

Title: Theory of entanglement phase transitions and natural error correction in quantum circuits with measurement

Speakers: Ehud Altman

Collection: Quantum Matter: Emergence & Entanglement 3

Date: April 23, 2019 - 3:00 PM

URL: <http://pirsa.org/19040101>

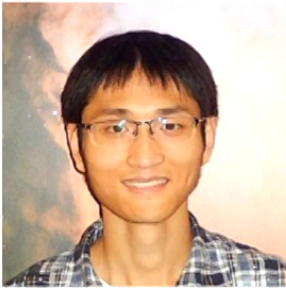
# UNIVERSAL OPERATOR GROWTH AND EMERGENT HYDRODYNAMICS IN QUANTUM SYSTEMS

Ehud Altman – Berkeley

Xiangyu Cao,

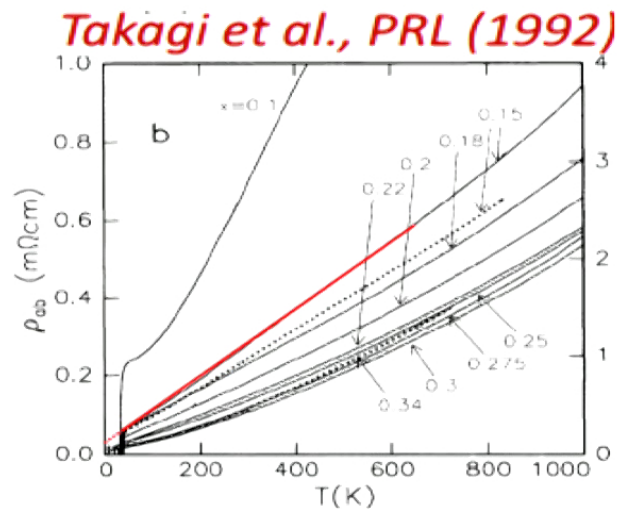
Daniel Parker,

Thomas Scaffidi

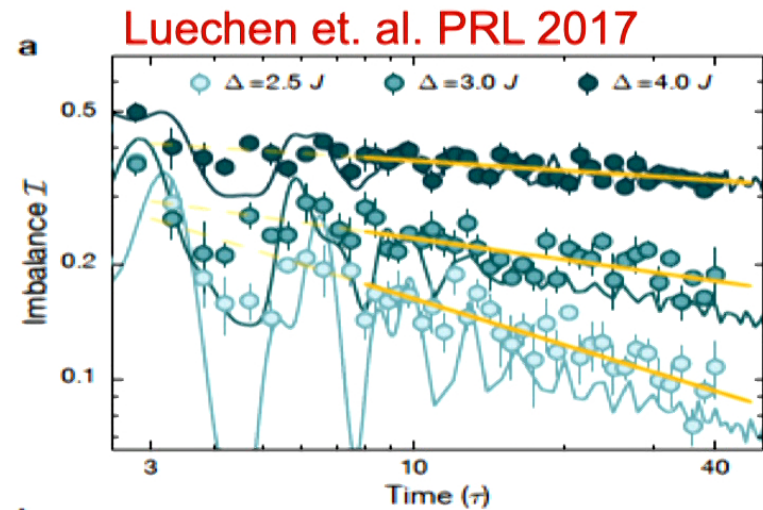


# How to compute dynamical properties of strongly coupled quantum matter ?

Unconventional transport:  
(Strange metals)



Relaxation following a quench in cold atomic systems:



Often boils down to computing time dependent correlations:

$$C(t) = \langle \mathcal{O}(t)\mathcal{O} \rangle$$

## Quantum Mechanics

Microscopic description of the system.

**Example:** Chaotic Ising Model

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i$$

**Correlation functions:**

$$C(t) = \langle \mathcal{O}(t, x) \mathcal{O}(0) \rangle$$

**Hard Solution:** Hamiltonian dynamics

$$\mathcal{O}(t) = e^{-iHt} \mathcal{O} e^{iHt}.$$

Exact and **reversible** dynamics.

?



## Hydrodynamics

Macroscopic description of quantum systems as classical PDEs.

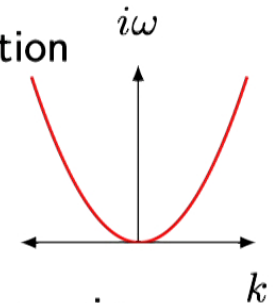
**Example:** Diffusion of energy

$$\frac{\partial}{\partial t} \varepsilon(t, x) = D \nabla^2 \varepsilon(t, x) + \nabla f,$$

with  $D$  diffusion,  $f$  thermal noise.

**Easy Solution:** Green's function

$$G(i\omega, k) = \frac{1}{i\omega + Dq^2}$$



Approximate & **irreversible** dynamics.

# Quantum evolution is hard to compute

Operators evolve in a huge Hilbert space:

$$-i \frac{d\hat{A}}{dt} = [H, \hat{A}] \quad \longleftrightarrow \quad -i \frac{d|A\rangle}{dt} = \mathcal{L}|A\rangle$$

$$(A|B) = \text{tr}(AB)$$

$$|A(t)\rangle = e^{-i\mathcal{L}t}|A\rangle$$

Spin-1/2 models:

$$H = \sum_{\langle ij \rangle} h_{\alpha\beta} \sigma_i^\alpha \sigma_j^\beta$$

Basis of “Pauli strings:

$$\sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \dots \otimes \sigma^{\alpha_N} \equiv |\alpha\rangle$$

$$\alpha_i = 0, 1, 2, 3$$

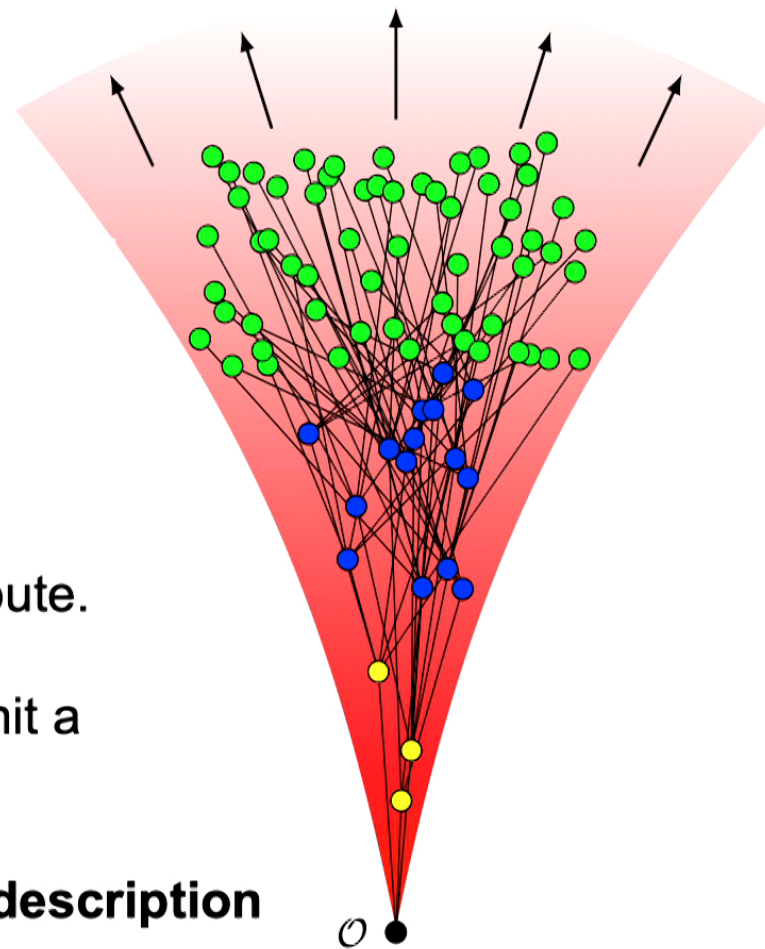
# Quantum evolution is hard to compute

$$C(t) = (\mathcal{O} | e^{i\mathcal{L}t} | \mathcal{O})$$

$$\mathcal{O}(t) = e^{i\mathcal{L}t} \mathcal{O} = \mathcal{O} + (it)\mathcal{L}\mathcal{O} + (it)^2\mathcal{L}^2\mathcal{O} + \dots$$

## The basic idea

- Operators flow from simple to complex eventually becoming too complex to compute.
- Sufficiently complex operators should admit a *universal statistical description*.
- **Our goal is to formulate this universal description**



# Outline

- Background: Krylov sub-space and operator complexity
- **A hypothesis for universal operator growth**
- Evidence for the hypothesis:
  - (i) Numerical (Spin chains)
  - (ii) Analytical (SYK models)
- Application: generalized notion of chaos and the bound
- Application: computation of transport coefficients

# Krylov basis: folding the graph on a line

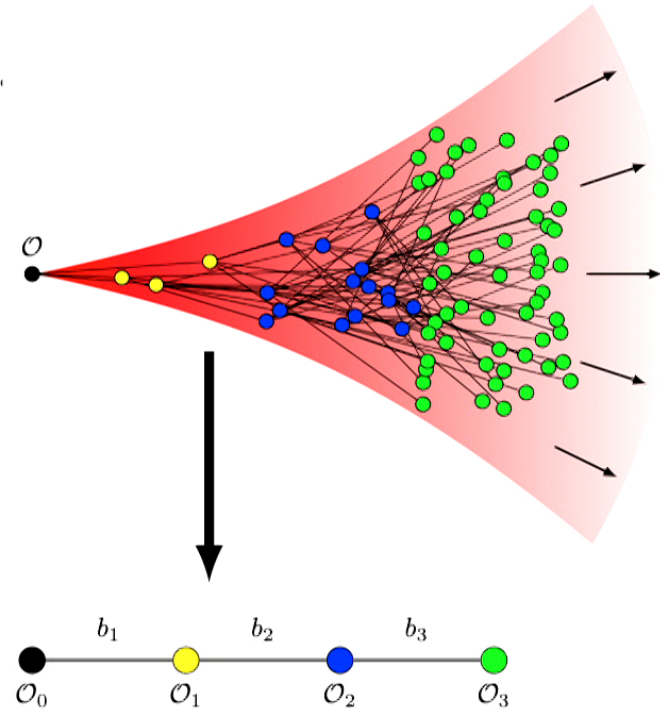
Generate orthonormal basis from successive application of  $\mathcal{L}$

$$|O\rangle \xrightarrow{\mathcal{L}} |O_1\rangle \xrightarrow{\mathcal{L}} |O_2\rangle \xrightarrow{\mathcal{L}} |O_3\rangle \dots$$

$$(\mathcal{O}_n | \mathcal{L} | \mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \dots \\ b_1 & 0 & b_2 & 0 & \dots \\ 0 & b_2 & 0 & b_3 & \dots \\ 0 & 0 & b_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

“Recursion Coefficients”

- Problem mapped to single-particle hopping on a semi-infinite chain !
- Krylov index  $\sim$  operator complexity

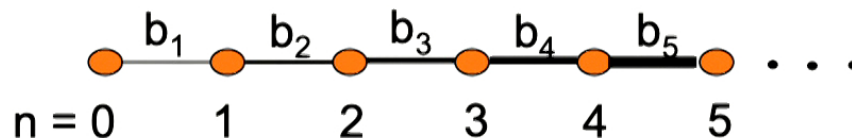


D.C. Mattis, 1981; Viswanath & Müller, *The Recursion Method*, 2008.

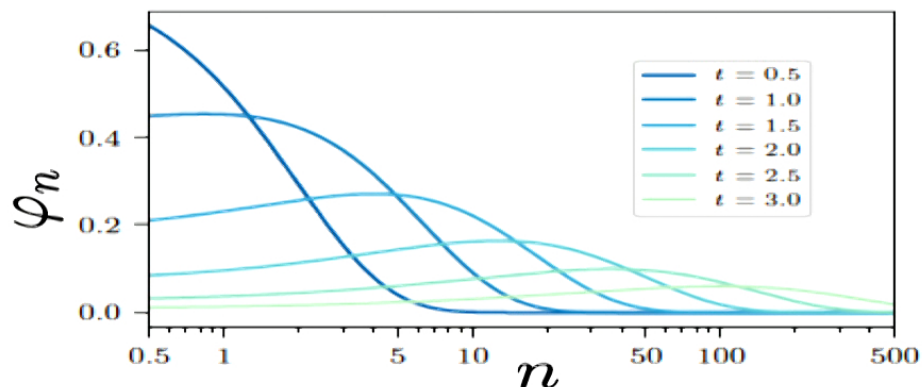


# “Operator wavefunction” in Krylov space

$$\varphi_n(t) = (\mathcal{O}_n | \mathcal{O}(t))$$



$$\partial_t \varphi_n = -b_{n+1} \varphi_{n+1} + b_n \varphi_{n-1}, \quad \varphi_n(0) = \delta_{n0}$$



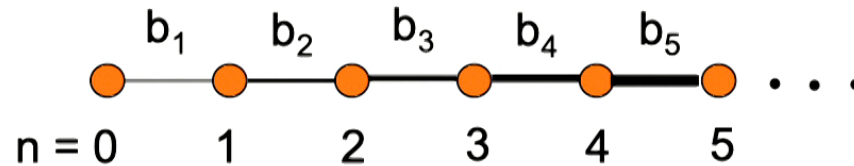
The autocorrelation function:

$$C(t) = \text{tr} [\mathcal{O}(t) \mathcal{O}] = \varphi_0(t)$$

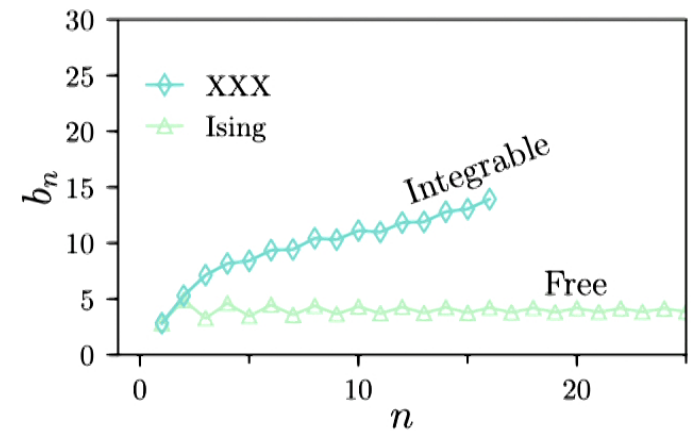
“Krylov-complexity”:

$$\langle n(t) \rangle = \sum_{n=0}^{\infty} |\varphi_n(t)|^2 n$$

# How do the recursion coefficients grow with $n$ ?



Asymptotic	Growth Rate	System Type
$b_n \sim O(1)$	constant	Free models
$b_n \sim O(\sqrt{n})$	square-root	Integrable models
$b_n \sim ???$	???	Chaotic models
$b_n \geq O(n)$	superlinear	Disallowed



D.C. Mattis, 1981; Viswanath & Müller, *The Recursion Method*, 2008.

# The hypothesis for generic models: linear growth of the reursion coefficients

Dan Parker, Xiangyu Cao, Thomas Scaffidi, EA arXiv:1812.08657

$$b_n = \alpha n + \beta, \quad n \rightarrow \infty$$

Logarithmic correction in 1d models:

$$b_n = \frac{\alpha n}{\log(n)} + \beta$$

(Theorem by Araki 1969 excludes faster growth. Thanks to Alex Advoshkin )

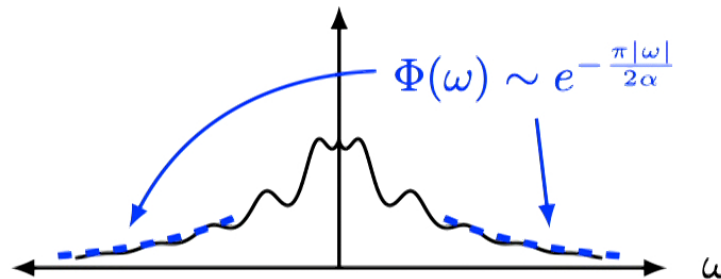
Faster asymptotic growth is not possible. We term  $\alpha$ , the “growth rate” of the operator for reasons that will become clear.

## Relation to spectral function

$$\Phi(\omega) = \int_{-\infty}^{\infty} dt C(t) e^{-i\omega t} = \int_{-\infty}^{\infty} dt \operatorname{tr} [\mathcal{O}(t) \mathcal{O}] e^{-i\omega t}$$

$$b_n = \alpha n + O(1) \iff \Phi(\omega) \sim e^{-\pi \frac{|\omega|}{2\alpha}}$$

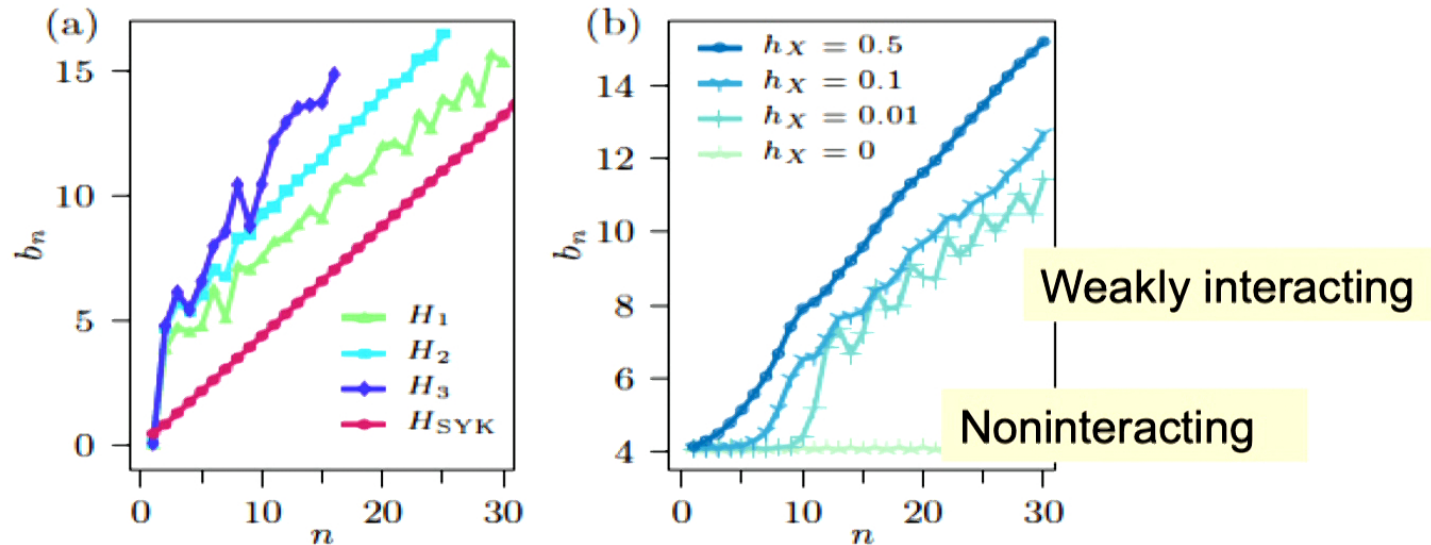
Or in 1d:  $b_n = \frac{\alpha n}{\log(n)} + O(1) \iff \Phi(\omega) \sim e^{-\pi \frac{|\omega|}{2\alpha} \log |\omega|}$



The operator “growth rate”  $\alpha$ , is directly related to the high frequency limit of the spectral function

# The evidence

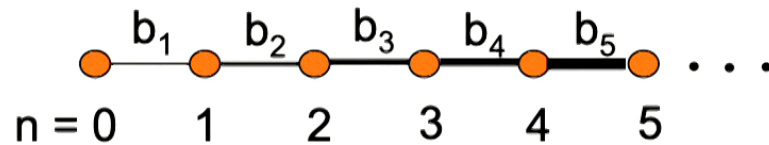
Numerical: Many distinct nonintegrable spin chains, SYK model



Analytical: SYK model in the limit of large  $q$  (infinite T)

$$b_n = J 2^{(1-q)/2} \sqrt{q n(n-1)} + O(1/q) \quad n \geq 2$$

# Phenomenology of the semi-infinite chain



Exactly solvable “universal” model:

$$\tilde{b}_n = \alpha \sqrt{n(n-1+\eta)} \xrightarrow{n \gg 1} \alpha n + \beta$$

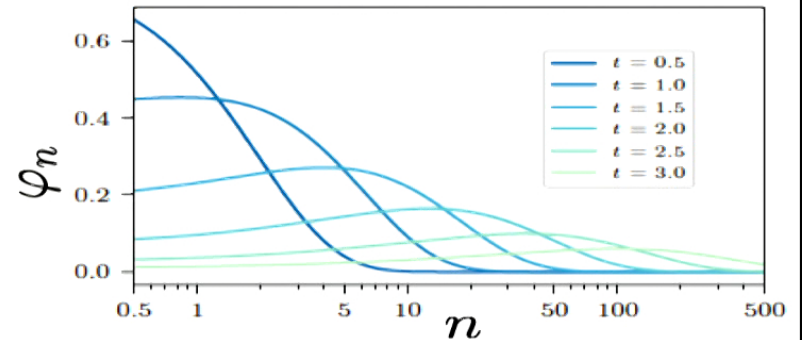
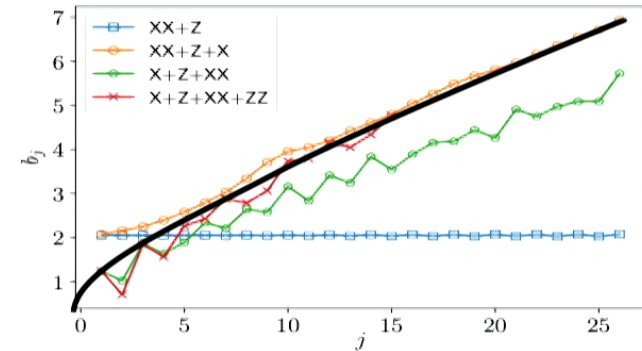


$$\langle n(t) \rangle = \eta \sinh(\alpha t)^2 \sim \eta e^{2\alpha t}$$

A general property of models with linear growth of recursion coeffs.

Operators grow as fast as they can.

What is the relation to quantum chaos ?




\* in 1d:  $\langle n(t) \rangle \sim e^{\sqrt{4\alpha t}}$

# Suggestive result from SYK model at infinite T

Both the recursion coefficients and the Lyapunov exponent can be computed exactly.  
(numerically for finite  $q$  and analytically in the large  $q$  limit)

$q$	2	3	4	7	10	$\infty$
$\alpha/\mathcal{J}$	0	0.461	0.623	0.800	0.863	1
$\lambda_L/(2\mathcal{J})$	0	0.454	0.620	0.799	0.863	1


$$\mathcal{J} = J 2^{(1-q)/2} \sqrt{q}$$

Taken from: Roberts, Stanford and Streicher JHEP 2018

We will now establish a precise connection.

## Define precise notion of complexity: Q-complexity

Positive semi-definite super-operator:  $\mathcal{Q} = \sum_a q_a |q_a\rangle\langle q_a|, q_a \geq 0$

Additional requirements :  $(q_b|\mathcal{L}|q_a) = 0$  if  $|q_a - q_b| > b$

$$(q_a|\mathcal{O}) = 0 \text{ if } q_a < d$$

(i.e.  $\mathcal{L}$  affects a bounded change of complexity and the initial operator complexity is small )

**Q-complexity:**  $(\mathcal{Q})_t := (\mathcal{O}(t)|\mathcal{Q}|\mathcal{O}(t))$



# Q-complexity - Examples

$$Q = \sum_a q_a |q_a\rangle\langle q_a|, \quad q_a \geq 0$$

$$(Q)_t := (\mathcal{O}(t) | Q | \mathcal{O}(t))$$

1. “Krylov” complexity

$$Q = \sum_{n=0}^{\infty} n |\mathcal{O}_n\rangle\langle \mathcal{O}_n|$$

2. Operator size

q-eigenvectors are Pauli strings

$$Q |IXYZI \dots\rangle = 3 |IXYZI \dots\rangle$$

3. OTOC

$$Q := \sum_i Q_i, \quad (A | Q_i | B) := ([V_i, A] | [V_i, B])$$

# A generalized bound of chaos

Theorem\*: the growth of any Q-complexity is bounded by the growth of the Krylov complexity

$$(Q)_t \leq C (n)_t$$

For proof see our paper [arXiv:1812.08657](https://arxiv.org/abs/1812.08657)

Implication: a generalized bound on chaos

$$\lambda_L \leq 2\alpha$$

Direct connection between the Lyapunov exponent and a property of an observable correlation function!

\*Extension of the notion and of the bound to finite T is still a conjecture

# Finite T Chaos bound

## Low T limit

High frequency spectral function dominated by thermal suppression:

$$\Phi_H(\omega) = \frac{\Phi(\omega)}{\sinh(\beta\omega/2)} \rightarrow e^{-\omega/\Lambda - \omega/2T} \sim e^{-\omega/2T}$$

→  $\lambda_L \leq 2\alpha = 2\pi T$

Recover the bound of  
Maldacena, Shenker & Stanford 2016

Tighter bound on chaos!  
SYK model saturates the tighter bound.

# Finite temperature

Freedom in defining the scalar product

To recover physical correlations:  $(A|B) = \text{tr}[\rho A^\dagger B] \equiv \langle A^\dagger B \rangle_\beta$

We will use a different scalar product:

$$(A|B)_H := \text{tr}[\sqrt{\rho} A^\dagger \sqrt{\rho} B] = \text{tr}[\rho A^\dagger B(i\beta/2)]$$

$$\rightarrow C_H(t) = \langle A^\dagger(t - i\beta/2) B \rangle_\beta = C(t - i\beta/2)$$

$$\rightarrow \Phi_H(\omega) = \frac{\Phi(\omega)}{\sinh(\beta\omega/2)}$$

# Finite T Chaos bound

## Low T limit

High frequency spectral function dominated by thermal suppression:

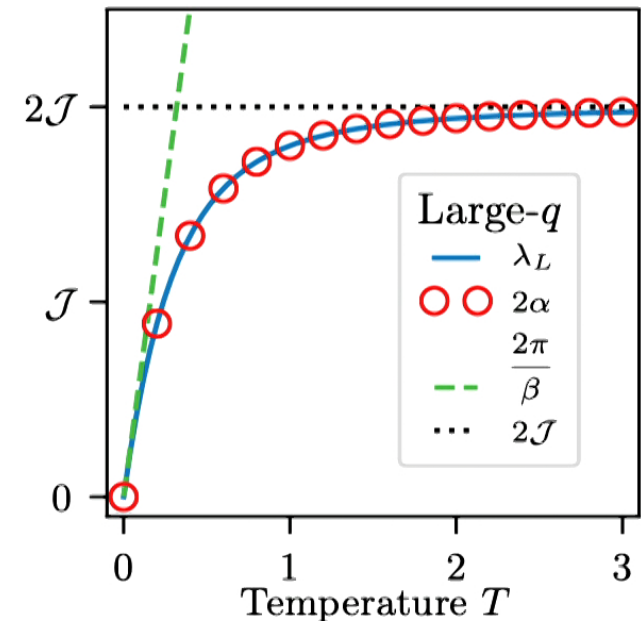
$$\Phi_H(\omega) = \frac{\Phi(\omega)}{\sinh(\beta\omega/2)} \rightarrow e^{-\omega/\Lambda - \omega/2T} \sim e^{-\omega/2T}$$

→  $\lambda_L \leq 2\alpha = 2\pi T$

Recover the bound of  
Maldacena, Shenker & Stanford 2016

From low to high T in the  
large q SYK model

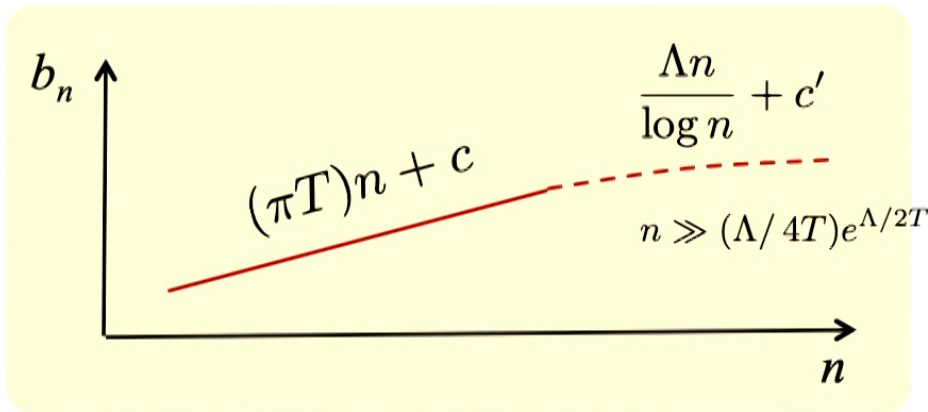
Tighter bound on chaos!  
SYK model saturates the tighter bound.



# Chaos bounds in the low T limit – 1d models

$$\Phi_H(\omega) \sim e^{-\frac{\omega}{\Lambda} \log\left(\frac{\omega}{\Lambda}\right) - \omega/2T}$$

Crossover frequency:  $\omega_* \sim \Lambda e^{\Lambda/2T}$



$$\langle n(t) \rangle \sim e^{2\pi T t}$$

for  $t < \Lambda/T^2$

So, even for 1d systems we recover:

$$\lambda_L \leq 2\alpha = 2\pi T$$

# Application to classical chaos

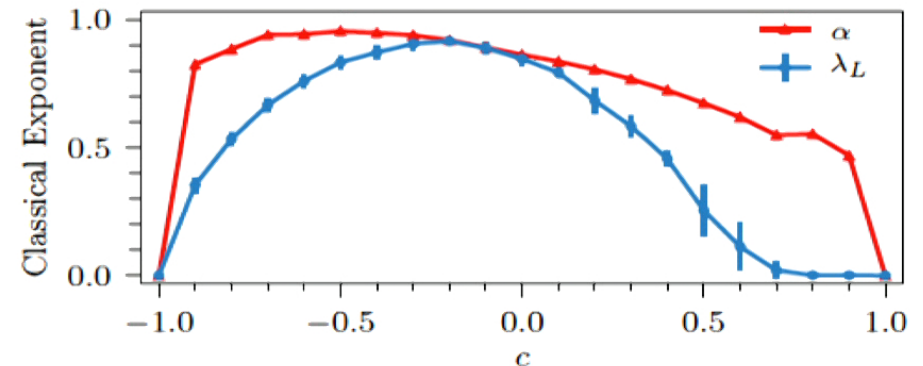
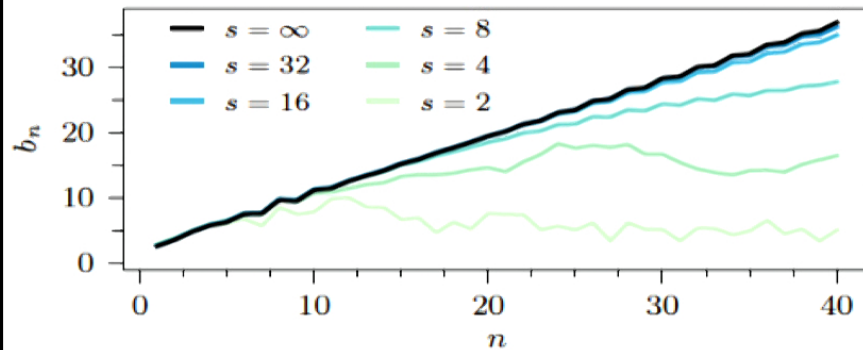
The framework carries over for classical dynamics with:

Liouvillian  $\longrightarrow \mathcal{L} = i\{\mathcal{H}, \cdot\}$

Operators  $\longrightarrow$  Functions on the classical phase space

Compare  $\alpha$  to  $\lambda_L$  Peres-Feingold model:

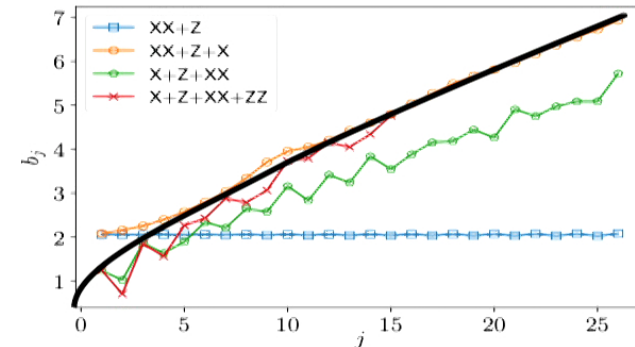
$$H_{\text{FP}} = (1 + c) [S_1^z + S_2^z] + 4s^{-1}(1 - c)S_1^x S_2^x$$



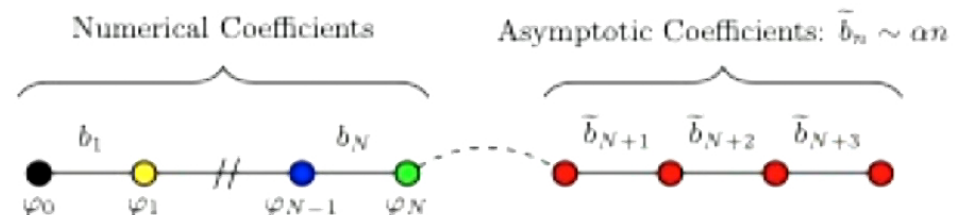
# Application: computing operator decay

## The basic idea:

1. Compute the first  $m$  recursion coefficients numerically.
2. Complete with the fitted “universal” model at larger values of  $n$



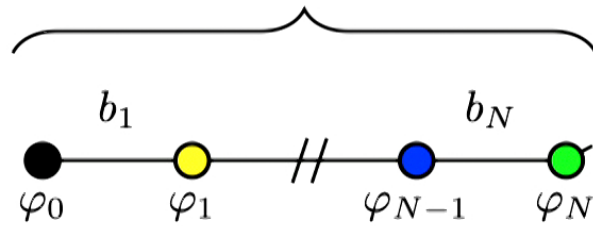
$$\tilde{b}_n = \alpha \sqrt{n(n-1+\eta)} \xrightarrow{n \gg 1} \alpha n + \beta$$



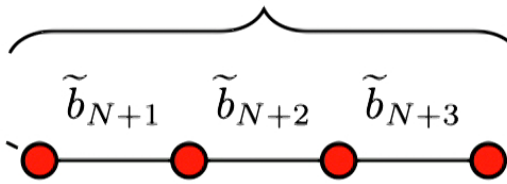
3. Stich the small to large  $n$  wavefunctions to get an approximation of the decay of  $C(t)$ .



Numerical Coefficients



Asymptotic Coefficients:  $\tilde{b}_n \sim \alpha n$



$$L = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & b_2 & 0 \\ 0 & b_2 & 0 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix} \approx \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & b_N \\ 0 & 0 & b_n & \widetilde{G^{(N)}}(z) \end{pmatrix}$$

$$G(z) = \int dt e^{izt} \langle \mathcal{O}(t) \mathcal{O}(0) \rangle \approx \frac{1}{z - \frac{b_1^2}{z - \frac{b_2^2}{z - \frac{b_2^2}{\ddots}}}}} = \frac{1}{z - b_N^2 \widetilde{G^{(N)}}(z)}$$

$$\widetilde{G^{(N)}}(z) = \Gamma(N+1) \Gamma\left(\frac{z+1}{2}\right) \times {}_2F_1\left(N+1, \frac{z+1}{2}, \frac{z+2N+3}{2}; -1\right)$$

## Diffusion in the Chaotic Ising Model

Chaotic Ising Model

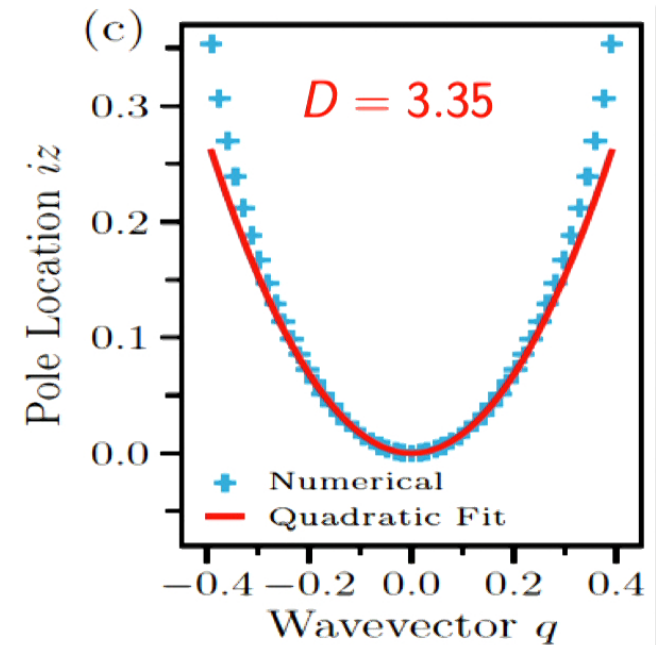
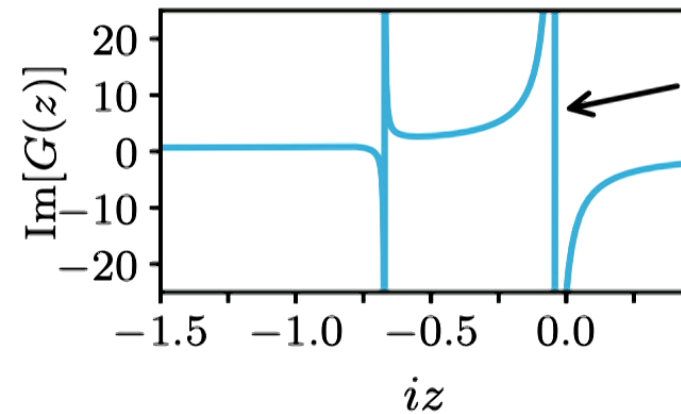
$$H = \sum_j X_j + 1.05Z_j Z_{j+1} + 0.5Z_j$$

Initial operator at wavevector  $k$ :

$$\mathcal{O}_k = \sum_j e^{ikj} (X_j + 1.05Z_j Z_{j+1} + 0.5Z_j)$$

We see the dispersion relation for diffusion

$$\frac{d}{dt}\epsilon(t, \mathbf{x}) = D\nabla^2\epsilon(t, \mathbf{x}).$$



# Summary

- Hypothesis for universal operator dynamics supported by extensive evidence. Linear growth of Lanczos coefficients

$$b_n = \alpha n + \beta + o(1), \quad n \rightarrow \infty$$

- Implies exponential growth in operator complexity:

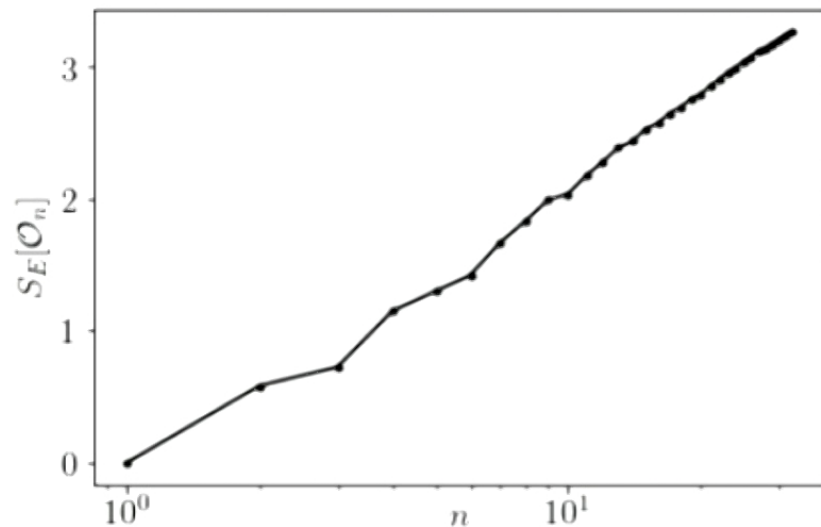
$$(n)_t \sim e^{2\alpha t} \quad \lambda_L \leq 2\alpha$$

- The hypothesis enables a new numerical scheme to compute dynamical correlations and transport coefficients.

# Extensions and outlook

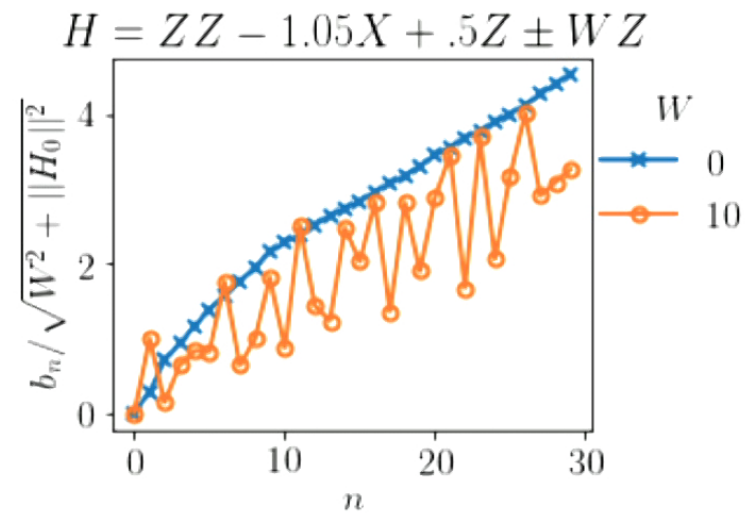
- Does K-complexity provide a more fundamental notion of chaos?  
**Measureable in a two point function (high frequency limit)!**
- MPO calculation of the Krylov vectors and recursion coefficients

**Operator entanglement  
appears to grow only  
logarithmically with  $n$ .  
Why ?**



# Extensions and outlook

- Behavior of Lanczos coefficients in integrable models?
- Growth of Lanczos coefficients in MBL?  
Systematic derivation of disorder averaged LIOMs
  - Non trivial violation of the hypothesis
  - Systematic derivation of disorder averaged LIOMs



# Extensions and outlook

- Finite size saturation
  - In a finite system, the exponential growth of K-complexity is saturated at a logarithmically long time.
  - Relation to Ehrenfest/Thouless time?

$$t^* \sim \ln L$$

Kos, Ljubotina, Prosen (2018);  
Chan, De Luca, Chalker (2018)

