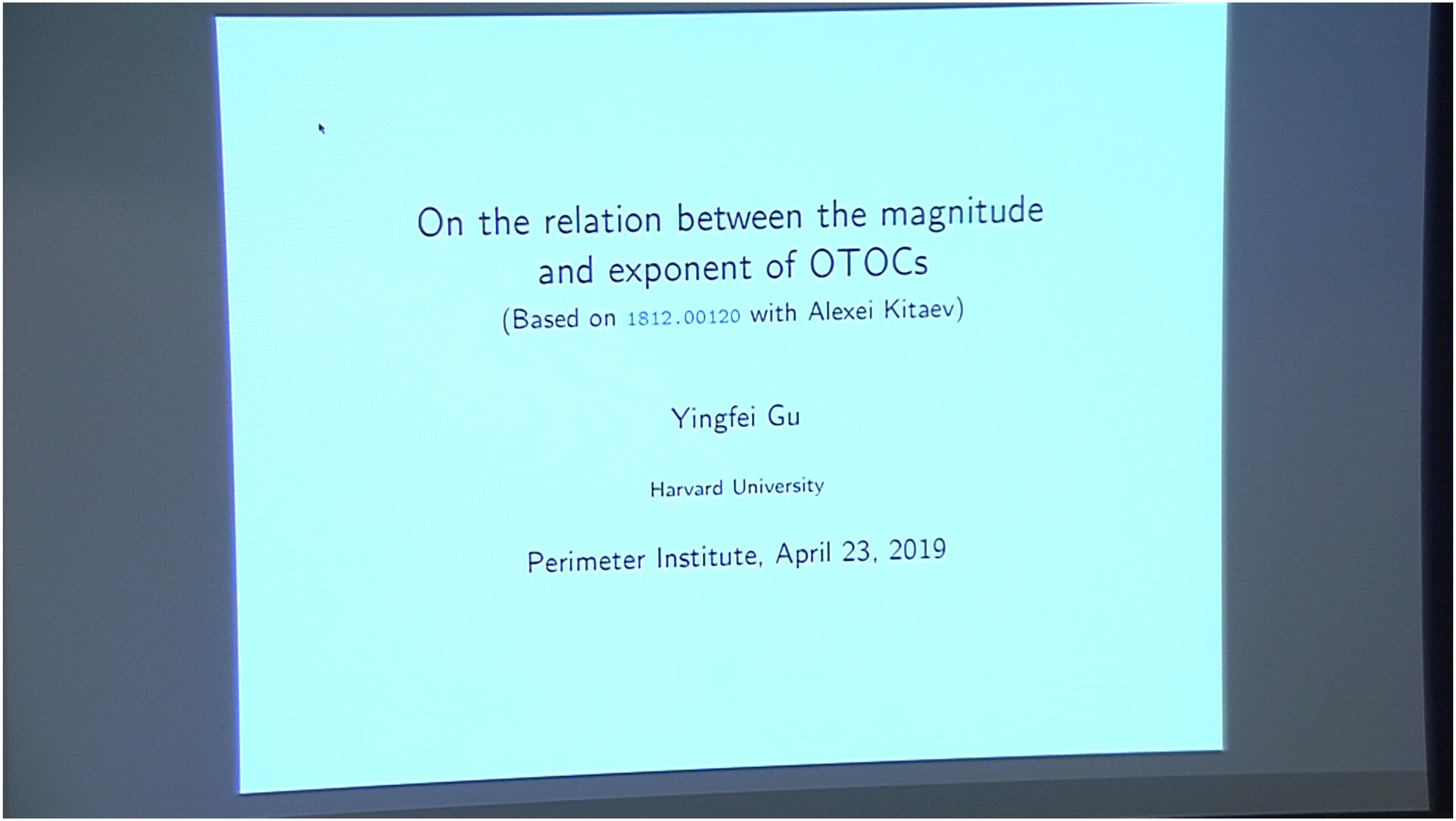
Title: On the relation between the magnitude and exponent of OTOCs

Speakers: Yingfei Gu

Collection: Quantum Matter: Emergence & Entanglement 3

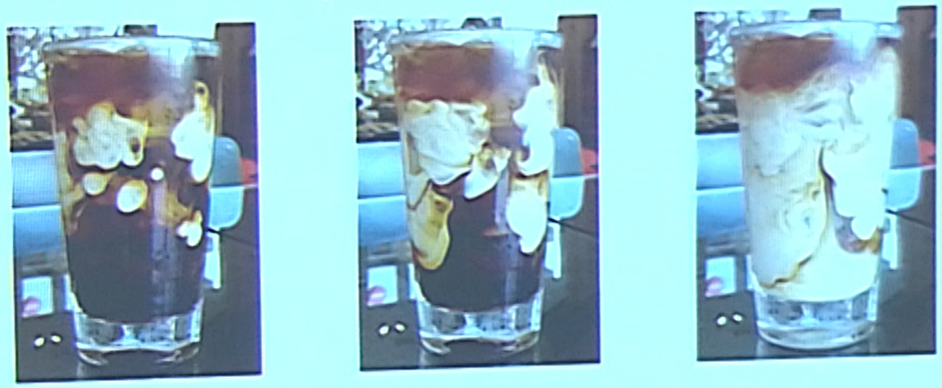
Date: April 23, 2019 - 9:30 AM

URL: http://pirsa.org/19040097


On the relation between the magnitude
and exponent of OTOCs
(Based on 1812.00120 with Alexei Kitaev)
Yingfei Gu
Harvard University
Perimeter Institute, April 23, 2019

Chaos

- Chaos in quantum many-body system: local information encoded in the whole system. (Entanglement.)



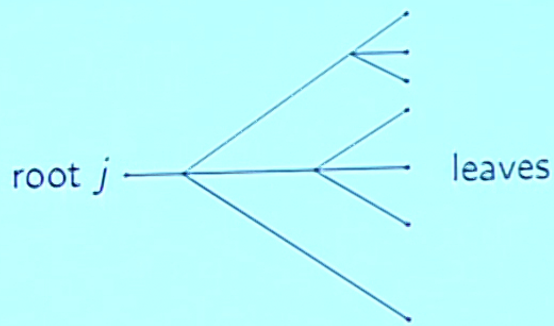
- Play with toy model $H_{\mathrm{SYK}} = \sum J_{jklm} \chi_j \chi_k \chi_l \chi_m$ [Sachdev-Ye, Kitaev].

- Check how operator grows:

$$\text{Rule of QM}: \quad \chi_j \to \chi_j + i \underbrace{[H, \chi_j]} \Delta t + \dots$$

Operator growth and OTOC

- Pictorially, $\chi_j(t)$: sum over trees



- "Size" of $\chi_j(t)$ grows exponentially: check $\langle \{\chi_j(t), \chi_k(0)\}^2 \rangle$. (Roberts-Stanford-Streicher, Qi-Streicher... 2018)
- Key ingredient: out-of-time-order correlator. (Larkin-Ovchinnikov 1969)

$$\mathrm{OTOC}(t): \quad \langle \chi_j(t)\chi_k(0)\chi_j(t)\chi_k(0) \rangle$$

SYK

What happens in SYK:

- In SYK (Maldacena-Stanford, 2016...)

$$\mathrm{OTOC}(t) \sim \frac{\beta J}{N} e^{\lambda_L t}$$

- OTOC is enhenced by βJ due to soft mode,

$$C \sim \frac{1}{\beta J}.$$

- $\lambda_L \approx \frac{2\pi}{\beta}$ near maximal chaos, small corrections due to the conformal matters,

$$1 - \frac{\lambda_L \beta}{2\pi} \sim \frac{1}{\beta J}.$$

- The following ratio has a finite limit at $\beta J \to \infty$

$$r = \frac{\cos \frac{\lambda_L \beta}{4}}{C}$$

Plan

What we will do in this talk:

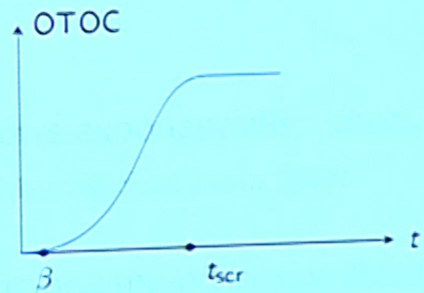
- Derive an expression for $\frac{\cos\frac{\lambda_L\beta}{4}}{C}$ which involves a new time scale: branching time t_B.
- Show two types of applications: (1) computational shortcut; (2) show exact maximal chaos in a 1D SYK-like model.

OTOC

- Generally, for systems with all-to-all interactions and large N (degrees of freedom), we expect:

$$\mathrm{OTOC}(t) \sim \frac{1}{N}\frac{e^{\lambda_L t}}{C}$$

in early time (before saturation/scrambling time).



- Goal today: understand the relation between C and λ_L.

Page 4 of 33

SYK

What happens in SYK:

- In SYK (Maldacena-Stanford, 2016...)

$$\mathrm{OTOC}(t) \sim \frac{\beta J}{N} e^{\lambda_L t}$$

- OTOC is enhenced by βJ due to soft mode,

$$C \sim \frac{1}{\beta J}.$$

- $\lambda_L \approx \frac{2\pi}{\beta}$ near maximal chaos, small corrections due to the conformal matters,

$$1 - \frac{\lambda_L \beta}{2\pi} \sim \frac{1}{\beta J}.$$

- The following ratio has a finite limit at $\beta J \to \infty$

$$r = \frac{\cos \frac{\lambda_L \beta}{4}}{C}$$

Plan

What we will do in this talk:

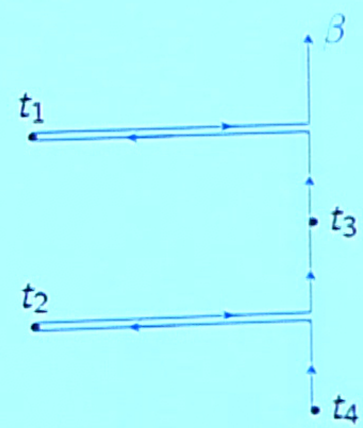
- Derive an expression for $\frac{\cos\frac{\lambda_L \beta}{4}}{C}$ which involves a new time scale: branching time t_B.
- Show two types of applications: (1) computational shortcut; (2) show exact maximal chaos in a 1D SYK-like model.

Conventions

- $\beta = 2\pi$, thus $0 \leqslant \lambda_L \leqslant 1$.
- We consider averaged (over j, k), connected OTOC

$$\langle \chi_j(t_1)\chi_k(t_3)\chi_j(t_2)\chi_k(t_4)\rangle , \quad t_1 \approx t_2 \gg t_3 \approx t_4$$

- Symmetric configuration s.t. OTOC is real



Denoted as $\mathrm{OTOC}(t_1, t_2, t_3, t_4)$

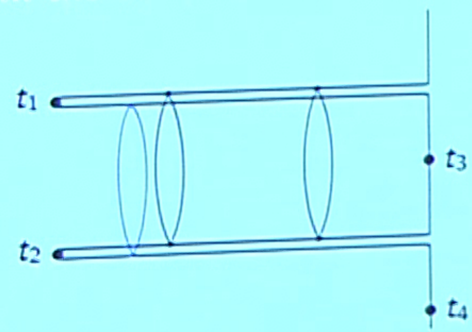
Kinetic equation

Lyapunov exponent λ_L can be determined by an integral equation.

- General four point function: ladder diagrams (Kitaev, Polchinski-Rosenhaus, Maldacena-Stanford ...):

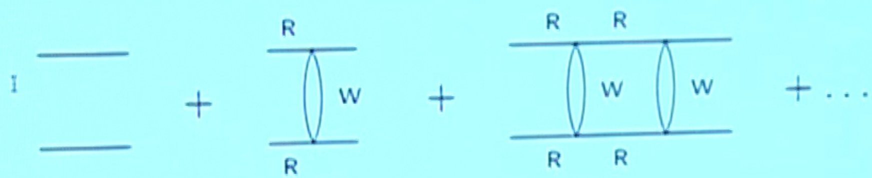
$$\mathcal{F} = \sum_n \mathcal{F}_n, \quad \mathcal{F}_n = [\text{rung}] \cdot \mathcal{F}_{n-1} \Rightarrow \mathcal{F} = \mathcal{F}_0 + K\mathcal{F}$$

- For OTOC: deform the contour to double Keldysh.



Retarded kernel

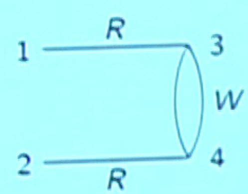
- OTOC ladder diagrams



$$F = F_0 + K^R \cdot F$$

- Kinetic equation (Kitaev 2015, Murugan-Stanford-Witten 2017...):

$$F \approx K^R \cdot F, \quad K^R(t_1, t_2, t_3, t_4):$$

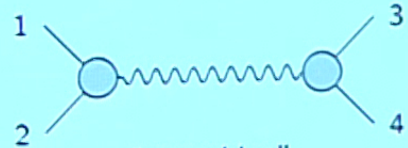


Retarded kernel

Single-mode ansatz

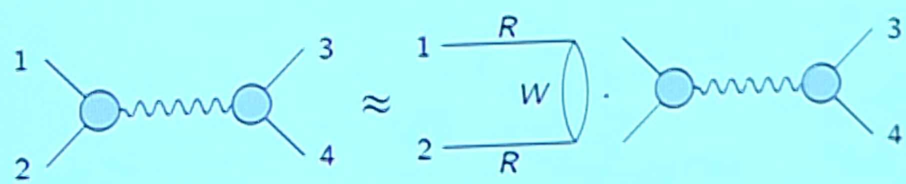
- Single-mode ansatz for early time regime (Kitaev-Suh, 2017)

$$\mathrm{OTOC}(t_1, t_2, t_3, t_4) \approx \frac{1}{N}\frac{e^{\lambda_L(t_1+t_2-t_3-t_4)/2}}{C}\Upsilon^R(t_{12})\Upsilon^A(t_{34})$$



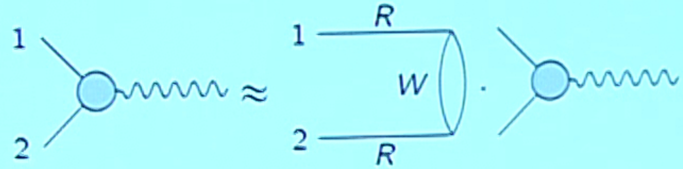
"scramblon"

- Plug into the equation $F \approx K^R \cdot F$



A variant of kernel

- K^R only acts on left half (retarded vertex):



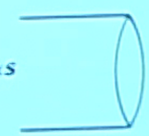
$$\Upsilon^R(t_{12})e^{\lambda_L(t_1+t_2)/2} = \int K^R(t_1, t_2, t_5, t_6)e^{\lambda_L(t_5+t_6)/2}\Upsilon^R(t_{56})dt_5dt_6$$

- Further simplification:

$$\Upsilon^R(t_{12}) = \int \left[\int K^R(t_1, t_2, t_5, t_6)e^{-\lambda_L\frac{t_1+t_2-(t_5+t_6)}{2}} d\frac{t_5+t_6}{2}\right]\Upsilon^R(t_{56})dt_{56}$$

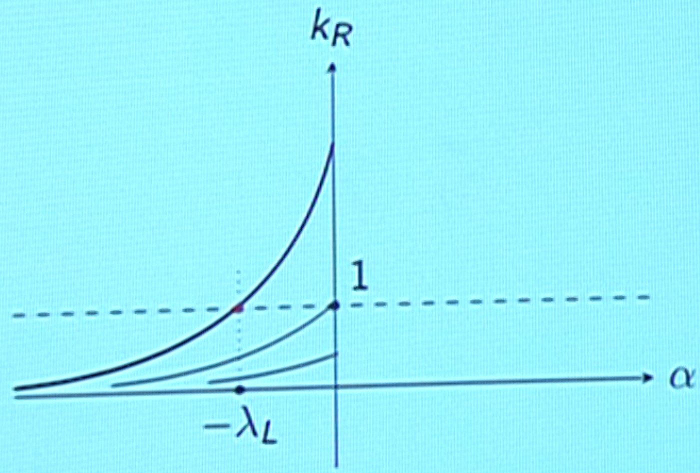
- In practice, solve by shooting: Define a variant of kernel for parameter $\alpha < 0$:

$$K^R_\alpha(t, t') = \int K^R\left(s+\frac{t}{2}, s-\frac{t}{2}, \frac{t'}{2}, -\frac{t'}{2}\right)e^{\alpha s}\,ds = \int dse^{\alpha s}$$



Eigenvalue $k_R(\alpha)$

- A continuous family of operators K_α^R.
- For general $\alpha < 0$, find largest eigenvalue of K_α^R, denoted as $k_R(\alpha)$



- Find solution of equation $k_R(\alpha) = 1$

$$\text{Lyapunov exponent } \lambda_L : k_R(-\lambda_L) = 1$$

Ladder identity

Kinetic equation is useful: (1) find chaos exponent: $k_R(-\lambda_L) = 1$; (2) find vertex functions: $\Upsilon^{R(A)} = \Upsilon^{R(A)}_{-\lambda_L}$.

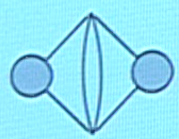
$$\mathrm{OTOC}(t_1, t_2, t_3, t_4) \approx \frac{1}{N} \frac{e^{\lambda_L (t_1 + t_2 - t_3 - t_4)/2}}{C} \Upsilon^R(t_{12}) \Upsilon^A(t_{34})$$

But prefactor C can not be determined by solving linear equation.

- Ladder identity:

$$\frac{2\cos\frac{\lambda_L \pi}{2}}{C} \cdot k_R'(-\lambda_L) \cdot (\Upsilon^A, \Upsilon^R) = 1\,.$$

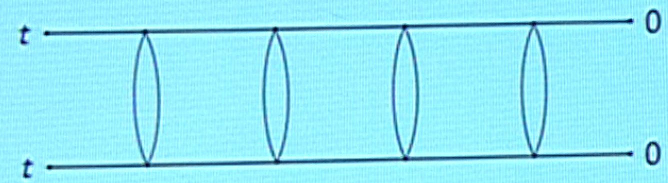
- (Υ^A, Υ^R): inner product of vertex functions:

$(\Upsilon^A, \Upsilon^R) =$  $= (q-1)J^2 \int dt\, \Upsilon^A(t) (G^W(t))^{q-2} \Upsilon^R(t)\,.$

- Branching time

$$t_B := k_R'(-\lambda_L)\,.$$

Branching time

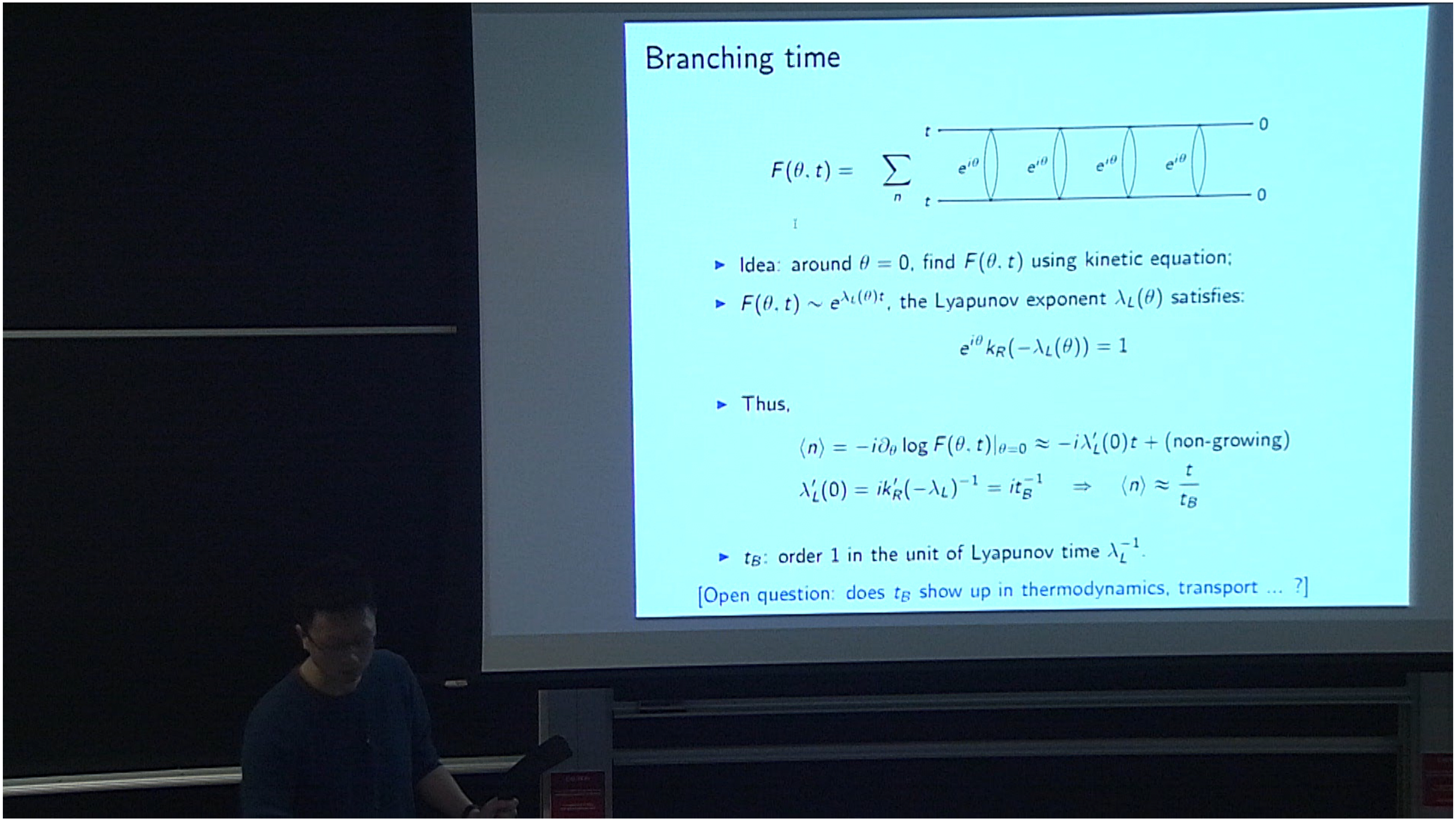


- Average distance between rungs $= t/\langle n\rangle$;
- Count number of rungs:

$$F(t) = \sum_n F_n(t)\,, \quad \langle n\rangle = \frac{\sum nF_n(t)}{\sum F_n(t)}\,.$$

- Introduce an auxiliary (generating) function:

$$F(\theta, t) := \sum_n F_n(t) e^{in\theta}\,, \quad \langle n\rangle = -i\partial_\theta \log F(\theta, t)|_{\theta=0}$$


Branching time
$F(\theta, t) = \sum_n$
t
t
$e^{i\theta}$
$e^{i\theta}$
$e^{i\theta}$
$e^{i\theta}$
0
0
Idea: around $\theta = 0$, find $F(\theta, t)$ using kinetic equation;
$F(\theta, t) \sim e^{\lambda_L(\theta)t}$, the Lyapunov exponent $\lambda_L(\theta)$ satisfies:
$e^{i\theta} k_R(-\lambda_L(\theta)) = 1$
Thus,
$\langle n \rangle = -i\partial_\theta \log F(\theta, t)|_{\theta=0} \approx -i\lambda_L'(0)t + \text{(non-growing)}$
$\lambda_L'(0) = ik_R'(-\lambda_L)^{-1} = it_B^{-1} \quad \Rightarrow \quad \langle n \rangle \approx \frac{t}{t_B}$
t_B: order 1 in the unit of Lyapunov time λ_L^{-1}.
[Open question: does t_B show up in thermodynamics, transport ... ?]

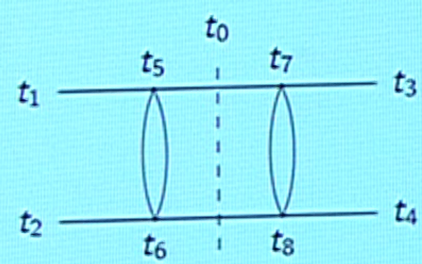
Derivation of the identity

Next, we sketch the derivation of

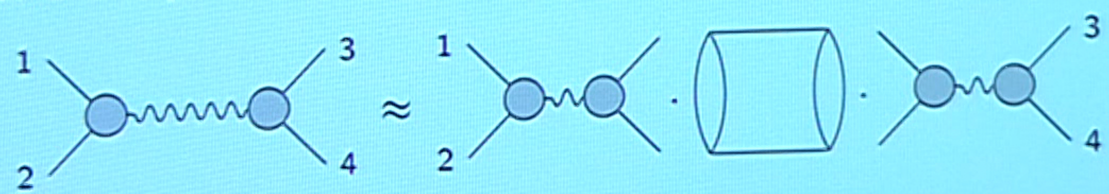
$$N \cdot \frac{2\cos\frac{\lambda_L \pi}{2}}{C} \cdot k_R'(-\lambda_L) \cdot (\Upsilon^A, \Upsilon^R) = 1\,.$$

Idea: cut a long ladder into pieces and find a consistency condition.

- Cut. Fix t_0, find adjacent $\frac{t_5+t_6}{2} < t_0 < \frac{t_7+t_8}{2}$



- Consistency condition

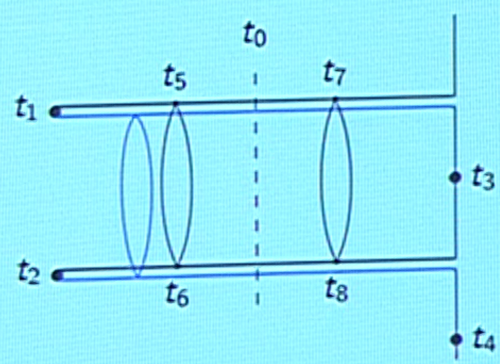


Derivation: $\cos\frac{\lambda_L \pi}{2}$ factor

- Naively, we would have a formula:

$$\text{OTOC} \approx \text{OTOC}_L \cdot \text{BOX} \cdot \text{OTOC}_R$$

- Subtlety: multiple choices on the double Keldysh contour

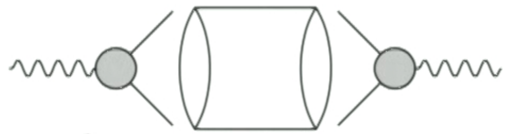
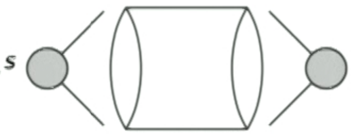


- Sum of two choices:

$$\text{OTOC} \approx \left(e^{i\frac{\lambda_L \pi}{2}} + e^{-i\frac{\lambda_L \pi}{2}}\right) \text{OTOC} \cdot \text{BOX} \cdot \text{OTOC}$$

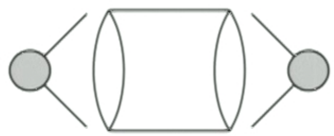
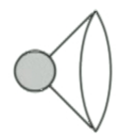
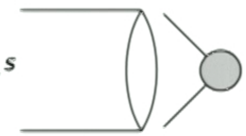
Derivation: box

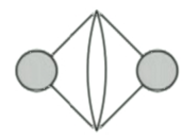
- Next, integrate over t_5, t_6, t_7, t_8

 : $\int ds dt_* e^{-\lambda_L s}$ 

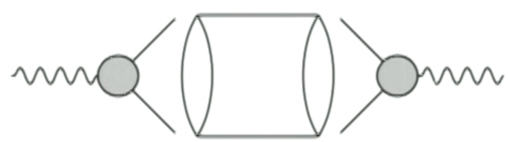
$s = \frac{t_5+t_6-t_7-t_8}{2}$: size of the box; t_*: center of mass time $\int dt_* = s$

- How does this term related to $k_R'(-\lambda_L)$?

$\int ds e^{-\lambda_L s}$  $=$  $\int ds e^{-\lambda_L s}$ 

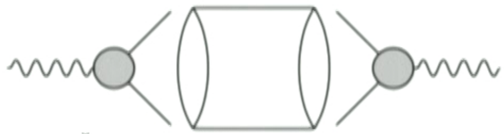
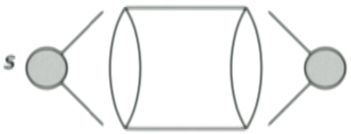
$= \quad k_R(-\lambda_L)$ 

- Take λ_L derivative:

$t_B\left(\Upsilon^A, \Upsilon^R\right) =$ 

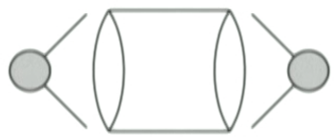
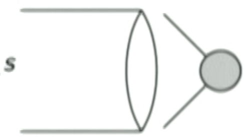
Derivation: box

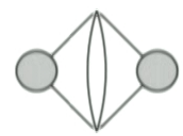
- Next, integrate over t_5, t_6, t_7, t_8

 : $\int ds dt_* e^{-\lambda_L s}$ 

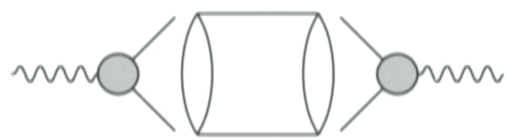
$s = \frac{t_5+t_6-t_7-t_8}{2}$: size of the box; t_*: center of mass time $\int dt_* = s$

- How does this term related to $k_R'(-\lambda_L)$?

$\int ds e^{-\lambda_L s}$  $=$  $\int ds e^{-\lambda_L s}$ 

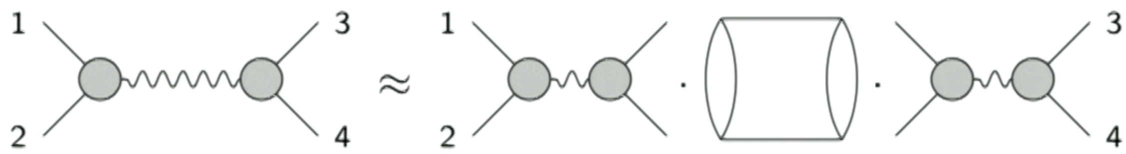
$= \quad k_R(-\lambda_L)$ 

- Take λ_L derivative:

$t_B\left(\Upsilon^A, \Upsilon^R\right) =$ 

Derivation: summary

- We start with consistency condition



- Compare two sides, we find

$$\frac{1}{C} = N \cdot \frac{2\cos\frac{\lambda_L \pi}{2}}{C^2} \cdot t_B \cdot \left(\Upsilon^A, \Upsilon^R\right)$$

Applications

The ladder identity:

$$N \cdot \frac{2\cos\frac{\lambda_L \pi}{2}}{C} \cdot t_B \cdot \left(\Upsilon^A, \Upsilon^R\right) = 1\,.$$

Next:

- Computational shortcuts: $C \Leftrightarrow \lambda_L$;
- In a 1D model, prove exact maximal chaos using the identity.

Computational shortcuts I: near maximal chaos

SYK at strong coupling $J \gg 1$, near maximal chaos $\lambda_L = 1 - \delta\lambda_L$.

- Schwarzian action:

$$I_{\text{Sch}}[\varphi] = -\frac{N\alpha_S}{J}\int_0^{2\pi} \text{Sch}\left(e^{i\varphi(\tau)}, \tau\right) d\tau$$

- Use Schwarzian action:

$$\text{OTOC} \approx \frac{Je^{(t_1+t_2-t_3-t_4)/2}}{2N\alpha_s} \cdot \frac{2\Delta b^{\Delta} J^{-2\Delta}}{\left(2\cosh\frac{t_{12}}{2}\right)^{2\Delta+1}} \cdot \frac{2\Delta b^{\Delta} J^{-2\Delta}}{\left(2\cosh\frac{t_{34}}{2}\right)^{2\Delta+1}}$$

- Find correction $\delta\lambda_L \approx 2\cos\frac{(1-\delta\lambda_L)\pi}{2}/\pi$ (Maldacena-Stanford, 2016)

$$\delta\lambda_L \approx \frac{C}{\pi t_B(\Upsilon^A, \Upsilon^R)} = \frac{6\alpha_S}{Jk_R'(-1)\Delta(1-\Delta)(1-2\Delta)\tan(\pi\Delta)}.$$

Computational shortcuts II: prefactor

- Large q SYK, fix $\mathcal{J} = \sqrt{2^{1-q}q}\,J$ (Maldacena-Stanford, 2016):

$$\frac{v}{2\cos\frac{\pi v}{2}} = \mathcal{J}, \quad 0 < v < 1\,.$$

- Exact correlation function at all coupling v;

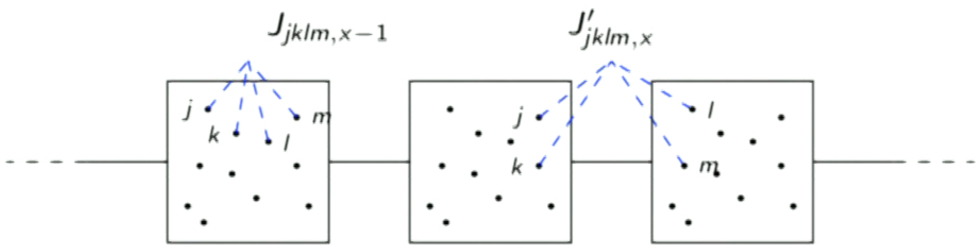
$$K^R = \theta(t_{13})\theta(t_{24})\frac{v^2}{2\cosh^2\frac{vt_{34}}{2}}, \quad \lambda_L = v$$

- Use the identity to find the prefactor (Qi-Streicher, 2018):

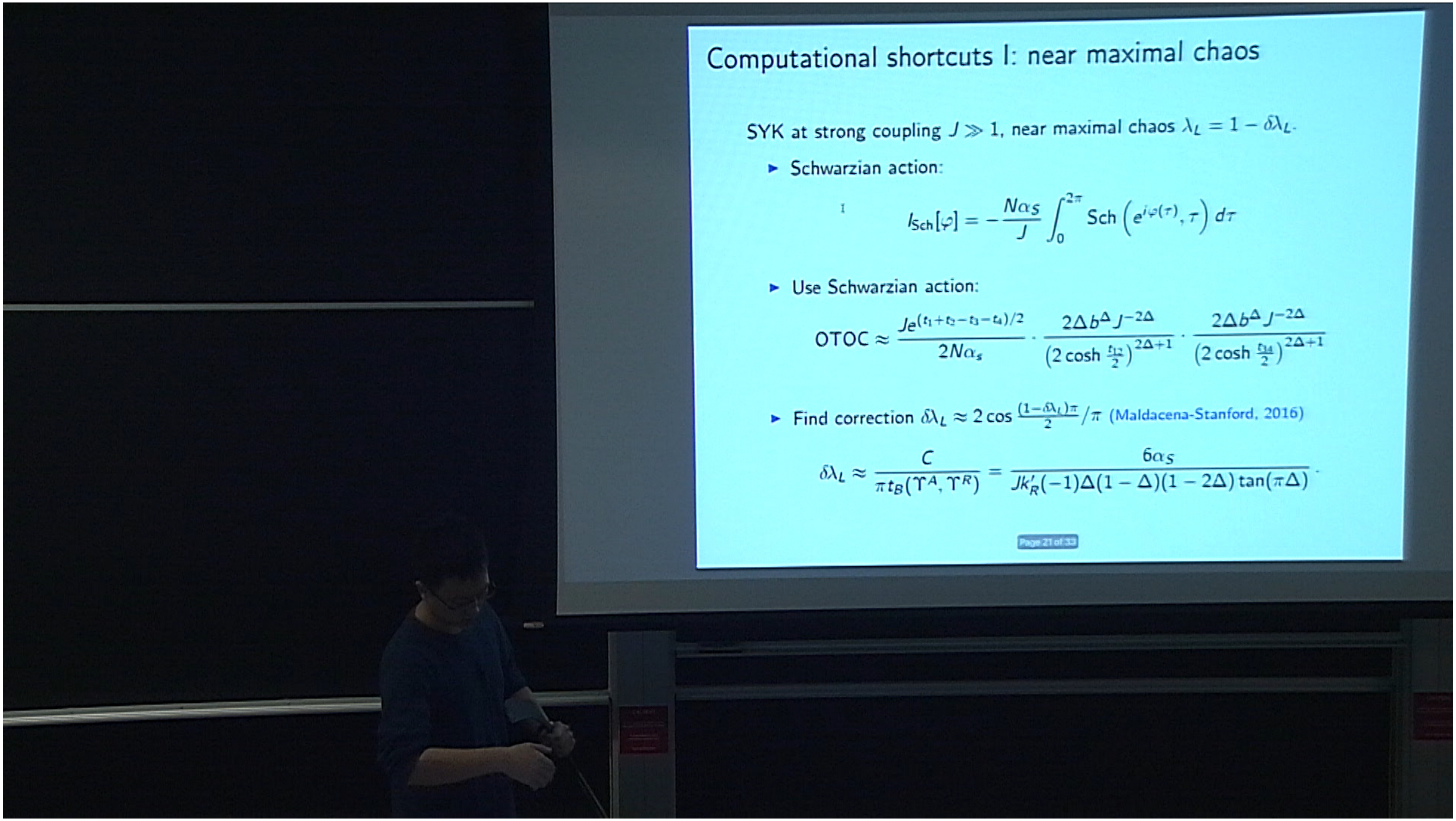
$$\mathrm{OTOC}(t_1, t_2; t_3, t_4) \approx \frac{1}{N\cos\frac{v\pi}{2}}\frac{e^{v(t_1+t_2-t_3-t_4)/2}}{\left(2\cosh\frac{vt_{12}}{2}\right)\left(2\cosh\frac{vt_{34}}{2}\right)}\,.$$

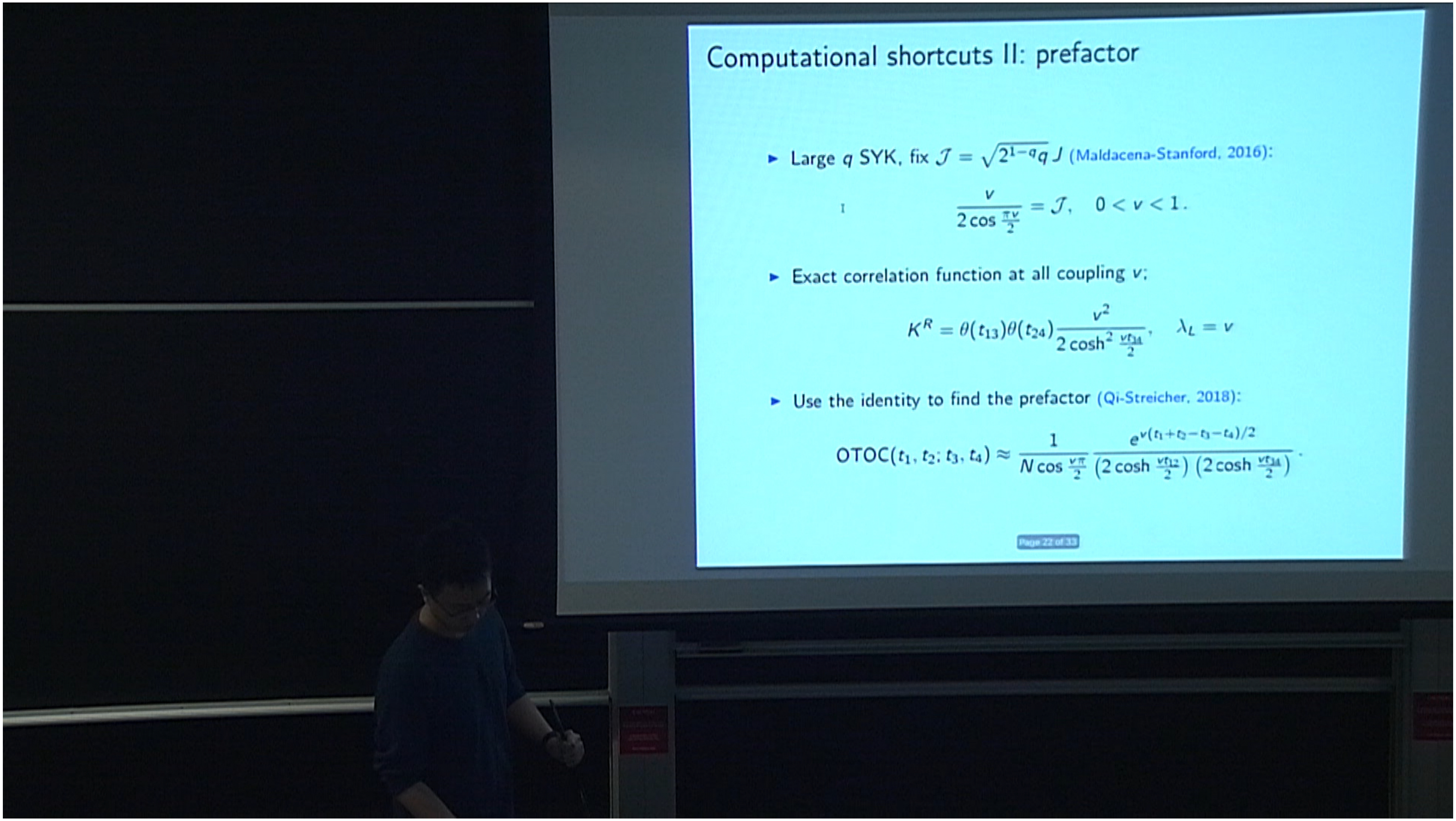
Maximal chaos in a 1D model

- Regard λ_L and C as analytic functions of some parameter, then the analytical properties of λ_L and C are locked by the ladder identity.
- A concrete example: SYK chain (YG-Qi-Stanford, 2016)



(also see [Guo-YG-Sachdev, 2019])


Computational shortcuts I: near maximal chaos
SYK at strong coupling $J \gg 1$, near maximal chaos $\lambda_L = 1 - \delta\lambda_L$.
Schwarzian action:
$I_{\text{Sch}}[\varphi] = -\frac{N\alpha_S}{J} \int_0^{2\pi} \text{Sch}\left(e^{i\varphi(\tau)}, \tau\right) d\tau$
Use Schwarzian action:
$\text{OTOC} \approx \frac{J e^{(t_1+t_2-t_3-t_4)/2}}{2N\alpha_s} \cdot \frac{2\Delta b^\Delta J^{-2\Delta}}{\left(2\cosh\frac{t_{12}}{2}\right)^{2\Delta+1}} \cdot \frac{2\Delta b^\Delta J^{-2\Delta}}{\left(2\cosh\frac{t_{34}}{2}\right)^{2\Delta+1}}$
Find correction $\delta\lambda_L \approx 2\cos\frac{(1-\delta\lambda_L)\pi}{2}/\pi$ (Maldacena-Stanford, 2016)
$\delta\lambda_L \approx \frac{C}{\pi t_B(\Upsilon^A, \Upsilon^R)} = \frac{6\alpha_S}{Jk_R'(-1)\Delta(1-\Delta)(1-2\Delta)\tan(\pi\Delta)}$.
Page 21 of 33


Computational shortcuts II: prefactor
Large q SYK, fix $\mathcal{J} = \sqrt{2^{1-q}q}\,J$ (Maldacena-Stanford, 2016):
$\frac{v}{2\cos\frac{\pi v}{2}} = \mathcal{J}, \quad 0 < v < 1.$
Exact correlation function at all coupling v:
$K^R = \theta(t_{13})\theta(t_{24})\frac{v^2}{2\cosh^2\frac{vt_{34}}{2}}, \quad \lambda_L = v$
Use the identity to find the prefactor (Qi-Streicher, 2018):
$\mathrm{OTOC}(t_1, t_2; t_3, t_4) \approx \frac{1}{N\cos\frac{v\pi}{2}} \frac{e^{v(t_1+t_2-t_3-t_4)/2}}{\left(2\cosh\frac{vt_{12}}{2}\right)\left(2\cosh\frac{vt_{34}}{2}\right)}.$
Page 22 of 33

Operators at two different locations

Operators at two different locations:

$$\mathrm{OTOC}_{x,0}(t_1, t_2, t_3, t_4) := \frac{1}{N^2} \sum_{j,k} \langle \chi_{j,x}(t_1) \chi_{k,0}(t_3) \chi_{j,x}(t_2) \chi_{k,0}(t_4) \rangle_{conn.}$$

- Fourier transform:

$$\mathrm{OTOC}_{x,0}(t_1, t_2, t_3, t_4) = \int \frac{dp}{2\pi} e^{ipx}\, \mathrm{OTOC}_p(t_1, t_2, t_3, t_4)$$

- Each OTOC_p: ladder diagrams dominate. Retarded kernel factorizes:

$$K^R(p) = s(p)\, K^R, \quad s(p) = 1 - 2a(1 - \cos p) \approx 1 - ap^2 \, .$$

$s(p)$: "band structure" of the bilocal fields $a = \frac{J_1^2}{3J^2} \in (0, 1/3)$.

Fourier Transform

- The ladder identity holds for each OTOC_p:

$$C(p) = 2\cos\frac{\lambda_L(p)\pi}{2} \cdot t_B \cdot (\Upsilon^A, \Upsilon^R)\,,$$

- The dependence of t_B and (Υ^A, Υ^R) on p is not important (analytic and do not vanish in the domain of interest).

$$\text{OTOC}_{x,0}(t_1, t_2, t_3, t_4) \sim \frac{1}{N} \underbrace{\int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{e^{\lambda_L(p)t + ipx}}{2\cos\frac{\pi\lambda_L(p)}{2}}}_{u(x,t)} \cdot \frac{\Upsilon^R(t_{12})\Upsilon^A(t_{34})}{t_B(\Upsilon^A, \Upsilon^R)}\,.$$

$t = \frac{t_1+t_2-t_3-t_4}{2}$. $\lambda_L(p) \approx \lambda_L(0) - t_B^{-1} a p^2$ when $|p| \ll 1$.

Butterfly wavefront

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{e^{\lambda_L(p)t + ipx}}{2\cos\frac{\lambda_L(p)}{2}}, \quad \lambda_L(p) \approx \lambda_L(0) - t_B^{-1} a p^2$$

- Butterfly wavefront: $u(x,t) \sim 1$.
- For large $x > 0$ and t, we can estimate by saddle point of the exponent:

$$\lambda_L'(p)t + ix = 0, \quad p = i|p|.$$

- Find a butterfly velocity

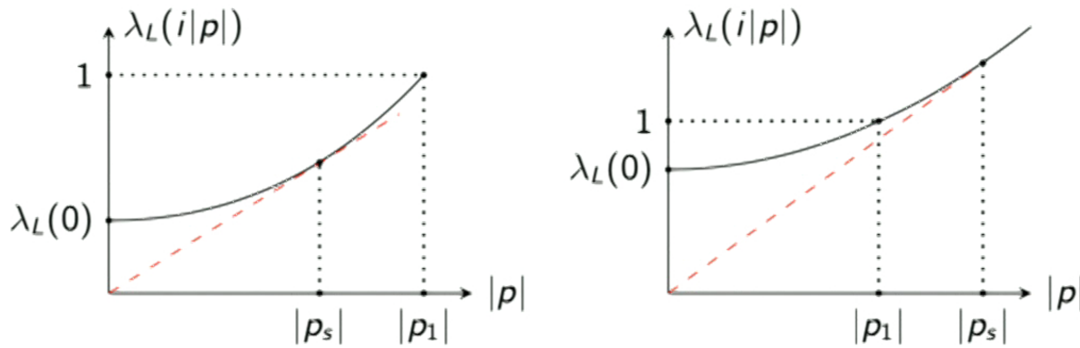
$$v_s = \frac{i\lambda_L(p_s)}{p_s} = i\lambda_L'(p_s)$$

Graphic solutions: two scenarios

- The relevant saddle point $p_s = i|p_s|$ is purely imaginary. Deform the integral contour to pass, might cross the pole:

$$\cos\frac{\lambda_L(p_1)\pi}{2} = 0, \quad \lambda_L(p_1) = 1, \quad p_1 = i\,|p_1|\,.$$

- Two scenarios:

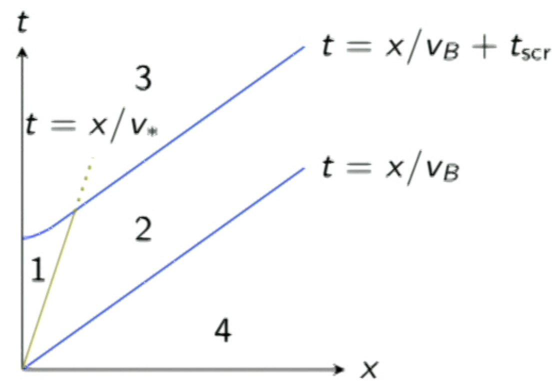


Pole contribution: maximal chaos

- In the second scenario, the pole dominates:

$$u_1(x,t) \approx \frac{e^{t-|p_1||x|}}{\pi i \lambda_L'(p_1)}, \quad v_1 = \frac{1}{|p_1|}$$

- In SYK at $J \gg 1$, $\delta\lambda_L = 1 - \lambda_L(0) \ll 1$, $|p_1| \approx \sqrt{t_B \delta\lambda_L / a} \ll |p_s|$, pole dominates. $v_B = v_1$.



Summary and discussion

- Ladder identity relates C and λ_L;

$$\frac{2\cos\frac{\lambda_L \pi}{2}}{C} \cdot t_B \cdot \left(\Upsilon^A, \Upsilon^R\right) = 1\,, \quad t_B = k_R'(-\lambda_L)$$

[Is t_B "useful"? (Related to thermodynamics or transport?)]

- Applications:
  - computational shortcuts, $\delta\lambda_L \propto t_B^{-1}$
  - maximal chaos [Why? Can we design $0+1$-d example?]