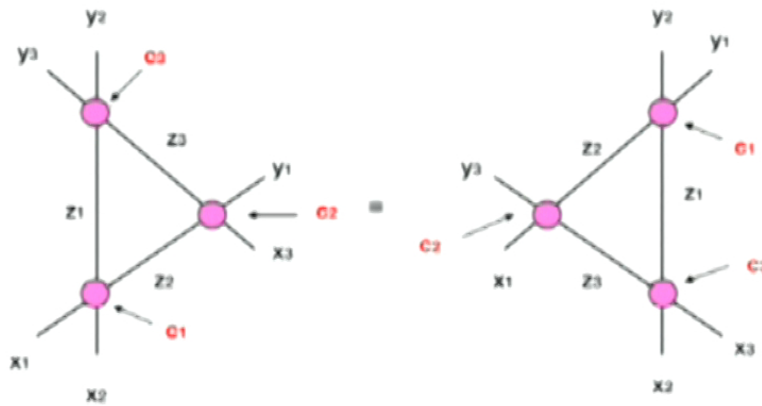
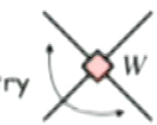


gaussian weights $e^{-\frac{1}{2}(\vec{x}^T M_x(c_i)\,\vec{x}+\vec{y}^T M_y(c_i)\,\vec{y}+2\vec{x}^T M(c_i)\,\vec{y})} = (\vec{x} \leftrightarrow \vec{y})$

$\vec{x} = (x_1, x_2, x_3)$
$\vec{y} = (y_1, y_2, y_3)$

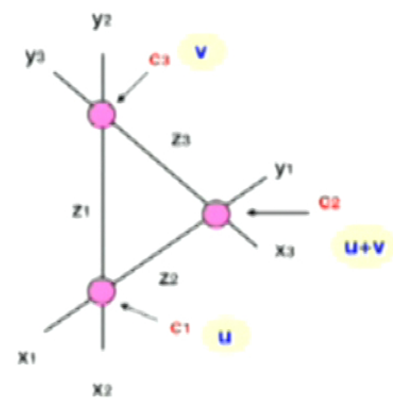
$$M_x = M_y\,,\quad M = M^T$$

180° symmetry



massless case $c_1 c_3 = 1 + c_2(c_1 + c_3)$

$c_1 = \cot u \qquad c_3 = \cot v \qquad c_2 = \cot(u+v)$



massive case **ALSO INTEGRABLE**

$$c_1 = \frac{cn(u,\kappa)}{sn(u,\kappa)dn(u,\kappa)} \qquad c_3 = \frac{cn(v,\kappa)}{sn(v,\kappa)dn(v,\kappa)}$$

$$c_2 = \frac{cn(u+v,\kappa)}{sn(u+v,\kappa)dn(u+v,\kappa)}$$

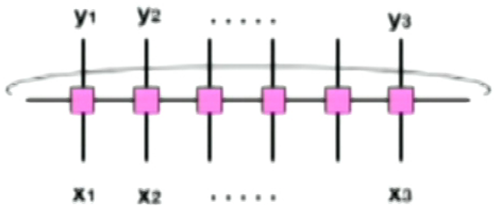
$$m_i^2 = \sqrt{\left(c_i + \frac{1}{c_i}\right)^2 - 4\kappa} - \left(c_i + \frac{1}{c_i}\right)$$

κ<0 for positive mass square $\lim_{c_i \to 0} m_i^2 = 0$ ⇨ T(0) = P

<u>smooth limit</u>

elliptic ⇨ trigonometric
massive $\kappa \to 0$ massless

EIGENSTATES $|\Psi\rangle = \int dx\, \Psi(x)|\vec{x}\rangle$



$$\int dx \langle \vec{y}|T(c)|\vec{x}\rangle \Psi(\vec{x}) = \Lambda(c)\Psi(\vec{y})$$

$e^{-\frac{1}{2}(\vec{x}\ \vec{y})M(c)(\vec{x}\ \vec{y})^T}$

independent of c

M(c): built out of 1, S, S^{-1} ⇨ momentum modes $\tilde{x}_k = \frac{1}{\sqrt{N}}\sum_{i=n}^{N} e^{2\pi i \frac{kn}{N}} x_n$

shift matrix

but integrability structure emerged from a local analysis

ground state: $\Psi_{gs}(\vec{x}) = e^{-\frac{1}{4}\sum_{k=1}^{N}\omega(k)\tilde{x}_k\tilde{x}_{N-k}}$

excited states: Hermite polynomials

energy

$$\omega(k) = \sqrt{4\sin^2\left(\frac{\pi k}{N}\right) - 4\kappa}$$

CONSERVED CHARGES

<u>ground state</u>

$$\Lambda_{gs}(c) = \prod_{k=1}^{N} \frac{1}{1+\frac{\omega(k)}{a(c)}}$$

⇨ conserved charges

$$\sum_{k=1}^{N} \omega(k)^j$$

$$a(c) = \sqrt{\left(c+\frac{1}{c}\right)^2 - 4\kappa} \to \frac{1}{c}$$

<u>excited states</u>

$$\Lambda(c) = \Lambda_{gs}(c) \prod_{k=1}^{N} \Lambda_k(c)^{n(k)}$$

$n(k)$ → occupation numbers

$$\Lambda_k(c) = e^{2\pi i \frac{k}{N}} \left(\frac{1+c^2 e^{-2\pi i \frac{k}{N}}}{a(c)c}\right)^2 \frac{1}{\left(1+\frac{\omega(k)}{a(c)}\right)^2}$$

$$\sum_{k=1}^{N} \left(n(k)+\frac{1}{2}\right)\omega(k)^j$$

$$\sum_{k=1}^{N} n(k)(\,p(k)+\ldots)$$

conclusions

adapted TRG protocol: lattice variables are fields

- numerically efficient, even in the massless limit
- RG flow: CDL IR fixed point
- starting point for perturbation theory

integrability in the bosonic lattice

- obtain local commuting charges
- link to Virasoro in the massless limit

THANKS!