Title: Dynamics of two-point correlation functions in quantum systems

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Abstract: We give rigorous analytical results on the temporal behavior of two-point correlation functions (also known as dynamical response functions or Greenâ€™s functions) in quantum many body systems undergoing unitary dynamics. Using recent results from large deviation theory, we show that in a large class of models the correlation functions factorize at late times -> , thus proving that dissipation emerges out of the unitary dynamics of the system. We also show that the fluctuations around this late-time value are bounded by the purity of the thermal ensemble, which generally decays exponentially with system size. This conclusion connects the behavior of correlation functions to that of the late-time fluctuations of quenched systems out of equilibrium.

For auto-correlation functions such as (as well as the symmetrized and anti-symmetrized versions) we provide an upper bound on the timescale at which they reach that factorized late time value. Remarkably, this bound is a function of local expectation values only, and does not increase with system size. As such it constraints, for instance, the behavior of current auto-correlation functions that appear in quantum transport. We give numerical examples that show that this bound is a good estimate in chaotic models, and argue that the timescale that appears can be understood in terms of an emergent fluctuation-dissipation theorem. Our study extends to further classes of two point functions such as the Kubo function of linear response theory, for which we give an analogous result.

Joint work with Luis Pedro Garcia-Pintos and Jonathon Riddell

Dynamics of 2-point correlation functions in quantum systems

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Correlation functions/dynamical response functions

$$\langle A(t)B\rangle_\beta \equiv \mathrm{tr}[\rho_\beta A(t)B]$$

$$\left\{\begin{array}{l} A(t) = e^{-itH}Ae^{itH} \\ H = \sum_i^N h_i \end{array}\right.$$

R Kubo 1966 *Rep. Prog. Phys.* **29** 255

Outline:

- At late times, "dissipation" occurs $\langle\overline{A(t)B}\rangle_\beta \to \langle A\rangle_\beta\langle B\rangle_\beta$
- For most times $\langle A(t)B\rangle_\beta \sim \langle A\rangle_\beta\langle B\rangle_\beta$ (small variance)
- Autocorrelation function timescales: after given $T_0 \sim \mathcal{O}(1)$

$$\langle A(t)A\rangle_\beta \sim \langle A\rangle_\beta\langle A\rangle_\beta$$

- Some extensions and applications

Correlation functions/dynamical response functions

$$\langle A(t)B\rangle_\beta \equiv \mathrm{tr}[\rho_\beta A(t)B]$$

$$\left\{\begin{array}{l} A(t) = e^{-itH}Ae^{itH} \\ H = \sum_i^N h_i \end{array}\right.$$

R Kubo 1966 *Rep. Prog. Phys.* **29** 255

This talk: rigorous analytical results on time evolution of these functions, which apply to non-integrable systems.

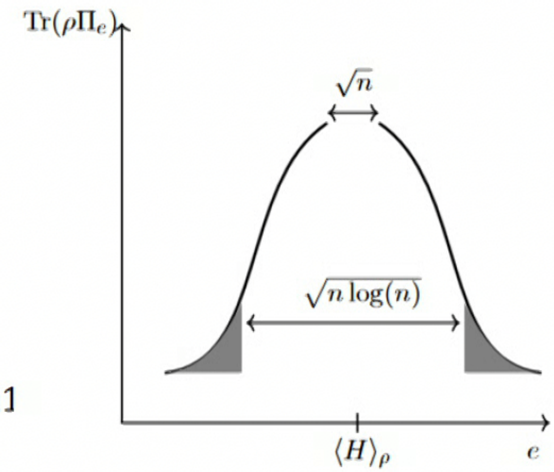
Factorization/dissipation

$$\langle A(t)B\rangle_\beta \longrightarrow \langle A\rangle_\beta \langle B\rangle_\beta$$

- Key assumption: Thermal states with finite correlation length on D-dim lattice

$$\max_{X\in\mathcal{X},Y\in\mathcal{Y}} \frac{|\langle X\otimes Y\rangle_\beta - \langle X\rangle_\beta\langle Y\rangle_\beta|}{||X||||Y||} \leq e^{-\frac{\mathrm{dist}(\mathcal{X},\mathcal{Y})}{\xi}}$$

- Their energy distribution follows "large deviation bounds"



$$\mathrm{tr}[\rho_\beta \Pi_{\geq\langle H\rangle_\beta + Na}] \leq \mathcal{O}(\xi) e^{-\frac{-(Na^2\xi)^{1/(D+1)}}{\mathcal{O}(1)\xi}}$$

Anurag Anshu 2016 *New J. Phys.* **18** 083011

Factorization/dissipation: "weak ETH"

$$\langle A(t)B\rangle_\beta \longrightarrow \langle A\rangle_\beta \langle B\rangle_\beta$$

- The key ingredient is a weak version of diagonal ETH:

$$A_{kk} \sim \langle A\rangle_\beta$$ "with high probability" or "for most eigenstates"

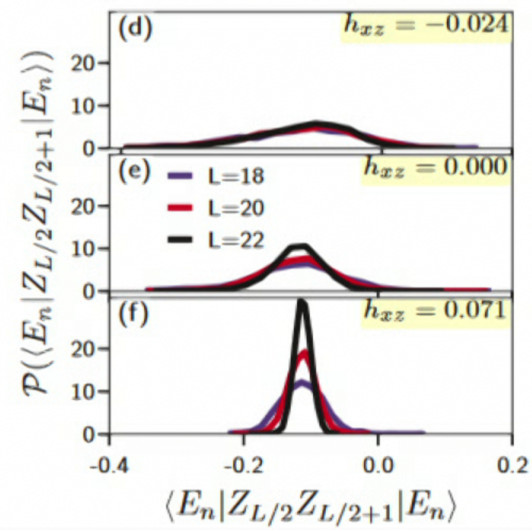
- Mori (1609.09776): weak ETH follows from large deviation bounds (like those of Anshu)
- Still true for systems that do not necessarily thermalize:

-Integrable? Biroli et al. PRL 105, 250401 (2010)

-Quantum scars Turner et al. Nature Physics 14, 745–749 (2018)

Khemani et al. PRB 99, 161101(R) (2019)

$$H = -\sum_i^N P_{i-1}X_iP_{i+1} - \sum_i^N h_{XZ}(P_{i-1}X_iP_{i+1}Z_{i+2} + Z_{i-2}P_{i-1}X_iP_{i+1})$$



Factorization/dissipation: “weak ETH”

- Brandao et al (1710.04631) put together Mori ’16 + Anshu ’16 to obtain explicit bounds

Let $\begin{cases} H & \text{translation-invariant on } N \text{ sites} \\ \rho_\beta & \text{exponential decaying correlations, with corr. length } \xi \\ A & \text{has support on } \mathcal{O}(1) \text{ sites} \end{cases}$

(1D case, but works for higher D)

$$\boxed{\Pr_{|E_k\rangle \in \rho_\beta}\left(|A_{kk} - \langle A\rangle_\beta| > \delta\right) \leq \exp(-c\delta N^{1/2}/\xi^{3/2})}$$

Factorization/dissipation: proof outline

Let
- H translation-invariant on N sites, **non-degenerate**
- ρ_β exponential decaying correlations
- A, B have support on $\mathcal{O}(1)$ sites

$$\lim_{T\to\infty} \int_0^T \frac{\mathrm{d}t}{T} \langle A(t) B \rangle_\beta$$

Factorization/dissipation: proof outline

Let
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- A, B have support on $\mathcal{O}(1)$ sites

$$\lim_{T\to\infty} \int_0^T \frac{\mathrm{d}t}{T} \langle A(t) B \rangle_\beta = \sum_k \rho_{kk} A_{kk} B_{kk}$$

Factorization/dissipation: proof outline

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- ρ_β exponential decaying correlations
- A, B have support on $\mathcal{O}(1)$ sites

$$\lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\langle A(t)B\rangle_\beta = \sum_k \rho_{kk}A_{kk}B_{kk} \qquad \delta_k^A \equiv A_{kk} - \langle A\rangle_\beta$$

$$= \langle A\rangle_\beta\langle B\rangle_\beta + \sum_k \rho_{kk}(\langle A\rangle_\beta\delta_k^B + \langle B\rangle_\beta\delta_k^A + \delta_k^A\delta_k^B)$$

$$= \langle A\rangle_\beta\langle B\rangle_\beta + C\sum_{\delta_k\geq\delta*}\rho_{kk}\delta_k + C\sum_{\delta_k<\delta*}\rho_{kk}\delta_k$$

Factorization/dissipation: proof outline

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$$= \langle A\rangle_\beta\langle B\rangle_\beta + C\sum_{\delta_k\geq\delta*}\rho_{kk}\delta_k + C\sum_{\delta_k<\delta*}\rho_{kk}\delta_k$$

Small due to weak ETH (for $\delta*$ large enough)

Small if $\delta*$ is chosen small enough

Factorization/dissipation: proof outline

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$$= \langle A\rangle_\beta\langle B\rangle_\beta + C\sum_{\delta_k\geq\delta*}\rho_{kk}\delta_k + C\sum_{\delta_k<\delta*}\rho_{kk}\delta_k$$

$$= \langle A\rangle_\beta\langle B\rangle_\beta + \mathcal{O}(\xi^{3/2}\frac{\log N}{\sqrt{N}})$$

Factorization/dissipation: result

Let
- H translation-invariant on N sites, **non-degenerate**
- ρ_β exponential decaying correlations
- A, B have support on $\mathcal{O}(1)$ sites

$$\lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\langle A(t)B\rangle_\beta = \langle A\rangle_\beta\langle B\rangle_\beta + \mathcal{O}(\xi^{3/2}\frac{\log N}{\sqrt{N}})$$

Factorization/dissipation: result

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This “dissipation” happens even if your system does not always thermalize (strong vs. weak ETH). Not the case for 4-point functions.

Average size of fluctuations

- Late-time average value (previous slide) $$\langle\overline{A(t)B}\rangle_\beta = \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\langle A(t)B\rangle_\beta$$
- Fluctuations around it $$\sigma_{AB} = \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}|\langle A(t)B\rangle_\beta - \langle\overline{A(t)B}\rangle_\beta|^2$$

Let H be such that : $$E_j - E_k = E_l - E_m \iff j = l, k = m$$

Then:

$$\boxed{\sigma_{AB} \leq ||A||||B||\mathrm{Tr}[\rho_\beta^2] \qquad \propto\ e^{-cN}}$$

Fluctuations in pure states/quenches

Let H be such that : $E_j - E_k = E_l - E_m \iff j = l, k = m$

Then: $\sigma_{AB} \leq ||A||||B||\mathrm{Tr}[\rho_\beta^2]$

There is a significant parallelism with results on equilibration of quenched out of eq. systems:

Linden et al PRE 79, 061103 (2009)
Reimann PRL 101, 190403 (2008)

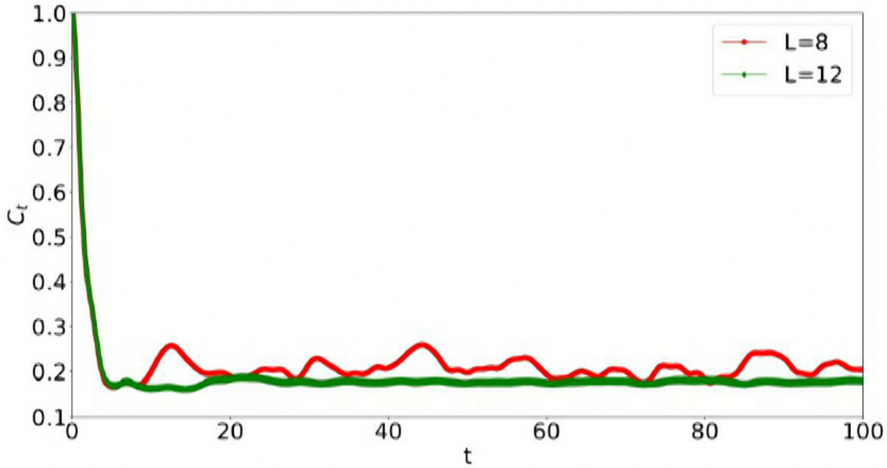
$$\overline{|\underbrace{\langle\psi(t)|A|\psi(t)\rangle - \overline{\langle\psi(t)|A|\psi(t)\rangle}}_{\text{Average size of out-of-equilibrium fluctuations}}|^2} \leq ||A||^2\mathrm{Tr}[\underbrace{\overline{|\psi(t)\rangle\langle\psi(t)|}^2}_{\text{Purity of diagonal ensemble (related to ergodicity of dynamics)}}]$$

See also: Srednicki 1999 *J. Phys. A: Math. Gen.* **32** 1163
Ritcher et al. 1805.11625

Timescales of autocorrelation functions

- The behavior of $\langle A(t)B\rangle_\beta$ might involve timescales depending on $\mathrm{dist}(A,B)$, N...
- How long until $\langle A(t)B\rangle_\beta \longrightarrow \langle A\rangle_\beta\langle B\rangle_\beta$?
- We focus on $A=B$

$\langle A(t)A\rangle_\beta$



$$\int_0^T \frac{\mathrm{d}t}{T}\frac{|\langle A(t)A\rangle_\beta - \overline{\langle A(t)A\rangle_\beta}|^2}{\langle A^2\rangle_\beta} \leq f(T)$$

Timescales of autocorrelation functions

$$\int_0^T \frac{\mathrm{d}t}{T} \frac{|\langle A(t)A\rangle_\beta - \langle \overline{A(t)A}\rangle_\beta|^2}{\langle A^2\rangle_\beta} \leq \frac{\pi a(\epsilon)}{\sigma_A T} + \delta(\epsilon)$$

If $\rho_{kk}|A_{jk}|$ smoothly distributed, and not too many degeneracies (both reasonable in non-integrable systems), it holds that

$$a(\epsilon) \sim \mathcal{O}(1) \qquad \delta(\epsilon) \ll 1$$

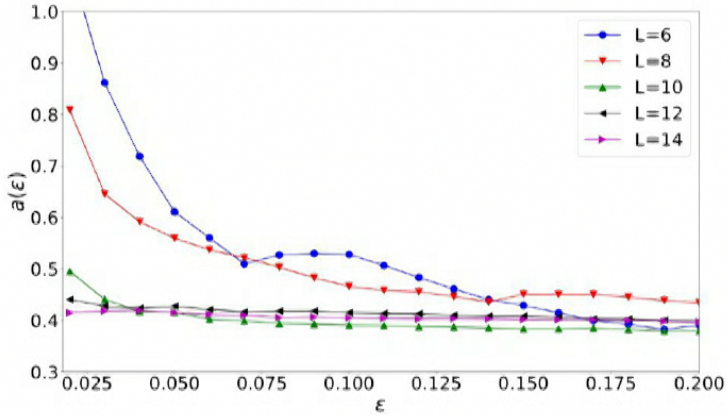
García-Pintos et al
Phys. Rev. X 7, 031027 (2017)

If so, a relevant timescale is given by σ_A

$$\sigma_A^2 = \frac{\langle [A,H][H,A]\rangle_\beta}{\langle A^2\rangle_\beta} - \frac{\langle [H,A],A\rangle_\beta^2}{\langle A^2\rangle_\beta^2}$$

Timescales of autocorrelation functions

$$\int_0^T \frac{\mathrm{d}t}{T} \frac{|\langle A(t)A\rangle_\beta - \langle\overline{A(t)A}\rangle_\beta|^2}{\langle A^2\rangle_\beta} \le \frac{\pi a(\epsilon)}{\sigma_A T} + \delta(\epsilon)$$



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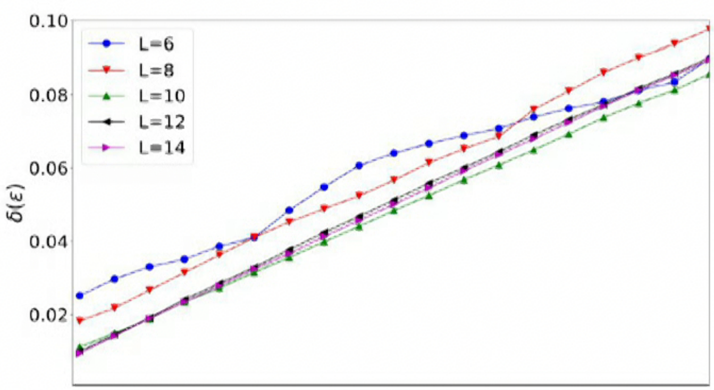


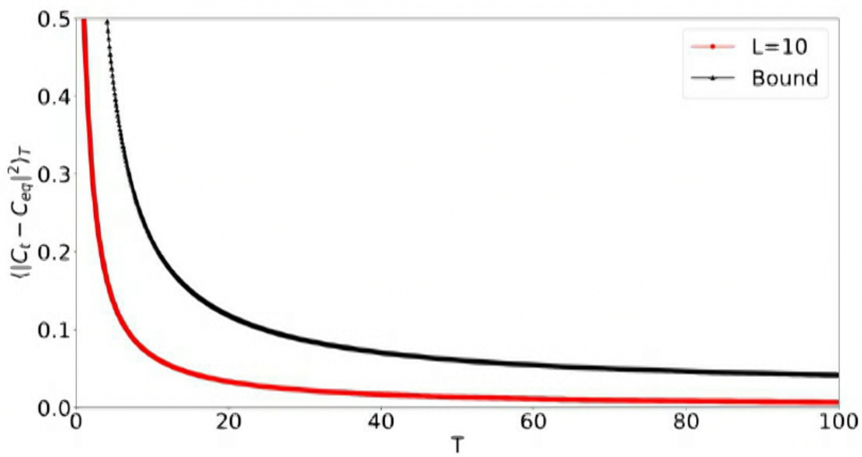
$$A = \sigma^x_{L/2}$$

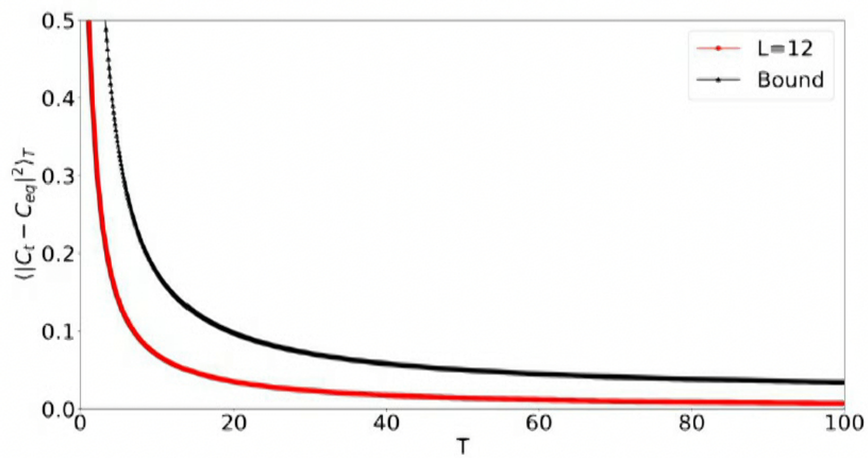
$$H = \sum_{j=1}^{L} (\gamma\sigma_j^X + \lambda\sigma_j^Z) + J\sum_{j=1}^{L-1} \sigma_j^Z\sigma_j^Z + \alpha\sum_{j=1}^{L-2}\sigma_j^Z\sigma_{j+2}^Z$$

Timescales of autocorrelation functions

$$\int_0^T \frac{\mathrm{d}t}{T} \frac{|\langle A(t)A\rangle_\beta - \langle \overline{A(t)A}\rangle_\beta|^2}{\langle A^2\rangle_\beta} \leq \frac{\pi a(\epsilon)}{\sigma_A T} + \delta(\epsilon) \qquad A = \sigma^x_{L/2}$$

Non-integrable model: $$H = \sum_{j=1}^{L}(\gamma\sigma_j^X + \lambda\sigma_j^Z) + J\sum_{j=1}^{L-1}\sigma_j^Z\sigma_j^Z + \alpha\sum_{j=1}^{L-2}\sigma_j^Z\sigma_{j+2}^Z$$





Timescales of (symmetric) autocorrelation functions

$$\int_0^T \frac{\mathrm{d}t}{T} \frac{|\langle\{A(t),A\}\rangle_\beta - \langle\overline{\{A(t),A\}}\rangle_\beta|^2}{\langle A^2\rangle_\beta} \leq \frac{\pi a(\epsilon)}{\sigma_A T} + \delta(\epsilon)$$

If $(\rho_{kk}+\rho_{jj})|A_{jk}|^2$ smoothly distributed, and not too many degeneracies (both reasonable in non-integrable systems), it holds that

$$a(\epsilon) \sim \mathcal{O}(1) \qquad \delta(\epsilon) \ll 1$$

If so, a relevant timescale is given by σ_A

$$\sigma_A^2 = \frac{\langle [A,H][H,A]\rangle_\beta}{\langle A^2\rangle_\beta}$$

Timescales of (symmetric) autocorrelation functions

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If so, a relevant timescale is given by σ_A

$$\sigma_A^2 = \frac{\langle [A,H][H,A]\rangle_\beta}{\langle A^2\rangle_\beta}$$

Proportionality between timescale of "dissipation" of perturbation A and size of fluctuations at equilibrium

$$\langle A^2\rangle_\beta$$

Emergent "fluctuation-dissipation theorem"?

Timescales of autocorrelation functions: a suggestion

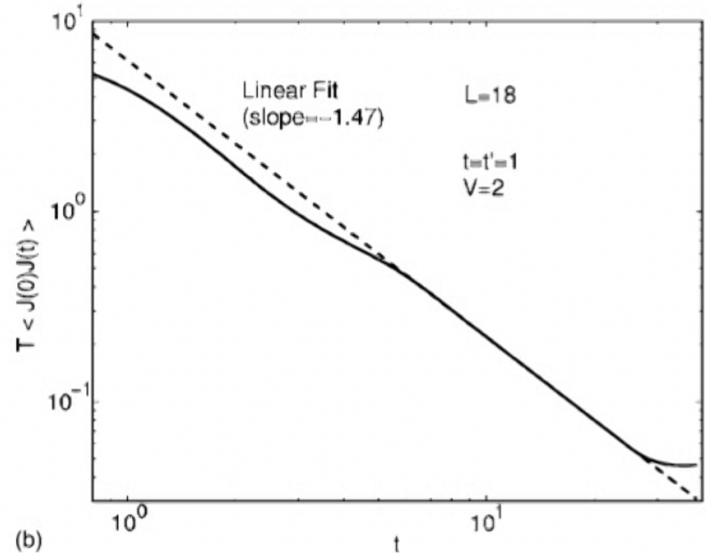
Example: **Mukerjee et al PRB 73, 035113 (2006)**

$$H = -\lambda \sum_j c_j^\dagger c_{j+1} - \lambda' \sum_j c_j^\dagger c_{j+2} + \text{h.c.} + V \sum_j n_j n_{j+1}$$

Total particle current autocorrelation, decays at "late times", and signals diffusive behavior:

$$\langle J(t) J \rangle_\beta \sim t^{-3/2} \qquad t \geq T_{\text{diff}}$$

Idea: If before this "late time" the decay is not as fast, our result can be used to bound that "late time" T_{diff}



Mukerjee et al PRB 73, 035113 (2006)

If:

$$\langle J(T_{\text{diff}}) J \rangle_\beta \sim k \langle J^2 \rangle_\beta$$
$$\delta(\epsilon) \ll 1$$

Then:

$$\boxed{T_{\text{diff}} \leq \frac{\pi a(\epsilon)}{k \sigma_J}}$$

$$\sigma_J = \frac{\langle [J,H][H,J] \rangle_\beta}{\langle J^2 \rangle_\beta} - \frac{\langle [H,J] J \rangle_\beta^2}{\langle J^2 \rangle_\beta^2}$$

Summary and open questions:

- If H local, translation-invariant, non-degenerate, and ρ_β has finite correlation length, then

$$\lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\langle A(t)B\rangle_\beta = \langle A\rangle_\beta\langle B\rangle_\beta + \mathcal{O}(\xi^{3/2}\frac{\log N}{\sqrt{N}})$$

- Moreover, if H has non-degenerate energy gaps, the fluctuations around that average are suppressed by $\mathrm{tr}[\rho_\beta^2]$

- Timescale for $\langle A(t)A\rangle_\beta \sim \langle A\rangle_\beta^2$ bounded by a constant in system size σ_A

- When is σ_A a good estimate? Is the bound tight? In which models/observables?

- Timescales for $\langle A(t)B\rangle_\beta$ and more fine-grained information about the different regimes ($\propto N, N^2, \ldots$)