

Title: On the relation between the magnitude and exponent of OTOCs

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Collection: Quantum Matter: Emergence & Entanglement 3

Date: April 23, 2019 - 9:30 AM

URL: <http://pirsa.org/19040097>

# On the relation between the magnitude and exponent of OTOCs

(Based on [1812.00120](#) with Alexei Kitaev)

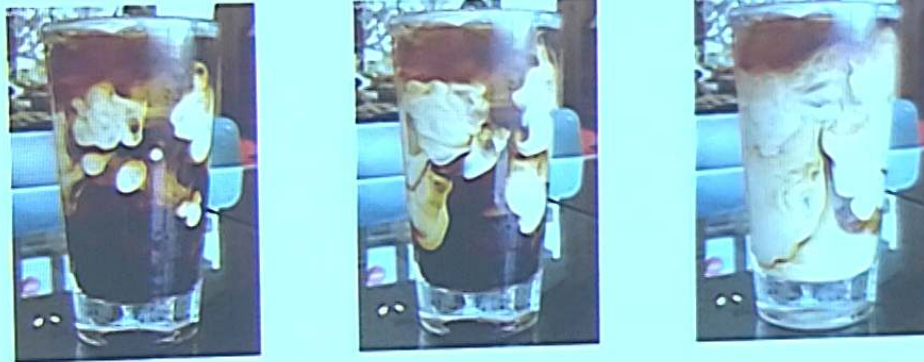
Yingfei Gu

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Perimeter Institute, April 23, 2019

# Chaos

- ▶ Chaos in quantum many-body system: local information encoded in the whole system. (Entanglement.)

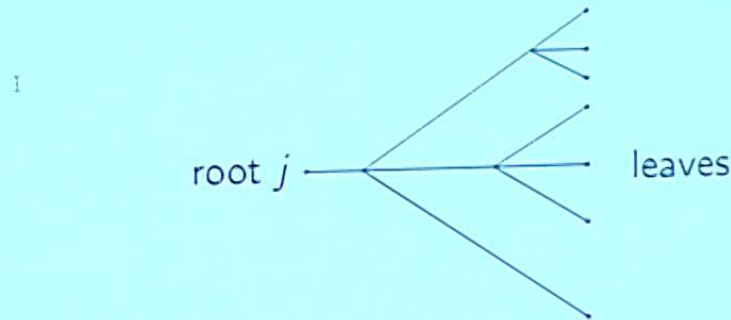


- ▶ Play with toy model  $H_{\text{SYK}} = \sum J_{jklm} \chi_j \chi_k \chi_l \chi_m$  [Sachdev-Ye, Kitaev].
- ▶ Check how operator grows:

$$\text{Rule of QM: } \chi_j \rightarrow \chi_j + i \underbrace{[H, \chi_j]}_{\text{local operator}} \Delta t + \dots$$

# Operator growth and OTOC

- ▶ Pictorially,  $\chi_j(t)$ : sum over trees



- ▶ "Size" of  $\chi_j(t)$  grows exponentially: check  $\langle \{\chi_j(t), \chi_k(0)\}^2 \rangle$ .  
(Roberts-Stanford-Streicher, Qi-Streicher... 2018)
- ▶ Key ingredient: out-of-time-order correlator. (Larkin-Ovchinnikov 1969)

$$\text{OTOC}(t) : \langle \chi_j(t) \chi_k(0) \chi_j(t) \chi_k(0) \rangle$$

# SYK

What happens in SYK:

- ▶ In SYK (Maldacena-Stanford, 2016...)

$$\text{OTOC}(t) \sim \frac{\beta J}{N} e^{\lambda_L t}$$

- ▶ OTOC is enhanced by  $\beta J$  due to soft mode,

$$C \sim \frac{1}{\beta J}.$$

- ▶  $\lambda_L \approx \frac{2\pi}{3}$  near maximal chaos, small corrections due to the conformal matters,

$$1 - \frac{\lambda_L \beta}{2\pi} \sim \frac{1}{\beta J}.$$

- ▶ The following ratio has a finite limit at  $\beta J \rightarrow \infty$

$$r = \frac{\cos \frac{\lambda_L \beta}{4}}{C}$$

## Plan

What we will do in this talk:

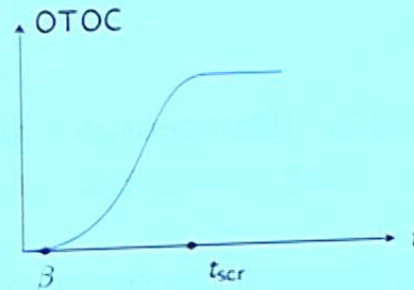
- ▶ Derive an expression for  $\frac{\cos \frac{\lambda_L \beta}{4}}{C}$  which involves a new time scale: branching time  $t_B$ .
- ▶ Show two types of applications: (1) computational shortcut; (2) show exact maximal chaos in a 1D SYK-like model.

## OTOC for growth and OTOC

- ▶ Generally, for systems with all-to-all interactions and large  $N$  (degrees of freedom), we expect:

$$\text{OTOC}(t) \sim \frac{1}{N} \frac{e^{\lambda_L t}}{C}$$

in early time (before saturation/scrambling time).



- ▶ Goal today: understand the relation between  $C$  and  $\lambda_L$ .

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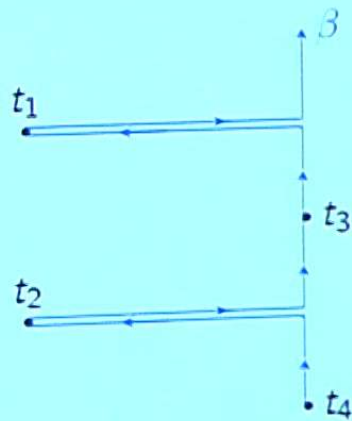
## Conventions

▶  $\beta = 2\pi$ , thus  $0 \leq \lambda_L \leq 1$ .

▶ We consider averaged (over  $j, k$ ), connected OTOC

$$i \langle \chi_j(t_1) \chi_k(t_3) \chi_j(t_2) \chi_k(t_4) \rangle, \quad t_1 \approx t_2 \gg t_3 \approx t_4$$

▶ Symmetric configuration s.t. OTOC is real



Denoted as  $\text{OTO}C(t_1, t_2, t_3, t_4)$

## Kinetic equation

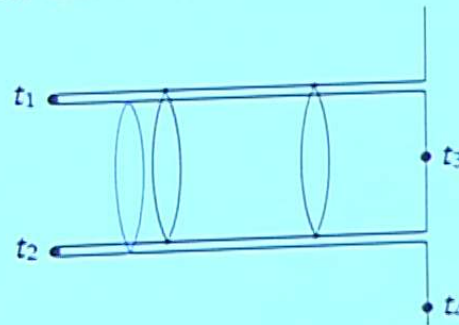
Lyapunov exponent  $\lambda_L$  can be determined by an integral equation.

- ▶ General four point function: ladder diagrams (Kitaev, Polchinski-Rosenhaus, Maldacena-Stanford ...):



$$\mathcal{F} = \sum_n \mathcal{F}_n, \quad \mathcal{F}_n = \text{[loop diagram]} \cdot \mathcal{F}_{n-1} \Rightarrow \mathcal{F} = \mathcal{F}_0 + K\mathcal{F}$$

- ▶ For OTOC: deform the contour to double Keldysh.



# Retarded kernel

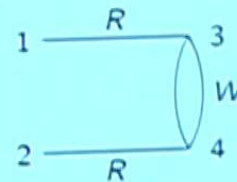
- ▶ OTOC ladder diagrams



$$F = F_0 + K^R \cdot F$$

- ▶ Kinetic equation (Kitaev 2015, Murugan-Stanford-Witten 2017...):

$$F \approx K^R \cdot F, \quad K^R(t_1, t_2, t_3, t_4) :$$

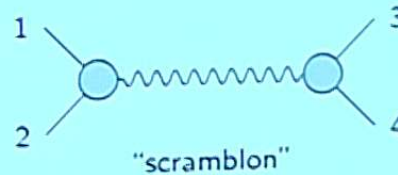


Retarded kernel

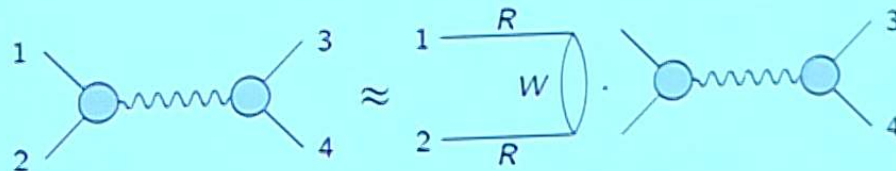
## Single-mode ansatz

- ▶ Single-mode ansatz for early time regime (Kitaev-Suh, 2017)

$$\text{OTOC}(t_1, t_2, t_3, t_4) \approx \frac{1}{N} \frac{e^{\lambda_L(t_1+t_2-t_3-t_4)/2}}{C} \Upsilon^R(t_{12}) \Upsilon^A(t_{34})$$

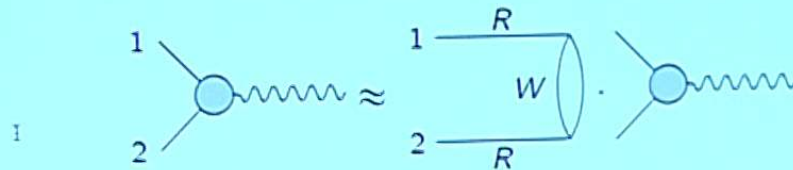


- ▶ Plug into the equation  $F \approx K^R \cdot F$



## A variant of kernel

- ▶  $K^R$  only acts on left half (retarded vertex):



$$\Upsilon^R(t_{12}) e^{\lambda_L(t_1+t_2)/2} = \int K^R(t_1, t_2, t_5, t_6) e^{\lambda_L(t_5+t_6)/2} \Upsilon^R(t_{56}) dt_5 dt_6$$

- ▶ Further simplification:

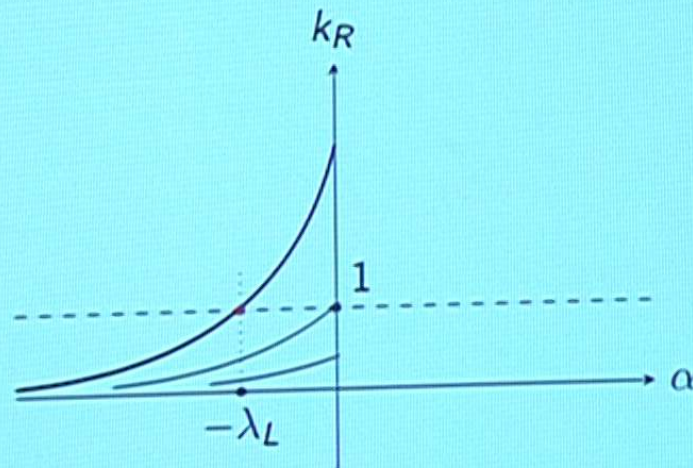
$$\Upsilon^R(t_{12}) = \int \left[ \int K^R(t_1, t_2, t_5, t_6) e^{-\lambda_L \frac{t_1-t_2-(t_5-t_6)}{2}} d \frac{t_5+t_6}{2} \right] \Upsilon^R(t_{56}) dt_{56}$$

- ▶ In practice, solve by shooting: Define a variant of kernel for parameter  $\alpha < 0$ :

$$K_\alpha^R(t, t') = \int K^R\left(s + \frac{t}{2}, s - \frac{t}{2}, \frac{t'}{2}, -\frac{t'}{2}\right) e^{\alpha s} ds = \int ds e^{\alpha s}$$

## Eigenvalue $k_R(\alpha)$

- ▶ A continuous family of operators  $K_\alpha^R$ .
- ▶ For general  $\alpha < 0$ , find largest eigenvalue of  $K_\alpha^R$ , denoted as  $k_R(\alpha)$



- ▶ Find solution of equation  $k_R(\alpha) = 1$

$$\text{Lyapunov exponent } \lambda_L : k_R(-\lambda_L) = 1$$

## Ladder identity

Kinetic equation is useful: (1) find chaos exponent:  $k_R(-\lambda_L) = 1$ ; (2) find vertex functions:  $\Upsilon^{R(A)} = \Upsilon_{-\lambda_L}^{R(A)}$ .

$$\text{OTOC}_I(t_1, t_2, t_3, t_4) \approx \frac{1}{N} \frac{e^{\lambda_L(t_1+t_2-t_3-t_4)/2}}{C} \Upsilon^R(t_{12}) \Upsilon^A(t_{34})$$

But prefactor  $C$  can not be determined by solving linear equation.

- ▶ Ladder identity:

$$\frac{2 \cos \frac{\lambda_L \pi}{2}}{C} \cdot k'_R(-\lambda_L) \cdot (\Upsilon^A, \Upsilon^R) = 1.$$

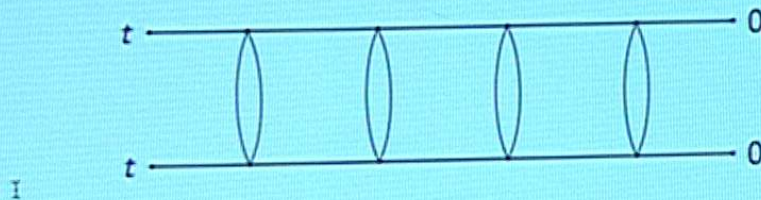
- ▶  $(\Upsilon^A, \Upsilon^R)$ : inner product of vertex functions:

$$(\Upsilon^A, \Upsilon^R) = \text{Diagram} = (q-1)J^2 \int dt \Upsilon^A(t) (G^W(t))^{q-2} \Upsilon^R(t).$$

- ▶ Branching time

$$t_B := k'_R(-\lambda_L).$$

## Branching time



▶ Average distance between rungs =  $t/\langle n \rangle$ ;

▶ Count number of rungs:

$$F(t) = \sum_n F_n(t), \quad \langle n \rangle = \frac{\sum_n n F_n(t)}{\sum_n F_n(t)}.$$

▶ Introduce an auxiliary (generating) function:

$$F(\theta, t) := \sum_n F_n(t) e^{in\theta}, \quad \langle n \rangle = -i\partial_\theta \log F(\theta, t)|_{\theta=0}$$

## Branching time

$$F(\theta, t) = \sum_n \int \text{[Diagram: A horizontal line with two parallel lines above and below, labeled 't' and '0'. Four vertical ovals are placed between the lines, each containing the expression 'e^{i\theta}'. The sum is over 'n' and the integral is over the space between the lines.]}$$

- ▶ Idea: around  $\theta = 0$ , find  $F(\theta, t)$  using kinetic equation;
- ▶  $F(\theta, t) \sim e^{\lambda_L(\theta)t}$ , the Lyapunov exponent  $\lambda_L(\theta)$  satisfies:

$$e^{i\theta} k_R(-\lambda_L(\theta)) = 1$$

- ▶ Thus,

$$\langle n \rangle = -i\partial_\theta \log F(\theta, t)|_{\theta=0} \approx -i\lambda'_L(0)t + (\text{non-growing})$$

$$\lambda'_L(0) = ik'_R(-\lambda_L)^{-1} = it_B^{-1} \Rightarrow \langle n \rangle \approx \frac{t}{t_B}$$

- ▶  $t_B$ : order 1 in the unit of Lyapunov time  $\lambda_L^{-1}$ .

[Open question: does  $t_B$  show up in thermodynamics, transport ... ?]

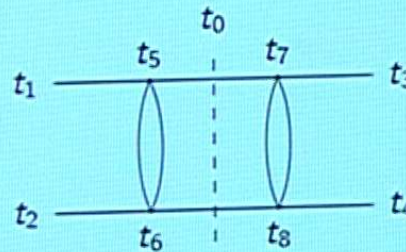
## Derivation of the identity

Next, we sketch the derivation of

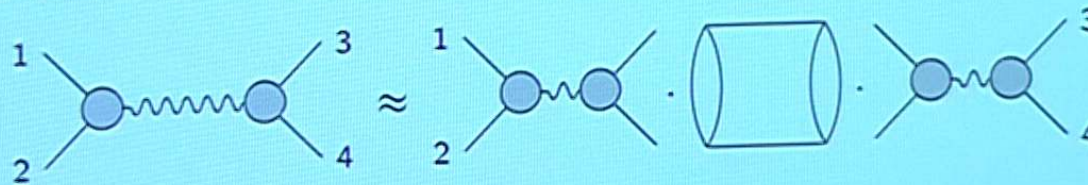
$$N \cdot \frac{2 \cos \frac{\lambda_L \pi}{2}}{C} \cdot k'_R(-\lambda_L) \cdot (\gamma^A, \gamma^R) = 1.$$

Idea: cut a long ladder into pieces and find a consistency condition.

- ▶ Cut. Fix  $t_0$ , find adjacent  $\frac{t_5+t_6}{2} < t_0 < \frac{t_7+t_8}{2}$



- ▶ Consistency condition

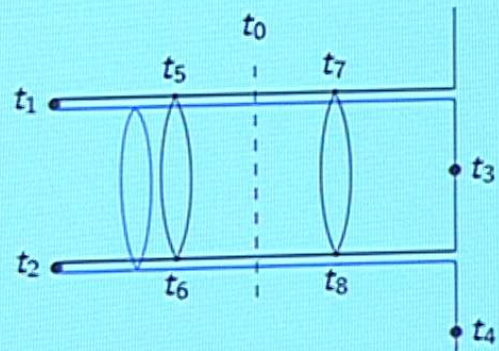


## Derivation: $\cos \frac{\lambda_L \pi}{2}$ factor

- ▶ Naively, we would have a formula:

$$\text{OTOC} \approx \text{OTOC}_L \cdot \text{BOX} \cdot \text{OTOC}_R$$

- ▶ Subtlety: multiple choices on the double Keldysh contour



- ▶ Sum of two choices:

$$\text{OTOC} \approx \left( e^{i \frac{\lambda_L \pi}{2}} + e^{-i \frac{\lambda_L \pi}{2}} \right) \text{OTOC}_L \cdot \text{BOX} \cdot \text{OTOC}_R$$

## Derivation: box

- ▶ Next, integrate over  $t_5, t_6, t_7, t_8$

$$\begin{array}{c}
 \text{wavy line} \text{---} \text{circle} \text{---} \text{box} \text{---} \text{circle} \text{---} \text{wavy line} \quad : \quad \int ds dt_* e^{-\lambda_L s} \text{---} \text{circle} \text{---} \text{box} \text{---} \text{circle} \\
 \\
 s = \frac{t_5 + t_6 - t_7 - t_8}{2} : \text{ size of the box; } t_* : \text{ center of mass time } \int dt_* = s
 \end{array}$$

- ▶ How does this term related to  $k'_R(-\lambda_L)$ ?

$$\begin{array}{c}
 \int ds e^{-\lambda_L s} \text{---} \text{circle} \text{---} \text{box} \text{---} \text{circle} = \text{---} \text{circle} \text{---} \int ds e^{-\lambda_L s} \text{---} \text{box} \text{---} \text{circle} \\
 \\
 = k_R(-\lambda_L) \text{---} \text{circle} \text{---} \text{circle}
 \end{array}$$

- ▶ Take  $\lambda_L$  derivative:

$$t_B(\gamma^A, \gamma^R) = \text{wavy line} \text{---} \text{circle} \text{---} \text{box} \text{---} \text{circle} \text{---} \text{wavy line}$$

## Derivation: box

- ▶ Next, integrate over  $t_5, t_6, t_7, t_8$

$$\begin{aligned}
 & \text{Diagram: wavy line} \rightarrow \text{circle} \rightarrow \text{box} \rightarrow \text{circle} \rightarrow \text{wavy line} \quad : \quad \int ds dt_* e^{-\lambda_L s} \text{Diagram: circle} \rightarrow \text{box} \rightarrow \text{circle} \\
 & s = \frac{t_5 + t_6 - t_7 - t_8}{2} : \text{ size of the box; } t_* : \text{ center of mass time } \int dt_* = s
 \end{aligned}$$

- ▶ How does this term related to  $k'_R(-\lambda_L)$ ?

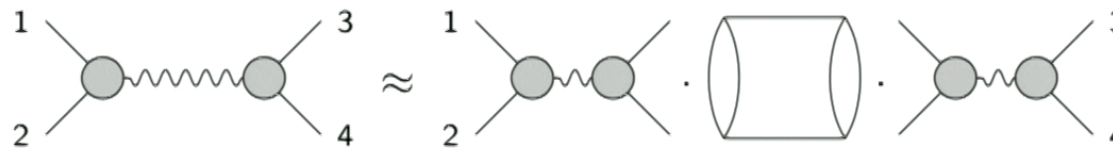
$$\begin{aligned}
 \int ds e^{-\lambda_L s} \text{Diagram: circle} \rightarrow \text{box} \rightarrow \text{circle} &= \text{Diagram: circle} \rightarrow \int ds e^{-\lambda_L s} \text{Diagram: box} \rightarrow \text{circle} \\
 &= k'_R(-\lambda_L) \text{Diagram: circle} \rightarrow \text{circle}
 \end{aligned}$$

- ▶ Take  $\lambda_L$  derivative:

$$t_B(\gamma^A, \gamma^R) = \text{Diagram: wavy line} \rightarrow \text{circle} \rightarrow \text{box} \rightarrow \text{circle} \rightarrow \text{wavy line}$$

## Derivation: summary

- ▶ We start with consistency condition



- ▶ Compare two sides, we find

$$\frac{1}{C} = N \cdot \frac{2 \cos \frac{\lambda_L \pi}{2}}{C^2} \cdot t_B \cdot (\gamma^A, \gamma^R)$$

## Applications

The ladder identity:

$$N \cdot \frac{2 \cos \frac{\lambda_L \pi}{2}}{C} \cdot t_B \cdot (\gamma^A, \gamma^R) = 1.$$

Next:

- ▶ Computational shortcuts:  $C \Leftrightarrow \lambda_L$ ;
- ▶ In a 1D model, prove exact maximal chaos using the identity.

## Computational shortcuts I: near maximal chaos

SYK at strong coupling  $J \gg 1$ , near maximal chaos  $\lambda_L = 1 - \delta\lambda_L$ .

- ▶ Schwarzian action:

$$I_{\text{Sch}}[\varphi] = -\frac{N\alpha_S}{J} \int_0^{2\pi} \text{Sch} \left( e^{i\varphi(\tau)}, \tau \right) d\tau$$

- ▶ Use Schwarzian action:

$$\text{OTOC} \approx \frac{J e^{(t_1+t_2-t_3-t_4)/2}}{2N\alpha_S} \cdot \frac{2\Delta b^\Delta J^{-2\Delta}}{(2 \cosh \frac{t_{12}}{2})^{2\Delta+1}} \cdot \frac{2\Delta b^\Delta J^{-2\Delta}}{(2 \cosh \frac{t_{34}}{2})^{2\Delta+1}}$$

- ▶ Find correction  $\delta\lambda_L \approx 2 \cos \frac{(1-\delta\lambda_L)\pi}{2} / \pi$  (Maldacena-Stanford, 2016)

$$\delta\lambda_L \approx \frac{C}{\pi t_B(\Upsilon^A, \Upsilon^R)} = \frac{6\alpha_S}{J k'_R(-1) \Delta(1-\Delta)(1-2\Delta) \tan(\pi\Delta)}.$$

## Computational shortcuts II: prefactor

- ▶ Large  $q$  SYK, fix  $\mathcal{J} = \sqrt{2^{1-q}q} J$  (Maldacena-Stanford, 2016):

$$I \quad \frac{v}{2 \cos \frac{\pi v}{2}} = \mathcal{J}, \quad 0 < v < 1.$$

- ▶ Exact correlation function at all coupling  $v$ ;

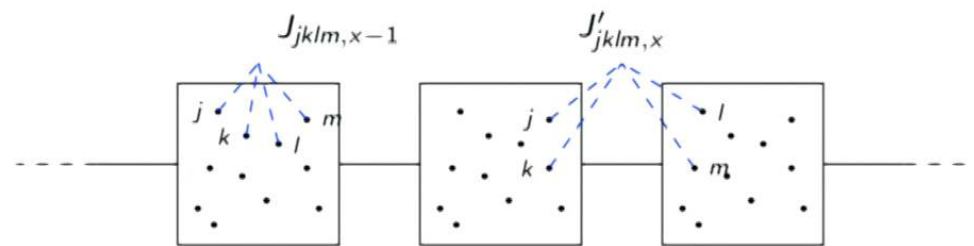
$$K^R = \theta(t_{13})\theta(t_{24}) \frac{v^2}{2 \cosh^2 \frac{vt_{34}}{2}}, \quad \lambda_L = v$$

- ▶ Use the identity to find the prefactor (Qi-Streicher, 2018):

$$\text{OTOC}(t_1, t_2; t_3, t_4) \approx \frac{1}{N \cos \frac{v\pi}{2}} \frac{e^{v(t_1+t_2-t_3-t_4)/2}}{(2 \cosh \frac{vt_{12}}{2}) (2 \cosh \frac{vt_{34}}{2})}.$$

## Maximal chaos in a 1D model

- ▶ Regard  $\lambda_L$  and  $C$  as analytic functions of some parameter, then the analytical properties of  $\lambda_L$  and  $C$  are locked by the ladder identity.
- ▶ A concrete example: SYK chain ([YG-Qi-Stanford, 2016](#))



(also see [[Guo-YG-Sachdev, 2019](#)] )

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Page 21 of 33

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$$K^R = \theta(t_{13})\theta(t_{24}) \frac{v^2}{2 \cosh^2 \frac{v t_{14}}{2}}, \quad \lambda_L = v$$

- ▶ Use the identity to find the prefactor (Qi-Streicher, 2018):

$$\text{OTOC}(t_1, t_2; t_3, t_4) \approx \frac{1}{N \cos \frac{v\pi}{2}} \frac{e^{v(t_1+t_2-t_3-t_4)/2}}{(2 \cosh \frac{v t_{12}}{2}) (2 \cosh \frac{v t_{34}}{2})}.$$

Page 22 of 33

## Operators at two different locations

Operators at two different locations:

$$\text{OTOC}_{x,0}(t_1, t_2, t_3, t_4) := \frac{1}{N^2} \sum_{j,k} \langle \chi_{j,x}(t_1) \chi_{k,0}(t_3) \chi_{j,x}(t_2) \chi_{k,0}(t_4) \rangle_{\text{conn.}}$$

- ▶ Fourier transform:

$$\text{OTOC}_{x,0}(t_1, t_2, t_3, t_4) = \int \frac{dp}{2\pi} e^{ipx} \text{OTOC}_p(t_1, t_2, t_3, t_4)$$

- ▶ Each  $\text{OTOC}_p$ : ladder diagrams dominate. Retarded kernel factorizes:

$$K^R(p) = s(p) K^R, \quad s(p) = 1 - 2a(1 - \cos p) \approx 1 - ap^2.$$

$s(p)$ : “band structure” of the bilocal fields  $a = \frac{J_1^2}{3J^2} \in (0, 1/3)$ .

## Fourier Transform

- ▶ The ladder identity holds for each  $\text{OTOC}_p$ :

$$C(p) = 2 \cos \frac{\lambda_L(p)\pi}{2} \cdot t_B \cdot (\Upsilon^A, \Upsilon^R),$$

- ▶ The dependence of  $t_B$  and  $(\Upsilon^A, \Upsilon^R)$  on  $p$  is not important (analytic and do not vanish in the domain of interest).

$$\text{OTOC}_{x,0}(t_1, t_2, t_3, t_4) \sim \frac{1}{N} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \underbrace{\frac{e^{\lambda_L(p)t+ipx}}{2 \cos \frac{\pi \lambda_L(p)}{2}}}_{u(x,t)} \cdot \frac{\Upsilon^R(t_{12})\Upsilon^A(t_{34})}{t_B(\Upsilon^A, \Upsilon^R)}.$$

$$t = \frac{t_1+t_2-t_3-t_4}{2}. \quad \lambda_L(p) \approx \lambda_L(0) - t_B^{-1}ap^2 \text{ when } |p| \ll 1.$$

## Butterfly wavefront

$$u(x, t) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{e^{\lambda_L(p)t + ipx}}{2 \cos \frac{\lambda_L(p)}{2}}, \quad \lambda_L(p) \approx \lambda_L(0) - t_B^{-1} ap^2$$

- ▶ Butterfly wavefront:  $u(x, t) \sim 1$ .
- ▶ For large  $x > 0$  and  $t$ , we can estimate by saddle point of the exponent:

$$\lambda'_L(p)t + ix = 0, \quad p = i|p|.$$

- ▶ Find a butterfly velocity

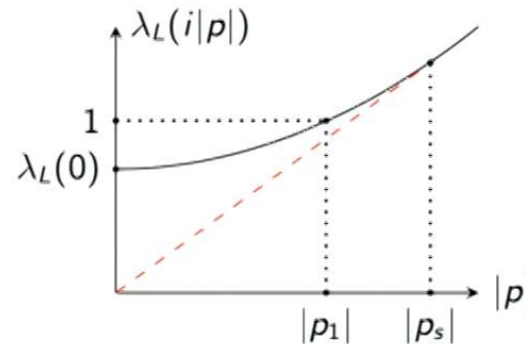
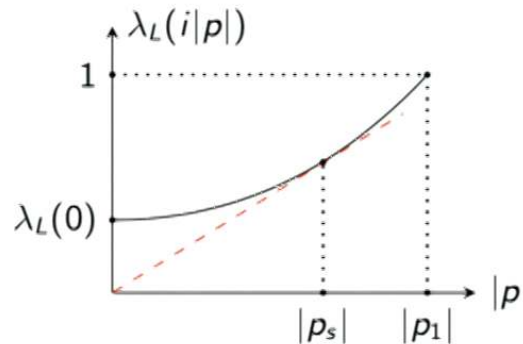
$$v_s = \frac{i\lambda_L(p_s)}{p_s} = i\lambda'_L(p_s)$$

## Graphic solutions: two scenarios

- ▶ The relevant saddle point  $p_s = i|p_s|$  is purely imaginary. Deform the integral contour to pass, might cross the pole:

$$\cos \frac{\lambda_L(p_1)\pi}{2} = 0, \quad \lambda_L(p_1) = 1, \quad p_1 = i|p_1|.$$

- ▶ Two scenarios:

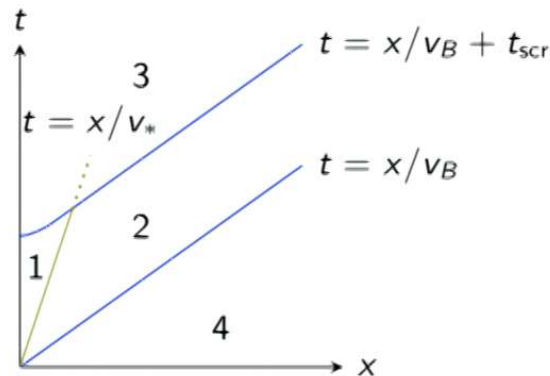


## Pole contribution: maximal chaos

- ▶ In the second scenario, the pole dominates:

$$u_1(x, t) \approx \frac{e^{t-|p_1|x}}{\pi i \lambda'_L(p_1)}, \quad v_1 = \frac{1}{|p_1|}$$

- ▶ In SYK at  $J \gg 1$ ,  $\delta\lambda_L = 1 - \lambda_L(0) \ll 1$ ,  $|p_1| \approx \sqrt{t_B \delta\lambda_L / a} \ll |p_s|$ , pole dominates.  $v_B = v_1$ .



## Summary and discussion

- ▶ Ladder identity relates  $C$  and  $\lambda_L$ ;

$$\frac{2 \cos \frac{\lambda_L \pi}{2}}{C} \cdot t_B \cdot (\gamma^A, \gamma^R) = 1, \quad t_B = k'_R(-\lambda_L)$$

[Is  $t_B$  “useful”? (Related to thermodynamics or transport?)]

- ▶ Applications:
  - ▶ computational shortcuts,  $\delta\lambda_L \propto t_B^{-1}$
  - ▶ maximal chaos [Why? Can we design 0 + 1-d example?]