Title: Multipole gauge theories and fractons

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Abstract: I will describe an infinite set of exotic gauge theories that have recently and simultaneously emerged in several a priori unrelated areas of condensed matter physics such as self-correcting quantum memory, topological order in 3+1 dimensions, spin liquids and quantum elasticity. In these theories the gauge field is a symmetric tensor (not to be confused with higher form, which is an anti-symmetric tensor), or in more exotic situations, the gauge fields do not have a well-defined transformation properties under rotations. I will discuss a few exotic features of these theories such as (i) corresponding Gauss law constraints (ii) failure of the gauge invariance in curved space, (iii) the nature of the gauge group, etc. I will also discuss the what kind of matter such theories can couple to. It turns out that the corresponding matter must conserve electric charge and various multipole moments of the electric charge (or number) density. The conservation laws of multipole moments lead to dramatic consequences for the dynamics. I will also discuss how such theories can be obtained by gauging a global symmetry. Finally, I will discuss non-local operators in this type of theories. Remarkably, in addition to more-or-less expected Wilson line and surface operators, such theories exhibit (at least upon discretization on a lattice) non-local operators supported on a space of fractional dimension (in between line and surfaces).

MULTIPOLE GAUGE THEORIES AND FRACTONS

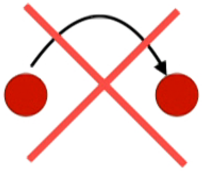
Andrey Gromov

Brown

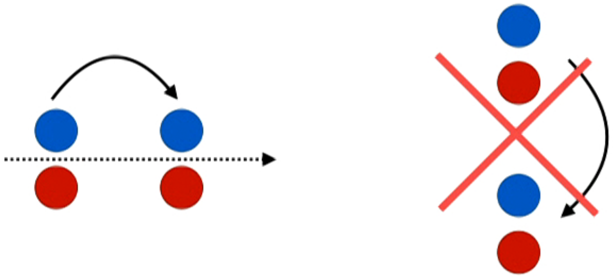
Perimeter, April 24, 2019

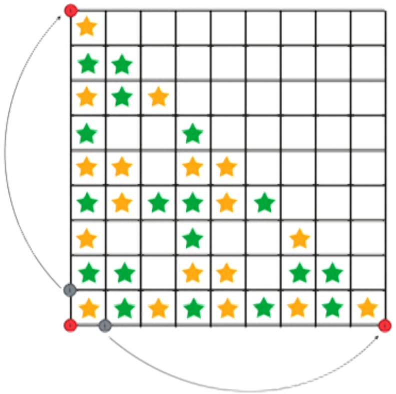
WHAT IS A FRACTON?

Fracton is a quasiparticle that cannot move

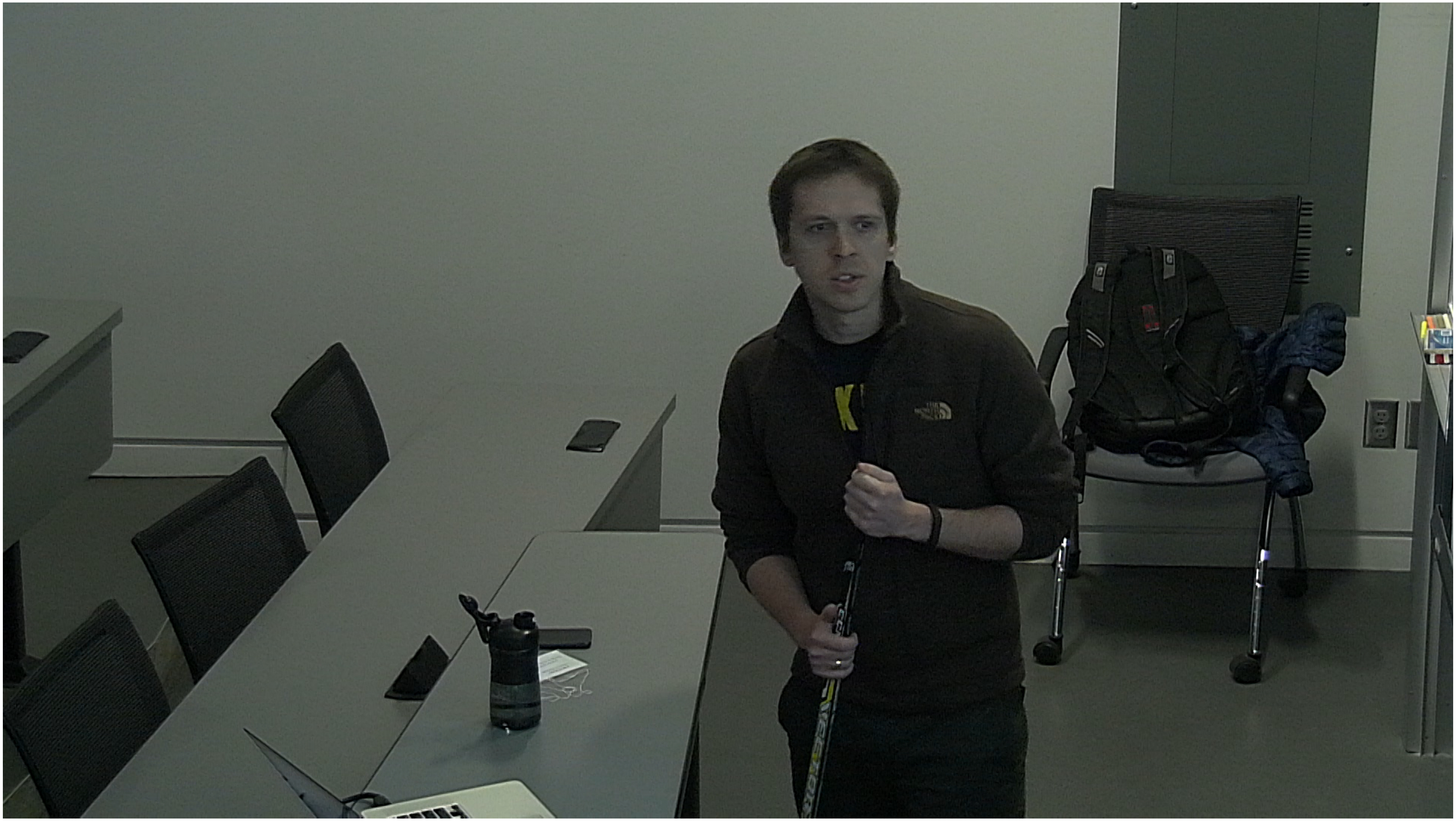


A combination of several fractons may move on a submanifold





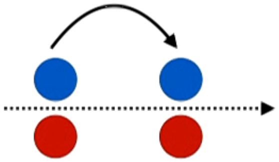
The submanifold may have fractional dimension



TWO TYPES OF FRACTON PHASES

There are two type of fracton models. ``type-I'' and ``type-II''. Precise mathematical definition of either phase is not known.

Presently, type-I means that there exists a combination of fractons that can move (either freely, or along lower dimensional manifolds).



Historically, the ``type-II'' models were discovered first. In such models no combination of fractons can move (except the trivial one).

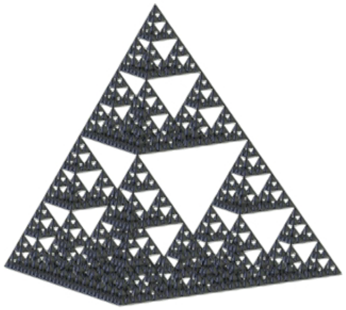
We would like to develop a field-theoretic approach to all such phases.

Chamon 2005 Haah 2011 Sagar Haah Fu 2016

HAAH'S CODE

Haah's code is the first discovered type-II model. All excitations are immobile

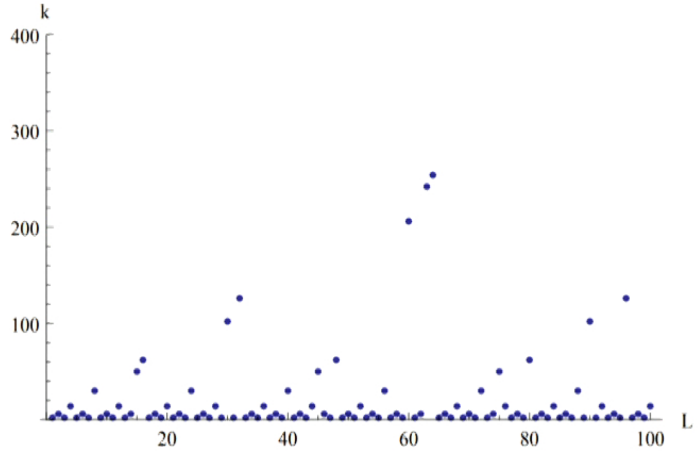
Excitations are $\mathbb{Z}_2$ charges created in quadruples at corners of a pyramid.
Haah's model is topologically ordered, however it does not appear to admit a description in terms of a TQFT



Topological order is often quantified by degeneracy without symmetry. Haah's code has such degeneracy. It equals 2^k, where k is given by

$$\frac{k+2}{4} = \begin{cases} 1 & \textit{if } L = 2^p+1, \\ L & \textit{if } L = 2^p, \\ L-2 & \textit{if } L = 4^p-1, \\ 1 & \textit{if } L = 2^{2p+1}-1. \end{cases}$$

L is a system size, and $p \in \mathbb{Z}$



Yoshida 2011 Haah 2011

A NOTE ON LANGUAGE

I will colloquially refer to both gapped and gapless phases that support fractons as fracton phases

The phases with U(1) charge will usually be gapless, while Higgsed phases with $\mathbb{Z}_k$ charge will be gapped.

Not all gapped phases that I discuss will be topological.

OUTLINE

- Particle-vortex duality and emergent gauge fields
- Symmetric tensor gauge fields, type-I phases
- Interlude I: Kinematics of elastic defects
- Interlude II: Symmetric tensor gauge theories in curved background
- Conservation of multipole moments as a global symmetry
- Multipole gauge theory, type-I and type-II phases
- A two-dimensional U(1) type-II fracton phase
- Fractal operators
- U(1) and $\mathbb{Z}_2$ Haah code
- Conclusions/open directions

References:
AG PRL 122, 076403
AG arXiv:1812.05104

PART I:

EFT FOR TYPE-I PHASES

PARTICLE-VORTEX DUALITY 2+1D

In condensed matter gauge fields can arise from conservation laws

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi$$

Global U(1) symmetry implies the conservation law $\partial_\mu J^\mu = 0$

This conservation law can be solved via $J^\mu = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho$

Such representation of the current is ambiguous up to $\delta A_\rho = \partial_\rho \alpha$

In terms of A_μ the Lagrangian takes Maxwell form

$$\mathcal{L} = F_{\mu\nu}F^{\mu\nu}$$

The Gauss law implies that charges for A_μ are vortices of $\Phi = |\Phi| e^{i\theta}$

$$\rho = \partial_i E^i = \epsilon^{ij}\partial_i J_j = \Omega = \epsilon^{ij}\partial_i\partial_j\theta$$

Peskin 1978

Dasgupta Halperin 1981

PARTICLE-VORTEX DUALITY 3+1D

In 3D the same logic leads to higher form gauge fields

$$J^\mu = \epsilon^{\mu\nu\rho\lambda}\partial_\nu A_{\rho\lambda}$$

The charges for the 2-form $A_{\rho\lambda}$ are the vortex *lines*

The gauge symmetry is 1-form symmetry

$$\delta A_{\rho\lambda} = \partial_\rho \alpha_\lambda - \partial_\lambda \alpha_\rho$$

Such gauge theories are common, but they do not lead to fractons

TENSOR GAUGE THEORY

Duality can lead to more exotic theories. Consider conservation of momentum

$$\partial_\mu T_i{}^\mu = \dot{P}_i + \partial_j T_i{}^j = 0$$

This conservation law appears in the theory of elastic medium. Solution

$$T_i{}^\mu = \epsilon^{\mu\nu\rho}\partial_\nu A_{i\rho}$$

In components

Momentum:

$$T_i{}^0 = P_i = \epsilon^{jk}\partial_j A_{ik} = B_i$$

Stress:

$$T_i{}^j = \epsilon^{jk} E_{ik}$$

$$E_{ik} = -\dot{A}_{ik} + \partial_k \underset{A_{i0}}{\underset{\uparrow}{C_i}}$$

Kleinert 1983

Pretko Radzihovsky 2017 **AG** 2017

TENSOR GAUGE THEORY

The gauge redundancy of the field redefinition is

$$\delta A_{i\mu} = \partial_\mu \alpha_i$$

In components

$$\delta A_{ij} = \partial_j \alpha_i \qquad \delta C_i = \dot{\alpha}_i$$

Symmetry of the stress tensor implies that E-field is traceless $E_i{}^i = 0$

The (flat space) action is two copies of Maxwell theory.

$$\mathcal{L} = E_{ij}E^{ij} + B_i B^i \qquad \partial_j E_{ij} = \rho_i$$

Elasticity is different from two-component superfluid: the index that labels components is spatial, and not internal. This is the defining feature of tensor gauge theories.

Kleinert 1983

Pretko Radzihovsky 2017 **AG** 2017

SYMMETRIC TENSOR GAUGE THEORY

Let’s take particular solution of the conservation law with a smaller gauge redundancy $\alpha_j = \partial_j \alpha$

$$\delta A_{ij} = \partial_i \partial_j \alpha \qquad \delta\phi = \dot{\alpha} \qquad A_{i0} = \partial_i \phi$$

$$\partial_i \partial_j E_{ij} = \rho = \partial_i \rho^i$$

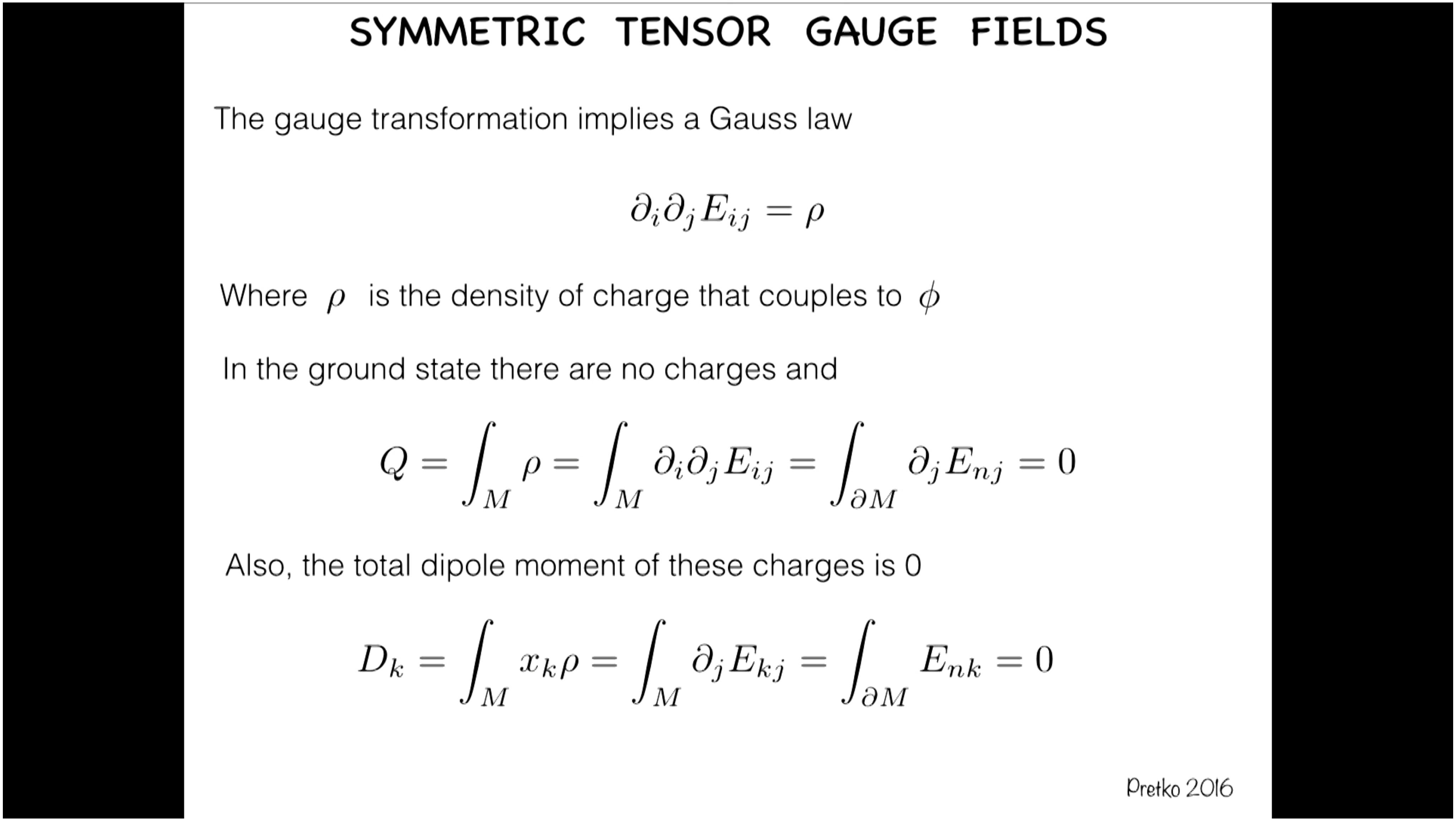
Thus we have assumed that there exists a density such that

$$\rho^i = x^i \rho$$

In ordinary elasticity such ρ happens to exist. It describes disclinations.

Kleinert 1983

Pretko Radzihovsky 2017 **AG** 2017



SYMMETRIC TENSOR GAUGE FIELDS

The gauge transformation implies a Gauss law

$$\partial_i \partial_j E_{ij} = \rho$$

Where ρ is the density of charge that couples to ϕ

In the ground state there are no charges and

$$Q = \int_M \rho = \int_M \partial_i \partial_j E_{ij} = \int_{\partial M} \partial_j E_{nj} = 0$$

Also, the total dipole moment of these charges is 0

$$D_k = \int_M x_k \rho = \int_M \partial_j E_{kj} = \int_{\partial M} E_{nk} = 0$$

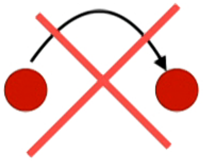
Pretko 2016

SYMMETRIC TENSOR GAUGE FIELDS

In ordinary Maxwell the dipole moment is not restricted

$$D_k = \int_M x_k \rho = \int_M x_k \partial_i E_i = \int_M E_k$$

Dipole constraint has dramatic consequences. Imagine a state with a charge



Charge cannot move because moving changes the dipole moment

Pretko 2016

SYMMETRIC TENSOR GAUGE FIELDS

However, a dipole can move, since quadrupole moment is not restricted

In elasticity there is another constraint

$$\int_M x^2\rho = \int_M E_i{}^i = \rho_{\text{defect}}$$

This constraint prohibits the second process

Pretko 2016

SYMMETRIC TENSOR GAUGE FIELDS

A hopping process is actually possible, but it is strange

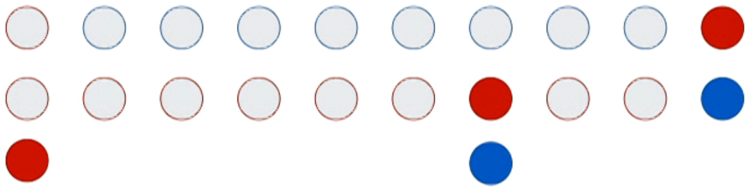
Say, I have a charge

SYMMETRIC TENSOR GAUGE FIELDS

Create a quadrupole from vacuum. (Not prohibited)

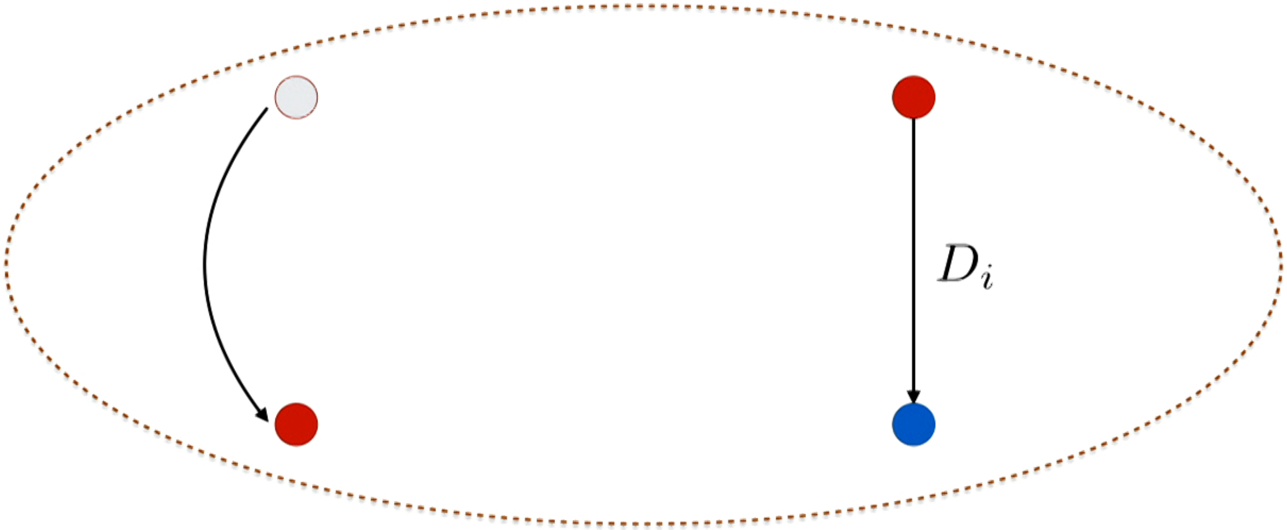
SYMMETRIC TENSOR GAUGE FIELDS

Repeat



SYMMETRIC TENSOR GAUGE FIELDS

Can hop by arbitrary distance, but will leave a ``scar'' of dipoles



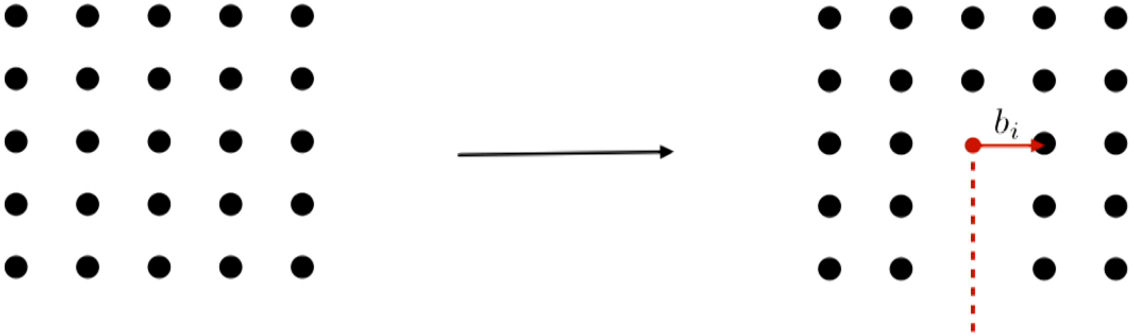
Dipole in the enclosed area is unchanged

Interlude I: Kinematics of elastic defects

Elasticity is very well studied. Surely we are not discovering anything new.

The r.h.s. of the Gauss law $\partial_i \partial_j E_{ij} = \rho$ describes the crystal defects

The dipoles are dislocations. They are characterized by a Burgers vector b_i

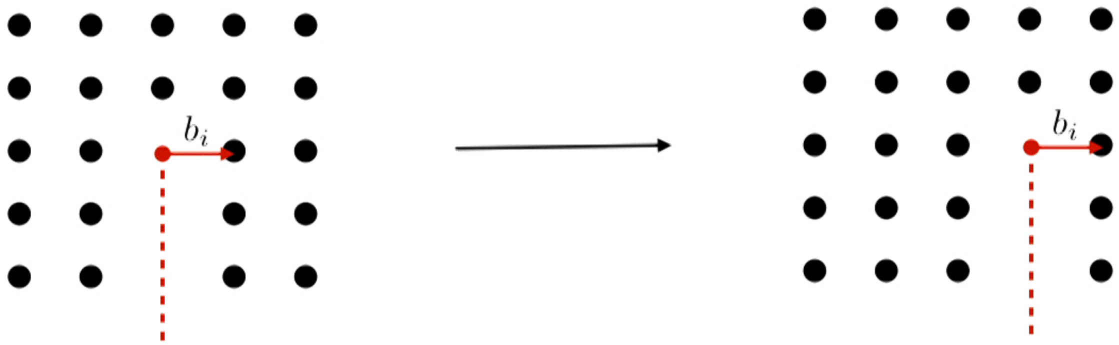


Dislocations can move only along b_i (aka glide).

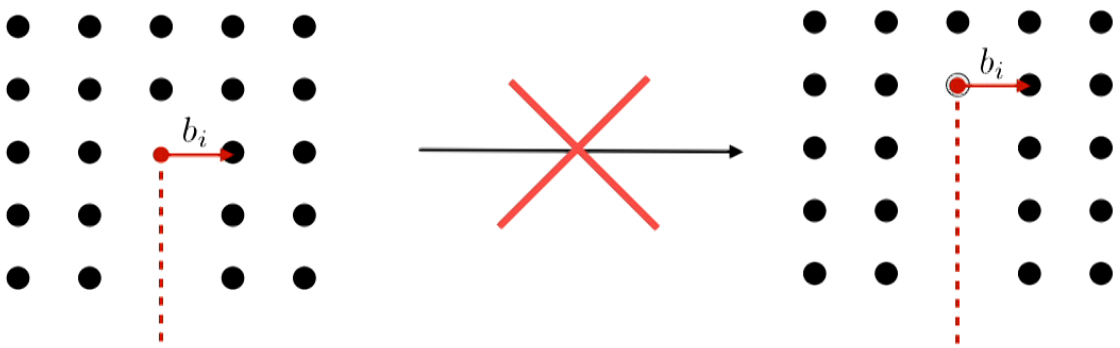
Moving perpendicular to b_i (aka climb) requires erasing lattice cites.

Interlude I: Kinematics of elastic defects

Glide:

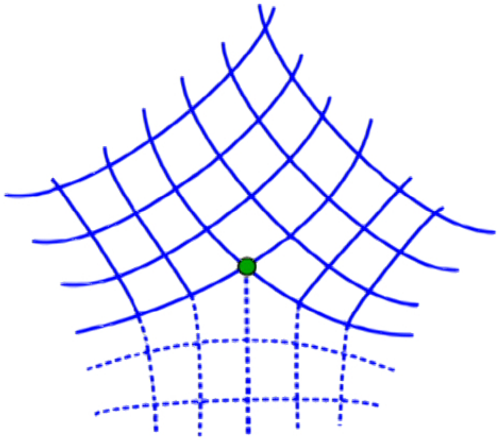


Climb:



Interlude I: Kinematics of elastic defects

The immobile fractons are disclinations



From Beekman et. al. 2017

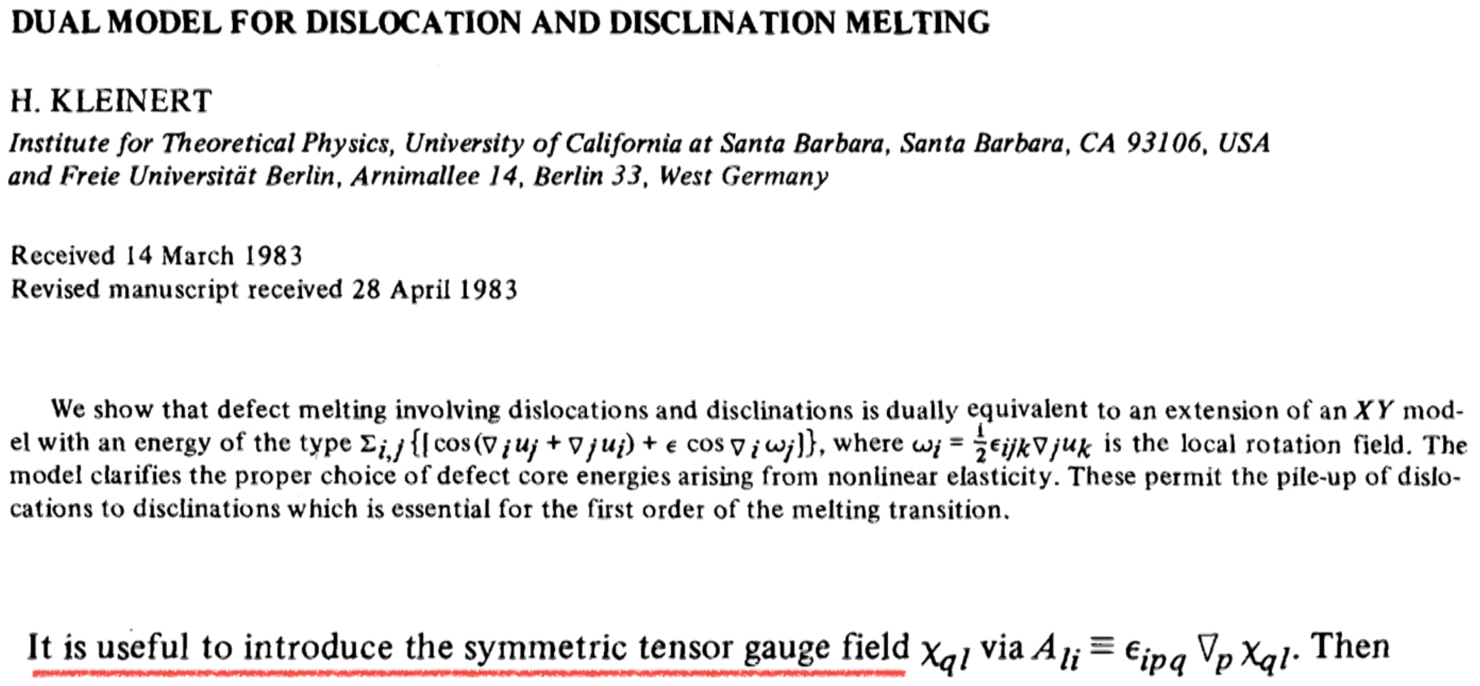
When a disclination moves, it leaves dislocations behind

$$\partial_\mu J^\mu_{\mathrm{i}} = \epsilon_{ij}\Theta^j$$

Dislocation can be viewed as a dipole of disclinations

Historic note

Elasticity has been studied using duality transformations is a series of papers by Kleinert in early 1980s, where he first introduced symmetric tensor gauge theories



DUAL MODEL FOR DISLOCATION AND DISCLINATION MELTING

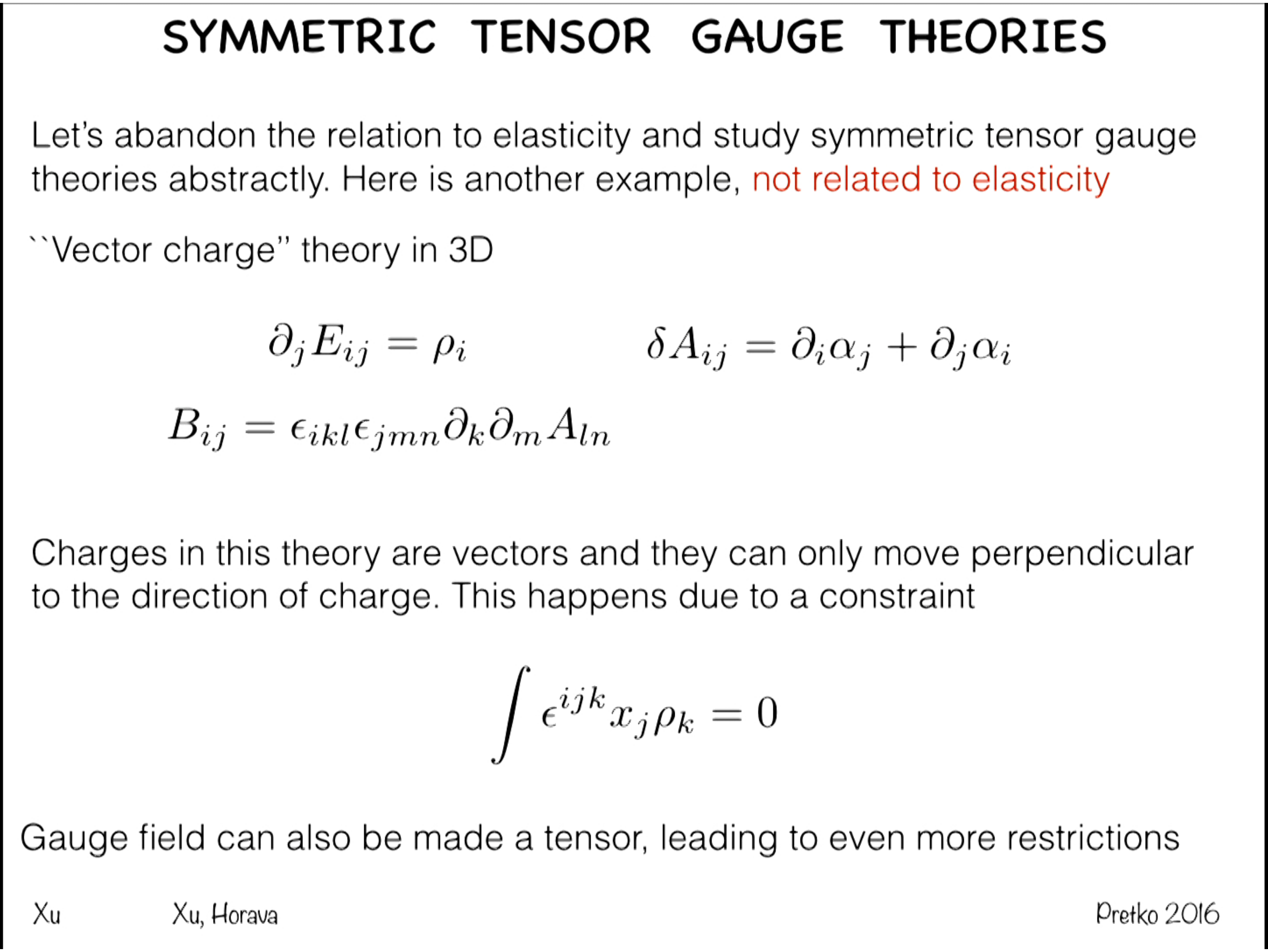
H. KLEINERT
Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, CA 93106, USA
and Freie Universität Berlin, Arnimallee 14, Berlin 33, West Germany

Received 14 March 1983
Revised manuscript received 28 April 1983

We show that defect melting involving dislocations and disclinations is dually equivalent to an extension of an XY model with an energy of the type $\Sigma_{i,j}\{[\cos(\nabla_i u_j + \nabla_j u_i) + \epsilon \cos \nabla_i \omega_j]\}$, where $\omega_i = \frac{1}{2}\epsilon_{ijk}\nabla_j u_k$ is the local rotation field. The model clarifies the proper choice of defect core energies arising from nonlinear elasticity. These permit the pile-up of dislocations to disclinations which is essential for the first order of the melting transition.

It is useful to introduce the symmetric tensor gauge field χ_{ql} via $A_{li} \equiv \epsilon_{ipq} \nabla_p \chi_{ql}$. Then

Kleinert 1983



SYMMETRIC TENSOR GAUGE THEORIES

Let's abandon the relation to elasticity and study symmetric tensor gauge theories abstractly. Here is another example, not related to elasticity

``Vector charge'' theory in 3D

$$\partial_j E_{ij} = \rho_i \qquad \delta A_{ij} = \partial_i \alpha_j + \partial_j \alpha_i$$

$$B_{ij} = \epsilon_{ikl}\epsilon_{jmn}\partial_k \partial_m A_{ln}$$

Charges in this theory are vectors and they can only move perpendicular to the direction of charge. This happens due to a constraint

$$\int \epsilon^{ijk} x_j \rho_k = 0$$

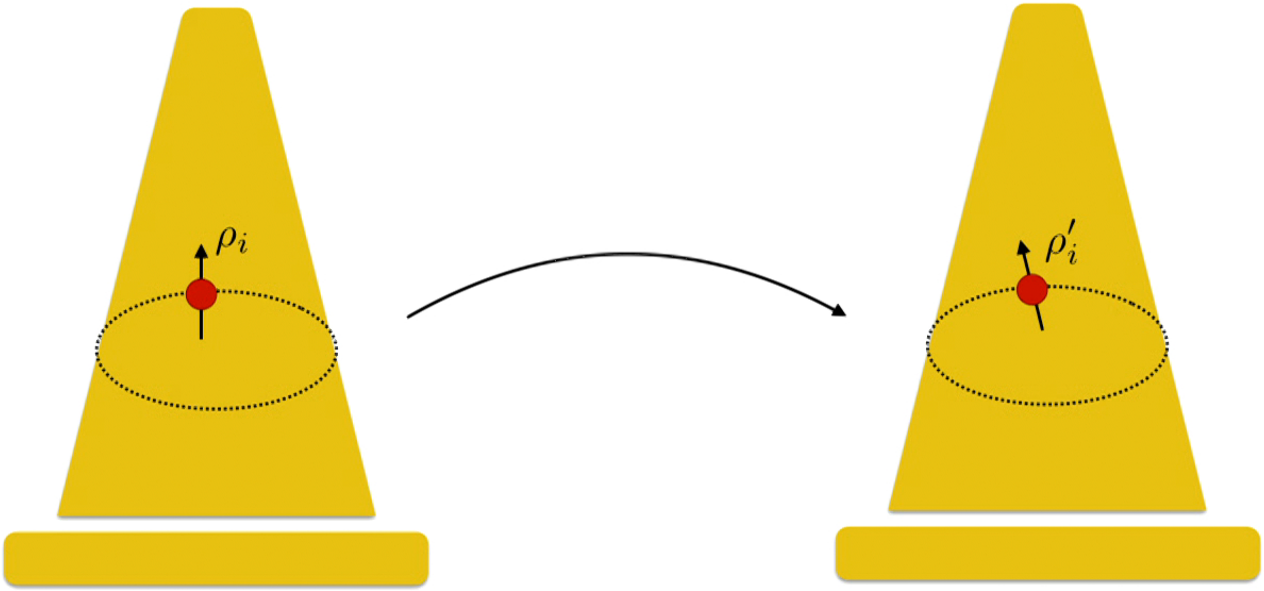
Gauge field can also be made a tensor, leading to even more restrictions

Xu Xu, Horava Pretko 2016

INTERLUDE II: STGTS IN CURVED SPACE

These theories share a common property. There is a conserved vector*.

We encounter an obvious problem in curved space.



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*Situation with CM theorem is not clear. These theories admit lattice realization and therefore exist (on a lattice).

INTERLUDE II: STGTS IN CURVED SPACE

Formally, the magnetic field ceases to be gauge invariant in curved space.

In curved space we replace all derivatives with covariant ones

$$B_i = \epsilon_{jk} \nabla_j A_{ki} \qquad \delta A_{ij} = \nabla_i \partial_j \alpha$$

Then

$$\delta B_i \propto [\nabla_i, \nabla_j] \partial_j \alpha \propto R_{ij} \partial_j \alpha$$

The general relationship between curved space and fractons is not simple. For example, in the 3D traceless ($E_i{}^i = 0$) scalar charge theory is doing fine on Einstein manifolds, while traceless vector charge theory is fine on Einstein manifolds of constant curvature.

Later I discuss more exotic theories, for which the relationship with curved space is presently unknown.

Slagle Prem Pretko 2018

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PART II:
TOWARDS EFT FOR TYPE-II PHASES

I have greatly benefited from the following works

Bulmash, Barkeshli arXiv:1806.01855

Pretko Phys. Rev. B 98, 115134

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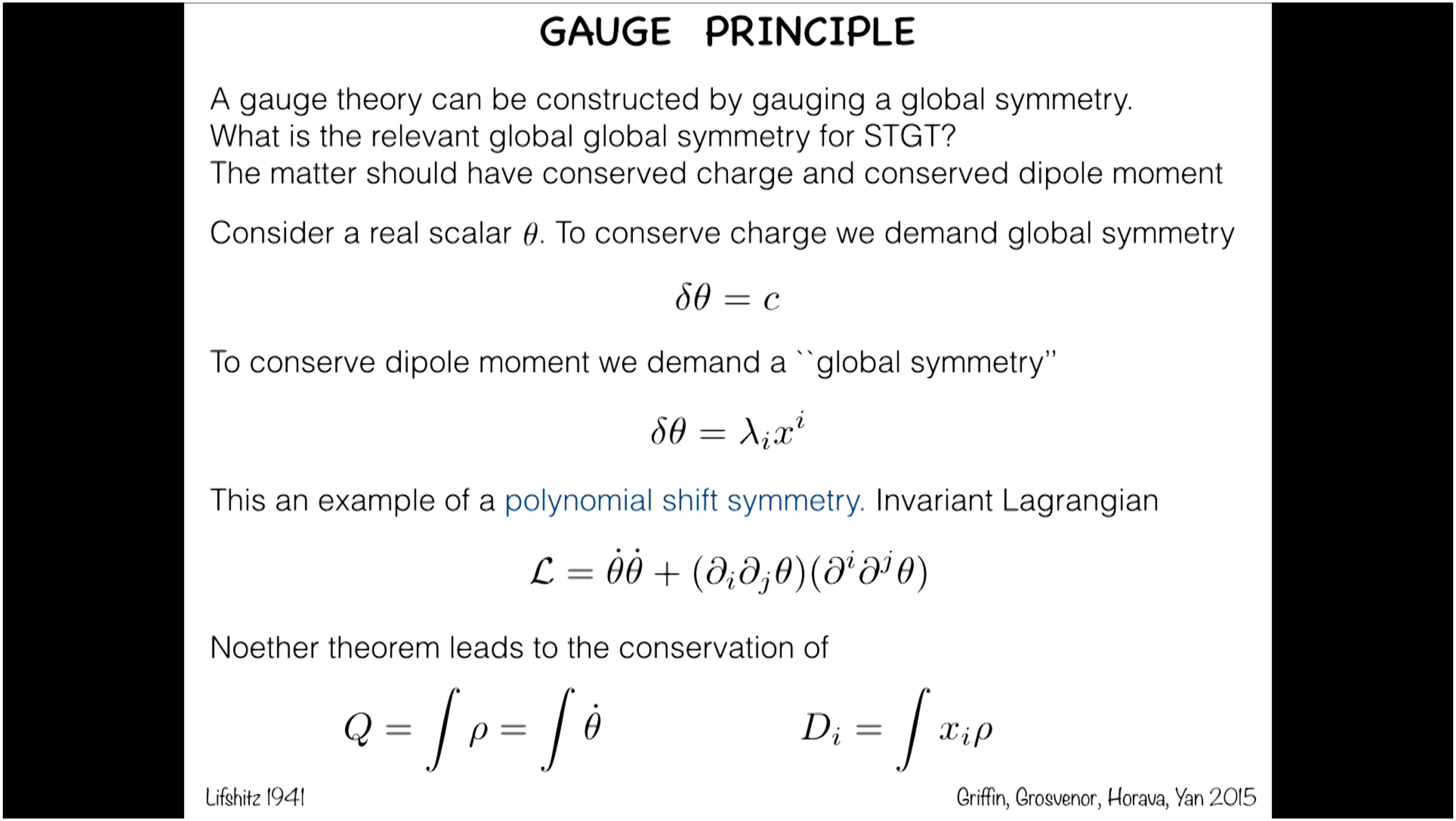


GAUGE PRINCIPLE

A gauge theory can be constructed by gauging a global symmetry.
What is the relevant global global symmetry for STGT?
The matter should have conserved charge and conserved dipole moment

Consider a real scalar θ. To conserve charge we demand global symmetry

$$\delta\theta = c$$

To conserve dipole moment we demand a ``global symmetry''

$$\delta\theta = \lambda_i x^i$$

This an example of a polynomial shift symmetry. Invariant Lagrangian

$$\mathcal{L} = \dot{\theta}\dot{\theta} + (\partial_i\partial_j\theta)(\partial^i\partial^j\theta)$$

Noether theorem leads to the conservation of

$$Q = \int \rho = \int \dot{\theta} \qquad\qquad D_i = \int x_i \rho$$

Lifshitz 1941

Griffin, Grosvenor, Horava, Yan 2015

GAUGE PRINCIPLE

Gauging such symmetry amounts to replacing

$$\delta\theta = c(x,t)$$

Introduce covariant derivatives

$$D\theta = \partial_i\partial_j\theta - A_{ij} \qquad \delta A_{ij} = \partial_i\partial_j c$$

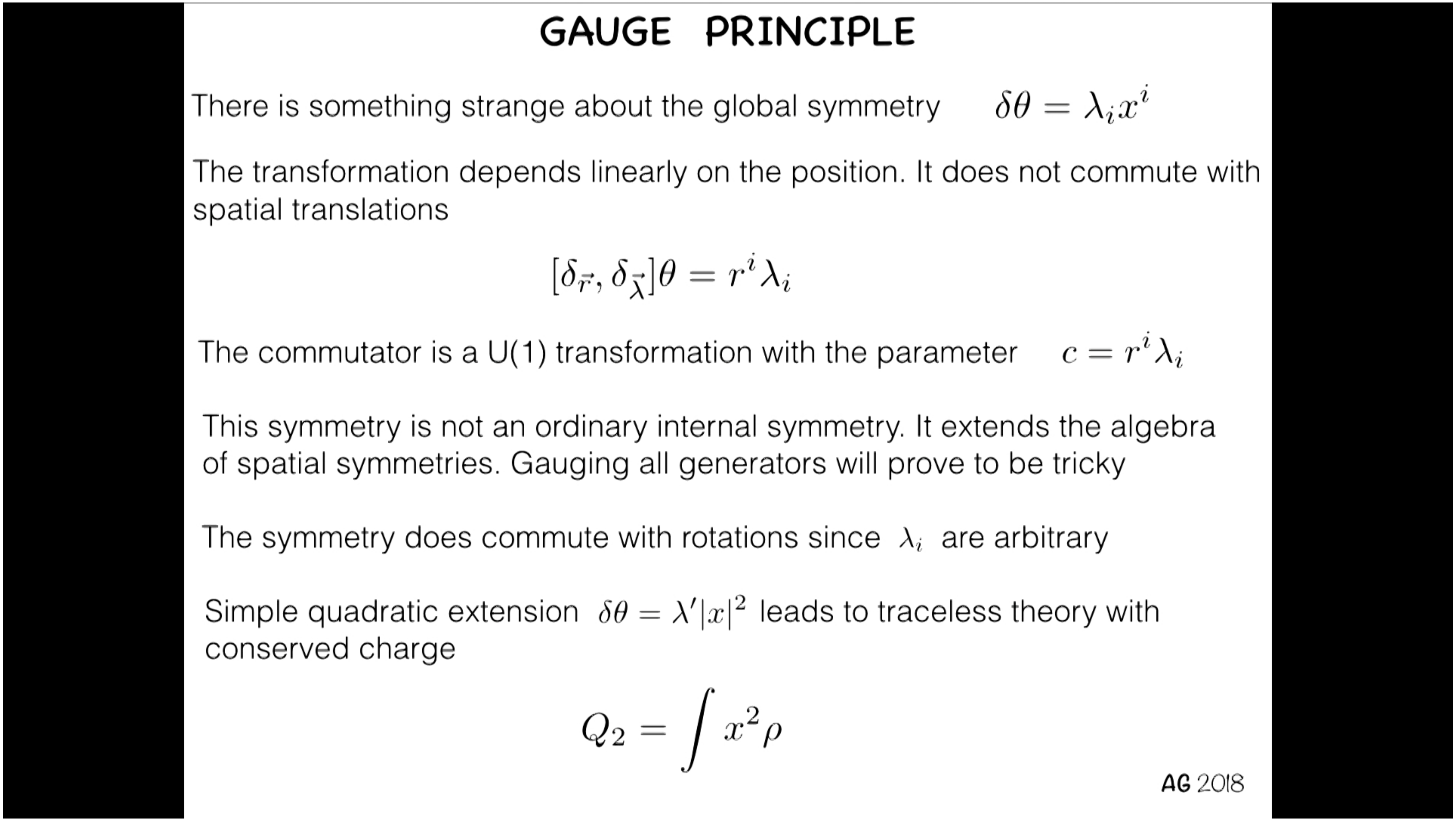
Allowing A_{ij} to fluctuate and providing generalized Maxwell terms we get

$$\mathcal{L} = (\dot{\theta} - \phi)^2 + (D\theta)^2 + E_{ij}E^{ij} + B_i B^i$$

This is what we called scalar charge theory. The Gauss law is

$$\partial_i\partial_j E_{ij} = \rho$$

Pretko 2018 **AG** 2018



GAUGE PRINCIPLE

There is something strange about the global symmetry $\delta\theta = \lambda_i x^i$

The transformation depends linearly on the position. It does not commute with spatial translations

$$[\delta_{\vec{r}}, \delta_{\vec{\lambda}}]\theta = r^i\lambda_i$$

The commutator is a U(1) transformation with the parameter $c = r^i\lambda_i$

This symmetry is not an ordinary internal symmetry. It extends the algebra of spatial symmetries. Gauging all generators will prove to be tricky

The symmetry does commute with rotations since λ_i are arbitrary

Simple quadratic extension $\delta\theta = \lambda'|x|^2$ leads to traceless theory with conserved charge

$$Q_2 = \int x^2\rho$$

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MULTIPOLE ALGEBRA

These symmetry algebra is an example of a more general multipole algebra

$$\delta\theta = \sum_{a,I_a} \lambda_{I_a} P_{I_a}(x)$$

Where $P_{I_a}(x)$ are homogeneous polynomials of degree a

These global symmetries lead to conservation of components of the multipole moments of the charge density

$$Q_{I_a} = \int P_{I_a}(x)\rho$$

If these polynomials are chosen ``at random'' then the symmetry will be incompatible with all spatial symmetries.

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CONSISTENCY WITH SPATIAL SYMMETRIES

Consistency with translation can be phrased as follows

$$P_{I_a}(x+r) - P_{I_a}(x) = \sum_{I_{a-1}} \alpha_{I_{a-1}} P_{I_{a-1}}(x)$$

This means that commutator of translation in the direction $\vec{r}$ and $P_{I_a}(x)$ must be a linear superposition of polynomials shifts of lower degree.
It may happen that only translations in some directions will satisfy that.

Similarly for rotations

$$P_{I_a}({R_i}^j x^i) = \sum_{I_a} \beta_{I_a} P_{I_a}(x)$$

Again, not all rotations will be compatible

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CONSISTENCY WITH SPATIAL SYMMETRIES

These consistency conditions are a consequence of the non-trivial transformation law of multipole moments under translations and rotations.

These laws take form

Dipole

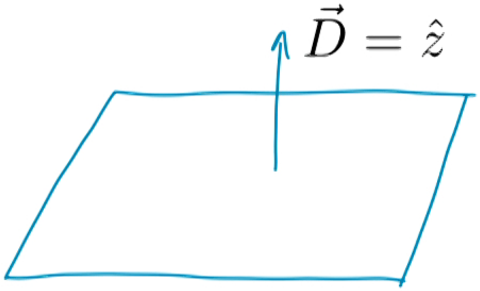
$$\delta_{\vec{r}} D_i = r_i Q$$

Quadrupole

$$\delta_{\vec{r}} D_{ij} = r_i r_j Q + r_i D_j + r_j D_i$$

Multipole moments are only translation invariant if all lower moments vanish

More subtle compromises are possible



$$\delta_{\vec{r}} D_{ij}\Big|_{x-y \text{ plane}} = 0$$

We have a 3D system with a 2D translation symmetry

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2D EXAMPLE

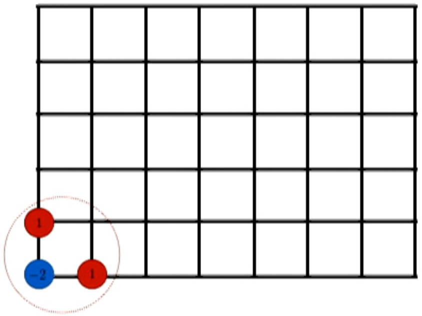
Consider a system where dipole moment in (1,-1) direction and the $Q_{11} + Q_{22} - 2Q_{12}$ component of the quadrupole moment is conserved

This means that the states with dipole $\vec{D} = D_0(1,1)$ are allowed. Note

$$\delta_{\vec{r}}(Q_{11} + Q_{22} - 2Q_{12}) = 0$$

for any translation $\vec{r}$. Translation (but not rotational) symmetry is retained.

This type of systems is easier to study on a lattice



The dipole is mobile, but separating the charges is hard

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2D EXAMPLE

The matter Lagrangian is

Dimension of length

$$\mathcal{L} = \dot{\theta}^2 + (D_1\theta)^2 + \lambda(D_2\theta)^2 + \lambda'(D_3\theta)^2$$

where the derivatives are

$$D_1\theta = \partial_x\theta + \partial_y\theta \qquad D_2\theta = (\partial_x^2 + \partial_x\partial_y)\theta \qquad D_3\theta = (\partial_y^2 + \partial_x\partial_y)\theta$$

These are invariant under the global symmetry

$$\delta\theta = c_0 + c_1(x-y) + c_2(x-y)^2$$

Dispersion relation is

$$\omega = |k_1 + k_2|\left[1 + \lambda\left(k_1^2 + \frac{\lambda'}{\lambda}k_2^2\right)\right]^{\frac{1}{2}}$$

Low energy physics is concentrated on a line $k_1 = -k_2$

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2D MULTIPOLE GAUGE THEORY

To gauge the symmetry I make the transformation local $\delta\theta = c(x)$

$$\mathcal{L} = \dot{\theta}^2 + (D_1\theta - A_1)^2 + \lambda(D_2\theta - A_2)^2 + \lambda'(D_3\theta - A_3)^2$$

The gauge transformations are

$$\delta A_1 = D_1 c \qquad \delta A_2 = D_2 c \qquad \delta A_3 = D_3 c$$

Gauss law

$$D_1^\dagger E_1 + D_2^\dagger E_2 + D_3^\dagger E_3 = \rho$$

where $D^\dagger$ is obtained from D via integrating by parts

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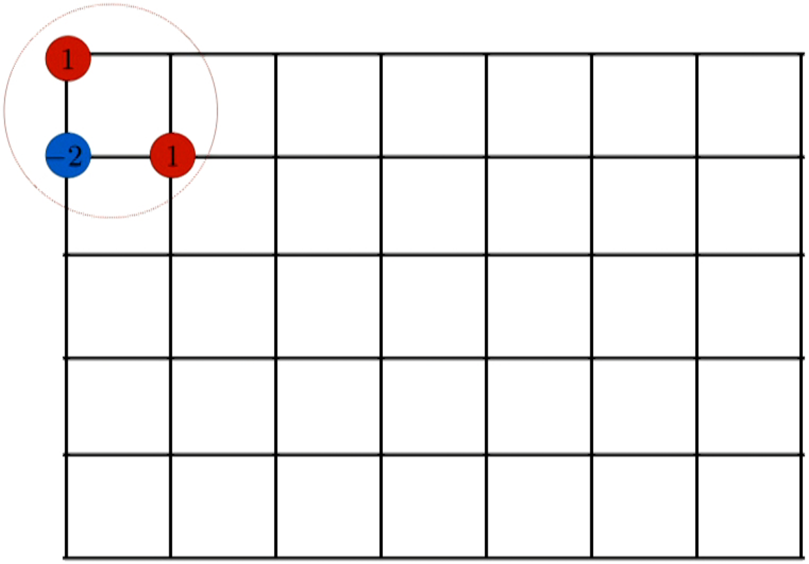
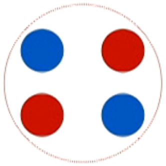
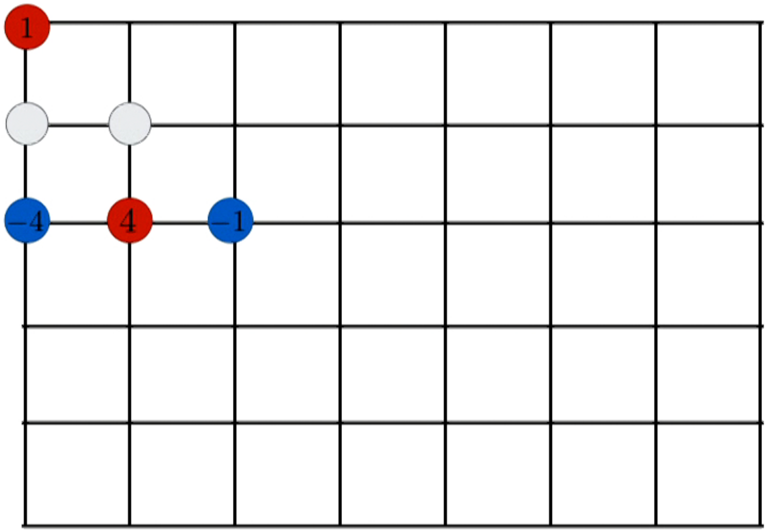
TYPE-II BEHAVIOR

Imagine creating a (1,1) ``particle”

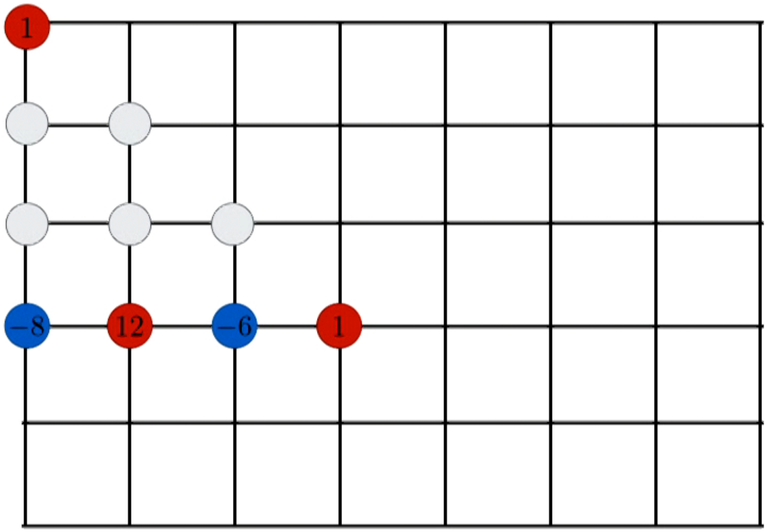


Analogue of  in dipole conserving theory

TYPE-II BEHAVIOR



TYPE-II BEHAVIOR



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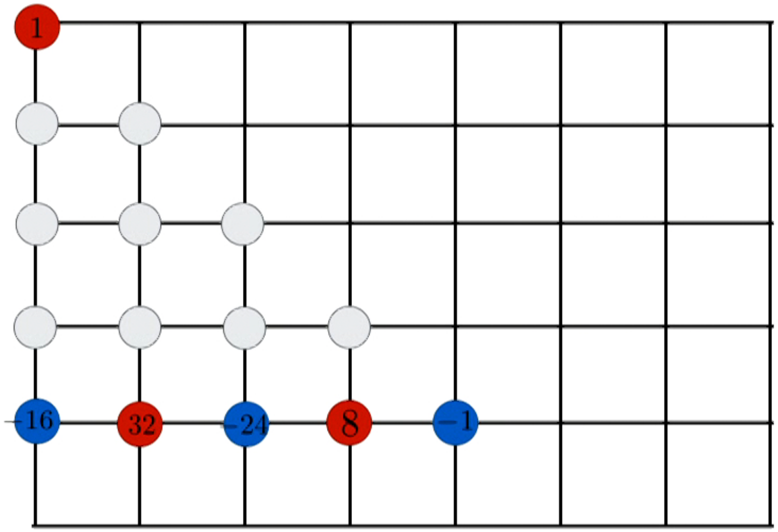

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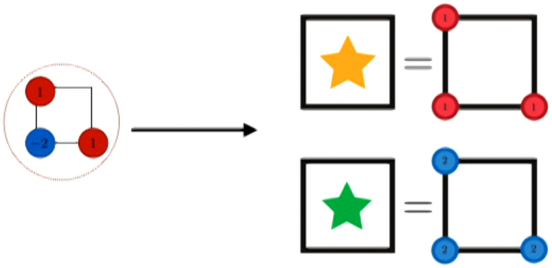
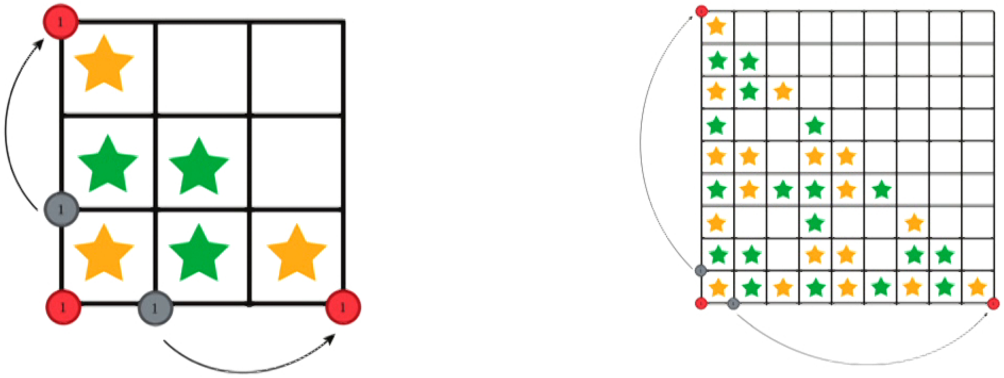
2D EXAMPLE WITH FRACTALS

Consider condensing charge 3 particles. In this case charge is defined mod 3.

In this case charge -2 particle is equivalent to charge 1



Which leads to fracton moving in a coherent fashion along a fractal space



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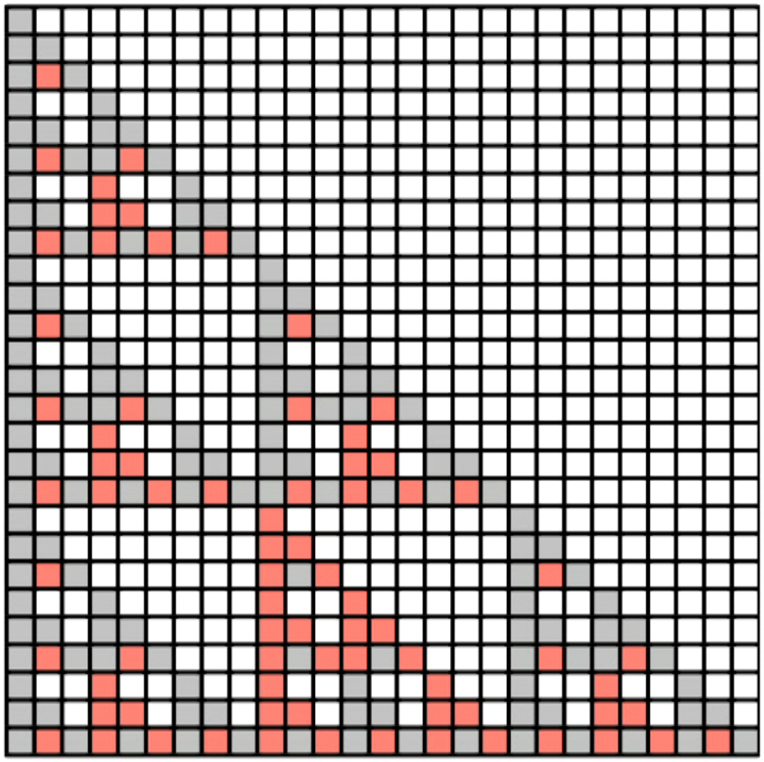
(GENERALIZED) SIERPINSKI TRIANGLE

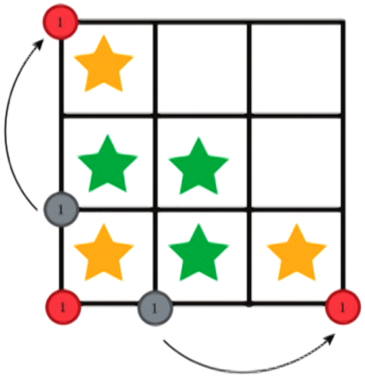


□ = 0 ▨ = 1 🟥 = 2

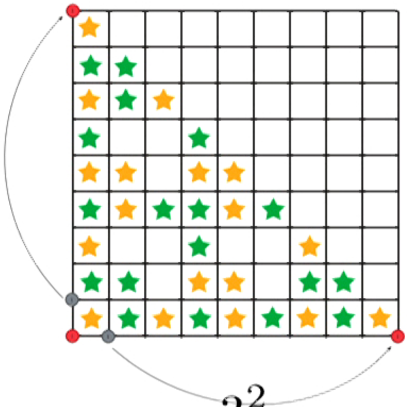
by Beni Yoshida

(GENERALIZED) SIERPINSKI TRIANGLE

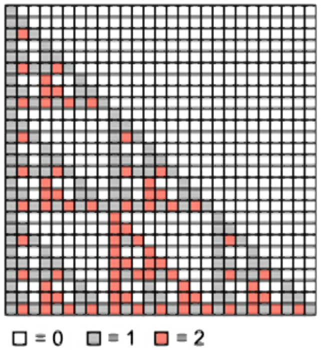
by Beni Yoshida



3^1



3^2



3^3

If we imagine periodic boundary conditions then a non-local ``Wilson fractal'' operator is only possible for the system sizes are $3^k \times 3^k$

This system is gapped, but not topologically ordered

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HAAH'S CODE

We consider a U(1) version first

$$\delta\theta = c_0 + c_1^1(x_1 - x_2) + c_2^1(x_1 + x_2 - 2x_3) + c_1^2(x_1 - x_2)(x_1 + x_2 - 2x_3) + c_2^2(2x_1 - x_2 - x_3)(x_2 - x_3)$$

Invariant Lagrangian

$$\mathcal{L} = \dot{\theta}^2 + (D_1\theta)^2 + \lambda(D_2\theta)^2 + \lambda'(D_3\theta)^2$$

Where the derivatives are

$$D_1\theta = (\partial_x + \partial_y + \partial_z)\theta \qquad D_2\theta = (\partial_x^2 + \partial_y^2 + \partial_z^2)\theta$$

$$D_3\theta = (\partial_x\partial_y + \partial_y\partial_z + \partial_x\partial_z)\theta$$

Slightly different theory found by

Barkeshli Bulmash 2017

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HAAH'S CODE

The theory has a whole zoo of symmetries

- Dipole(1,-1,0) and (1,1,-2) is conserved, and so are the quadrupoles

$$Q_{11} - Q_{22} - 2Q_{13} + 2Q_{23} \qquad Q_{33} - Q_{22} + 2Q_{12} - 2Q_{13}$$

- Translation invariance in all directions
- SO(2) Rotational invariance in (1,-1,0)-(1,1,2)-plane

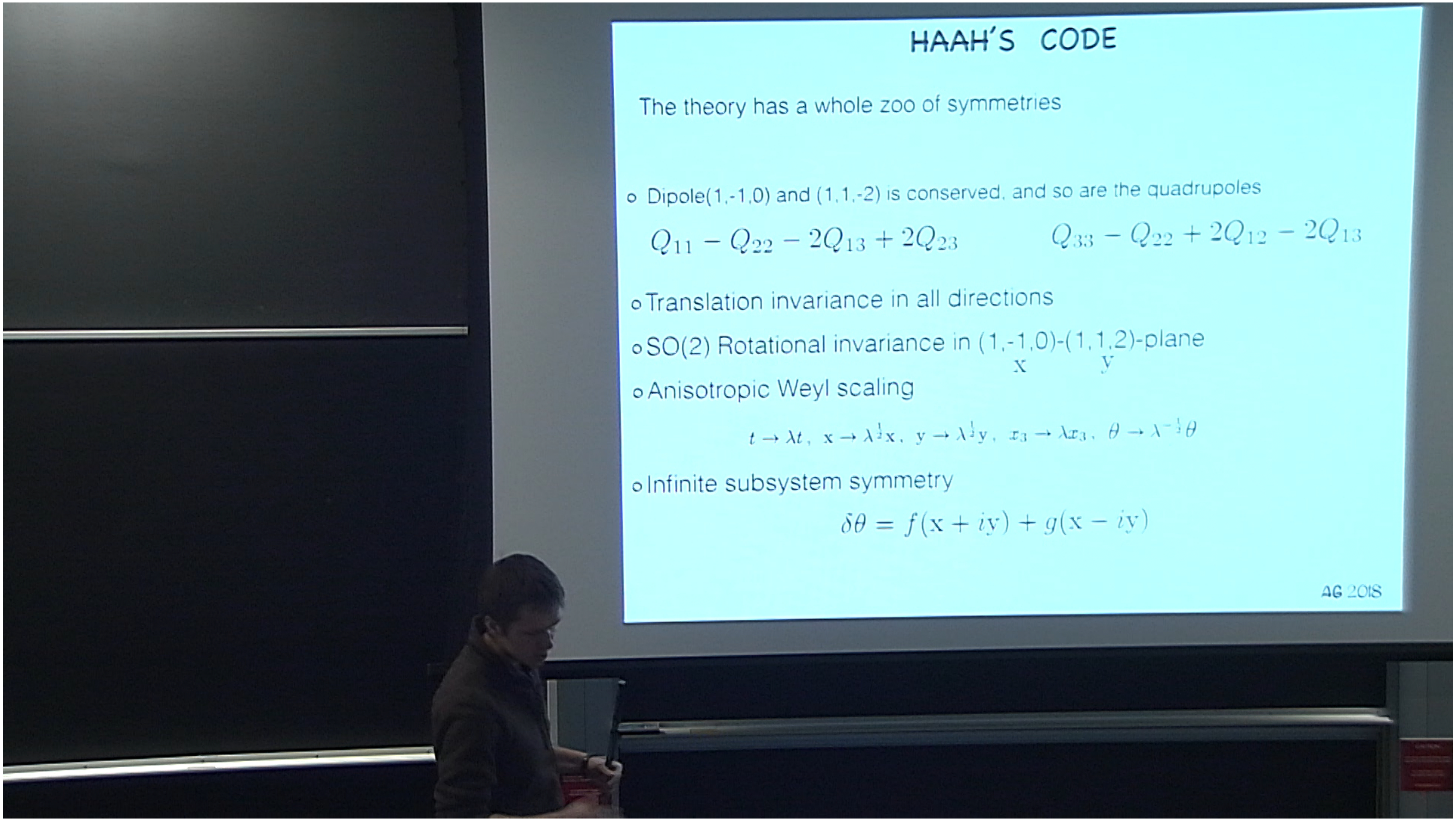
x y

- Anisotropic Weyl scaling

$$t \to \lambda t\,,\ \mathrm{x} \to \lambda^{\frac{1}{2}}\mathrm{x}\,,\ \mathrm{y} \to \lambda^{\frac{1}{2}}\mathrm{y}\,,\ x_3 \to \lambda x_3\,,\ \theta \to \lambda^{-\frac{1}{2}}\theta$$

- Infinite subsystem symmetry

$$\delta\theta = f(\mathrm{x} + i\mathrm{y}) + g(\mathrm{x} - i\mathrm{y})$$



HAAH'S CODE

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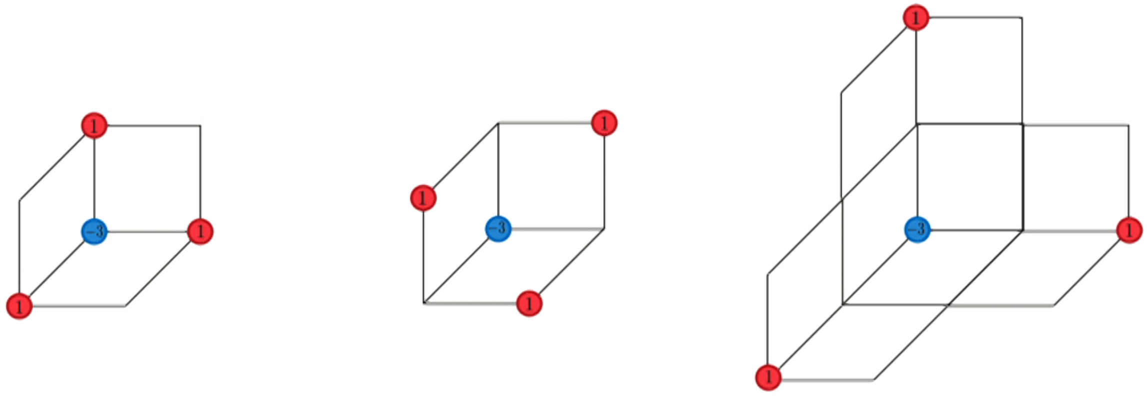
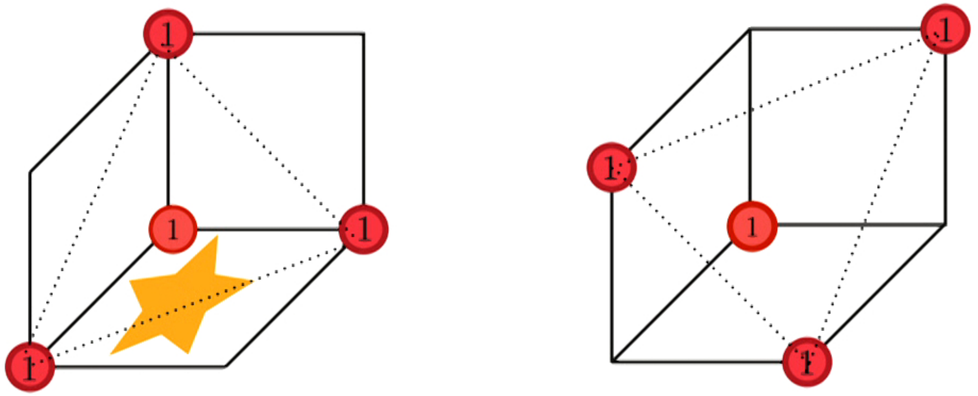
HAAH'S CODE



The original $\mathbb{Z}_2$ Haah code is obtained by condensing charge-2 objects



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RESUME

- Fractons are particles with restricted mobility
- Their dynamics is described by theories with conserved multipole moments
- Gauging such symmetries leads to gauge theories of symmetric tensors
- Physical realizations: quantum elasticity and spin liquids
- Type-II models can be described by generalized polynomial shift symmetries
- Such models are inherently anisotropic
- They also exhibit dimensional reduction at low energies
- Gauging the symmetries leads to multipole gauge theories
- Condensing charges leads to fractal operators