

Title: PSI 2018/2019 - Explorations in Cosmology - Lecture 10

Speakers: Kendrick Smith

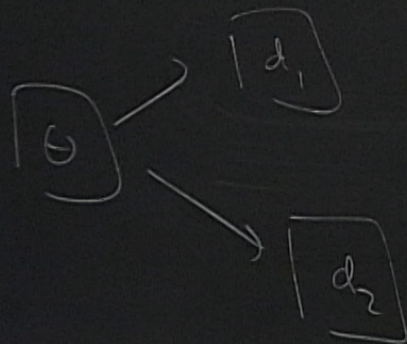
Collection: PSI 2018/2019 - Explorations in Cosmology (Smith)

Date: April 29, 2019 - 11:30 AM

URL: <http://pirsa.org/19040072>

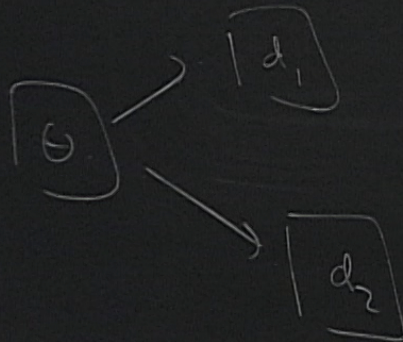
NO LECTURE TOMORROW

$$F_{ab} = \left\langle - \frac{\partial^2 \log p(d|\theta)}{\partial \theta^a \partial \theta^b} \right\rangle$$

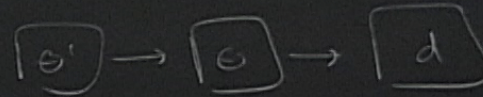


NO LECTURE TOMORROW

$$F_{ab} = \left\langle - \frac{\partial^2 \log p(d|\theta)}{\partial \theta^a \partial \theta^b} \right\rangle$$



$$F = F_1 + F_2$$



$$F'_{ab} = \frac{\partial \theta^c}{\partial \theta'^a} \frac{\partial \theta^d}{\partial \theta'^b} F_{cd}$$

LEMMA

F

LEMMA FISHER MATRIX IS GIVEN BY:

$$F_{ab} = \left\langle \left(\frac{\partial \log p(\mathbf{d}|\boldsymbol{\theta})}{\partial \theta^a} \right) \left(\frac{\partial \log p(\mathbf{d}|\boldsymbol{\theta})}{\partial \theta^b} \right) \right\rangle_{\mathbf{d}}$$

LEMMA. FISHER MATRIX IS GIVEN BY:

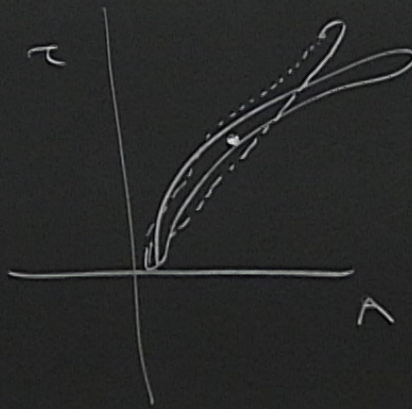
$$F_{ab} = \left\langle \left(\frac{\partial \log p(d|\theta)}{\partial \theta^a} \right) \left(\frac{\partial \log p(d|\theta)}{\partial \theta^b} \right) \right\rangle_d$$

PROOF. FIRST NOTE $\langle \dots \rangle_d = \int Dd \, p(d|\theta) [\dots]$

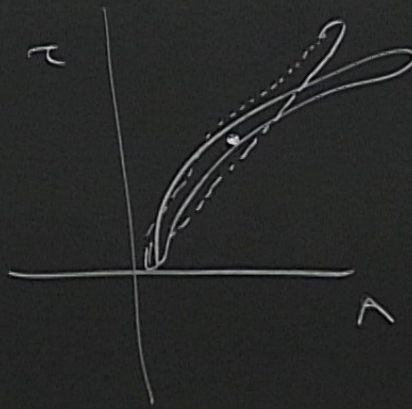
$$\begin{aligned} \text{THEN } F_{ab} &= \left\langle - \frac{\partial^2 \log p(d|\theta)}{\partial \theta^a \partial \theta^b} \right\rangle_d \\ &= \int Dd \, p(d|\theta) \left[- \frac{1}{p(d|\theta)} \frac{\partial^2 p(d|\theta)}{\partial \theta^a \partial \theta^b} + \frac{1}{p(d|\theta)^2} \frac{\partial p(d|\theta)}{\partial \theta^a} \frac{\partial p(d|\theta)}{\partial \theta^b} \right] \end{aligned}$$

$$\begin{aligned}
 F_{ab} &= \int da \, p(a|\theta) \left[-\frac{1}{p(a|\theta)} \frac{\partial^2 p(a|\theta)}{\partial \theta^a \partial \theta^b} + \frac{1}{p(a|\theta)^2} \frac{\partial p(a|\theta)}{\partial \theta^a} \frac{\partial p(a|\theta)}{\partial \theta^b} \right] \\
 &= \frac{\partial^2}{\partial \theta^a \partial \theta^b} \underbrace{\int da \, p(a|\theta)}_{=1} + \left\langle \left(\frac{\partial \log p(a|\theta)}{\partial \theta^a} \right) \left(\frac{\partial \log p(a|\theta)}{\partial \theta^b} \right) \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 F_{ab} &= \int da \, p(a|\theta) \left[-\frac{1}{p(a|\theta)} \frac{\partial^2 p(a|\theta)}{\partial \theta^a \partial \theta^b} + \frac{1}{p(a|\theta)^2} \frac{\partial p(a|\theta)}{\partial \theta^a} \frac{\partial p(a|\theta)}{\partial \theta^b} \right] \\
 &= \frac{\partial^2}{\partial \theta^a \partial \theta^b} \underbrace{\int da \, p(a|\theta)}_{=1} + \left\langle \left(\frac{\partial \log p(a|\theta)}{\partial \theta^a} \right) \left(\frac{\partial \log p(a|\theta)}{\partial \theta^b} \right) \right\rangle
 \end{aligned}$$



$$\frac{\partial^2 \mathcal{L}}{\partial \theta^a \partial \theta^b}$$



- THEOREM: F_{ab} IS ALWAYS POSITIVE SEMIDEFINITE.
(I.E. ALL EIGENVALUES ≥ 0)

FIRST RECALL (F IS POS. SEMIDEFINITE) $\Leftrightarrow (V^T F V \geq 0 \quad \forall \text{ VECTORS } V)$

$$\begin{aligned}\text{NOW } V^T F V &= V^a V^b F_{ab} \\ &= V^a V^b \left\langle \left(\frac{\partial \log p(a|b)}{\partial \theta^a} \right) \left(\frac{\partial \log p(a|b)}{\partial \theta^b} \right) \right\rangle_a \\ &= \left\langle \left(V^a \frac{\partial \log p(a|b)}{\partial \theta^a} \right)^2 \right\rangle_a \geq 0\end{aligned}$$

PROOF OF TRANSFORMATION LAW

$$\boxed{\theta'} \rightarrow \boxed{\theta} \rightarrow \boxed{d}$$

$$\begin{aligned} F'_{ab} &= \left\langle \left(\frac{\partial \log p(d|\theta)}{\partial \theta'^a} \right) \left(\frac{\partial \log p(d|\theta)}{\partial \theta'^b} \right) \right\rangle_d \\ &= \left\langle \left(\frac{\partial \theta^c}{\partial \theta'^a} \frac{\partial \log p(d|\theta)}{\partial \theta^c} \right) \left(\frac{\partial \theta^d}{\partial \theta'^b} \frac{\partial \log p(d|\theta)}{\partial \theta^d} \right) \right\rangle_d \\ &= \frac{\partial \theta^c}{\partial \theta'^a} \frac{\partial \theta^d}{\partial \theta'^b} F_{cd} \end{aligned}$$

TRANSFORMATION LAWS

"JEFFREYS PRIOR"

• PRIOR
$$p'(\theta') = \underbrace{\left(\frac{D^n \theta'}{D^n \theta} \right)^{-1}}_{\text{JACOBIAN DETERMINANT}} p(\theta) [\theta \rightarrow \theta']$$

$$\begin{aligned} \langle f \rangle &= \int D^n \theta \, p(\theta) f(\theta) \\ &= \int D^n \theta' \, p(\theta') f(\theta') \end{aligned}$$

"JEFFREYS PRIOR"

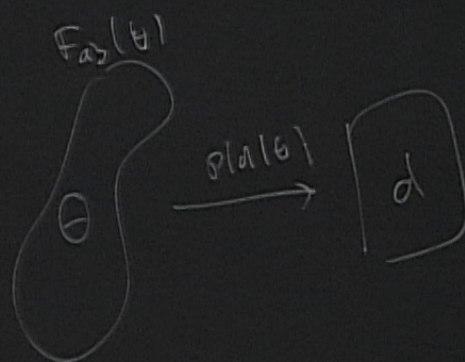
(6) $[\theta \rightarrow \theta']$

DOMINANT

(6) $f(\theta)$

$p(\theta') f(\theta')$

$$ds^2 = F_{ab}(\theta) d\theta^a d\theta^b$$



UNIFORM PRIOR $p(\theta) = 1$

IS NOT RE

"JEFFREYS PRIOR"

(5) $[\theta \rightarrow \theta']$

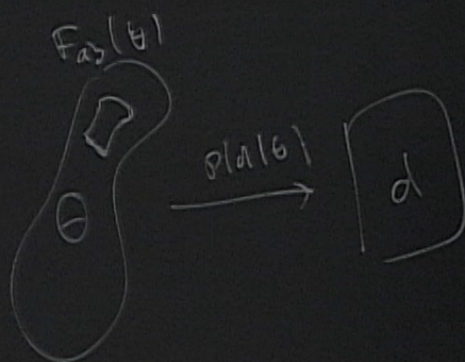
$$\int d^n \theta \sqrt{F(\theta)} [\dots]$$

DOMINANT

(6) $f(\theta)$

$p(\theta') f(\theta')$

$$ds^2 = F_{ab}(\theta) d\theta^a d\theta^b$$



UNIFORM PRIOR $p(\theta) = 1$

IS NOT RE

JEFF

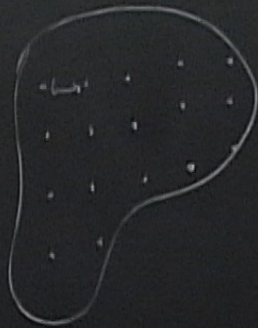
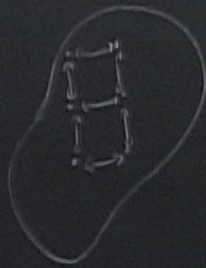
UNIFORM PRIOR $p(\theta) = 1$

IS NOT REPARAMETERIZATION INVARIANT

JEFFREYS PRIOR

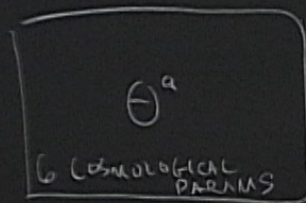
$$p(\theta) = \sqrt{\text{DET. } F_{\theta_j}(\theta)}$$

IS INVARIANT

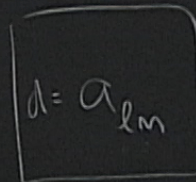


$$\sqrt{-g'} = \left(\frac{\partial u_{x'}}{\partial u_x} \right)^{-1} \sqrt{g}$$

MONTÉ CARLO MARKOV CHAINS



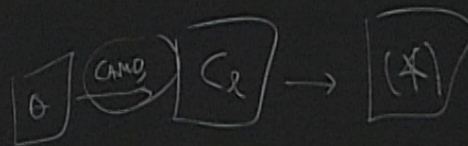
$p(a_{lm}|\theta)$



CMB MULTIPOLES
COSMIC VARIANCE
LIMITED

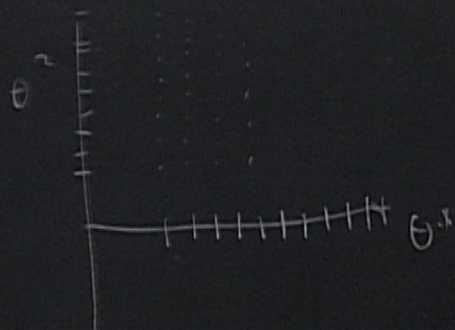
"THE" LIKELIHOOD

$$\mathcal{L}(\theta) = p(b|a_{lm}) \propto p(a_{lm}|\theta) \propto \prod_{\ell} C_{\ell}^{-(\ell+1)/2} \prod_{m=-\ell}^{\ell} \exp\left(-\frac{|a_{\ell m}|^2}{2C_{\ell}}\right) \quad (*)$$



1 CPU-SECOND

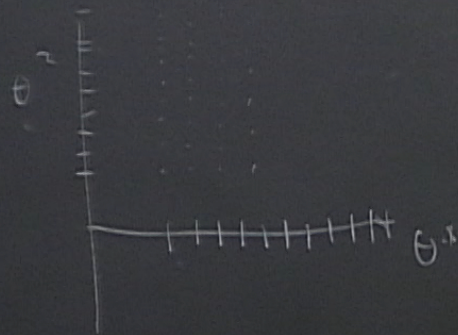
"EXPLORE" $\mathcal{L}(\theta) \iff$ "GRID" ON 20^6 POINTS ?



(*)

"EXPLORE" $L(\theta)$ $\Leftrightarrow$ "GRID" ON 20^6 POINTS ? X

20^6 CPU-SECONDS = 2 CPU-YEARS



$\Leftrightarrow$ MCMC

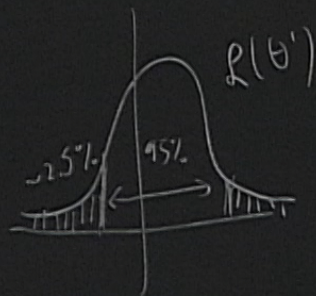
BEST REPRESENTATION IS NOT A "GRID"
BUT A SEQUENCE OF RANDOM SAMPLES

$\theta^{a(1)}, \theta^{a(2)}, \dots, \theta^{a(M)}$

$M \approx 10^5$

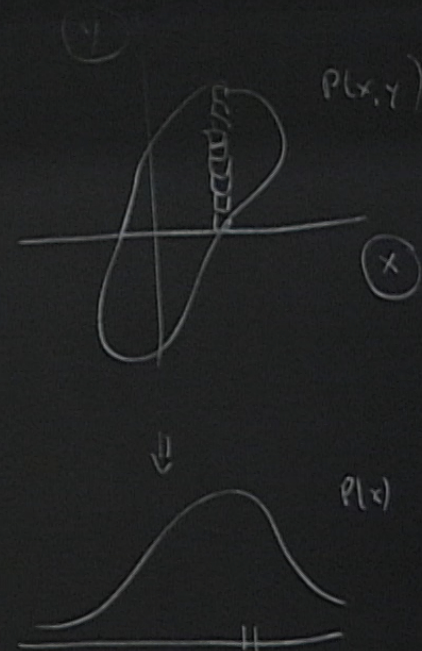
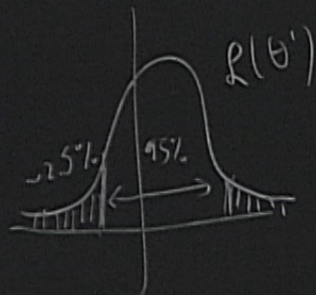
USUALLY WE'RE INTERESTED IN
"MARGINALIZED 1-D LIKELIHOOD"

$$L(\theta^1) = \int d\theta^2 \dots d\theta^N L(\theta^1, \theta^2, \dots, \theta^N)$$



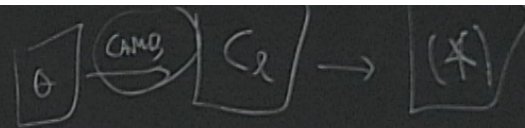
USUALLY WE'RE INTERESTED IN
"MARGINALIZED 1-D LIKELIHOOD"

$$\underbrace{L(\theta^1)} = \int d\theta^2 \dots d\theta^N \underbrace{L(\theta^1, \theta^2, \dots, \theta^N)}_{10^5}$$



$$\mathcal{L}(\theta^1, \dots, \theta^N) \approx \frac{1}{N} \sum_n \delta^{(n)}(\theta^1 - \theta^{1(n)})$$

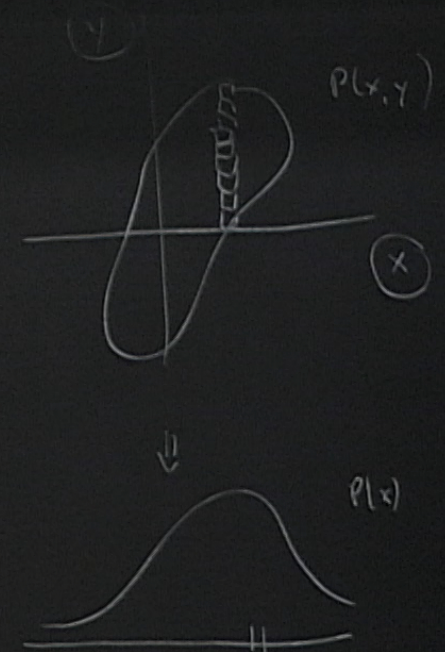
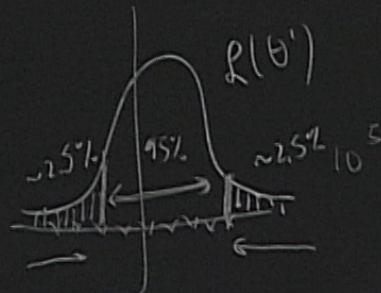
$$\mathcal{L}(\theta^1) = \int d\theta^2 \dots d\theta^N \mathcal{L}(\theta^1, \dots, \theta^N) \approx \frac{1}{N} \sum_n \delta^n(\theta^1 - \theta^{1(n)})$$

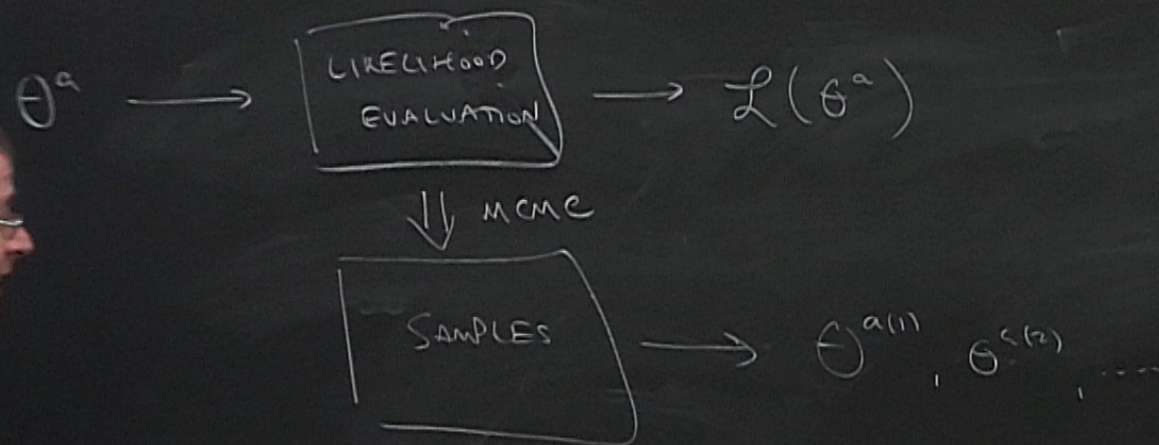


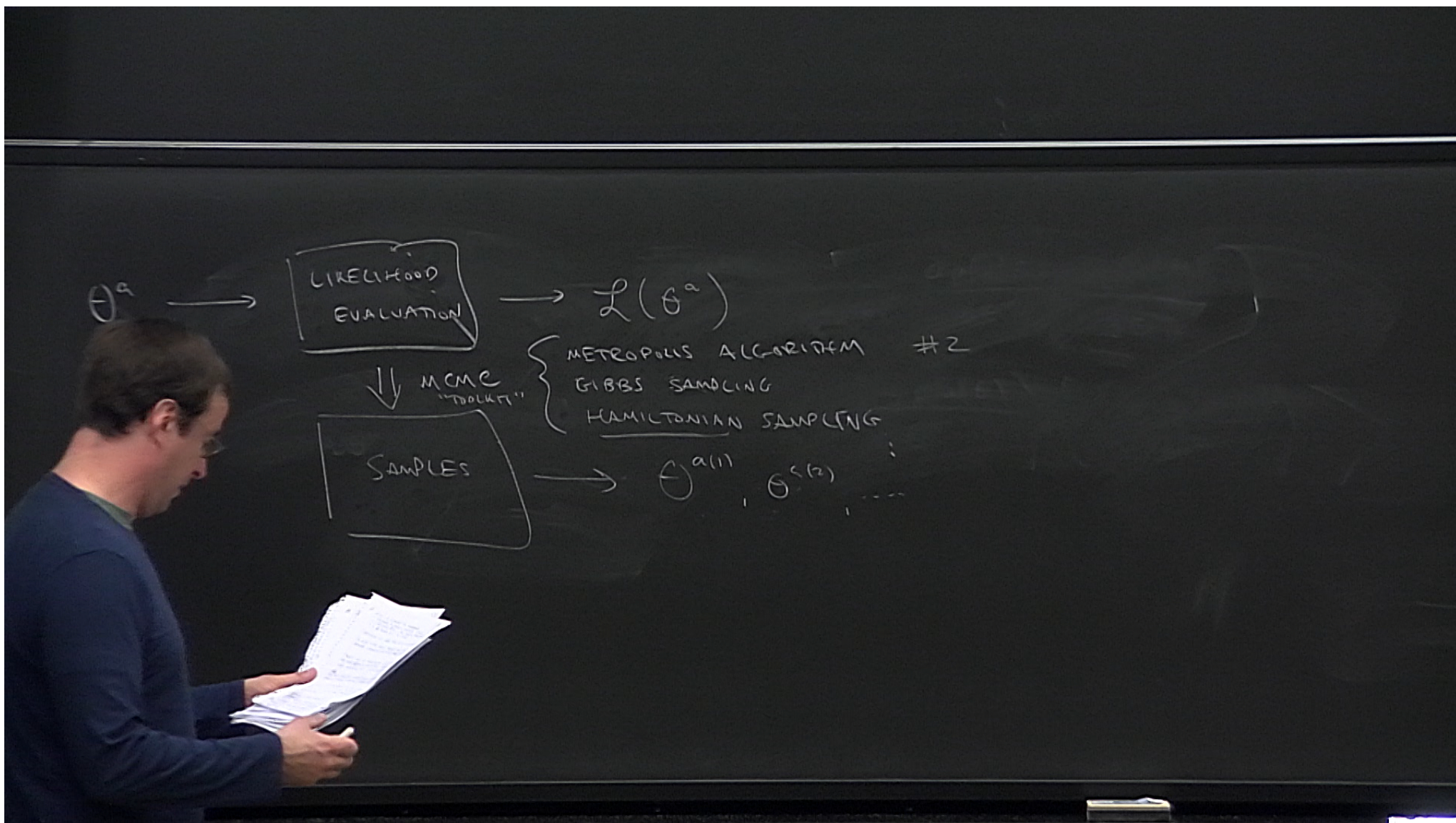
1 CPU-SECOND

USUALLY WE'RE INTERESTED IN
"MARGINALIZED 1-D LIKELIHOOD"

$$\mathcal{L}(\theta^1) = \int d\theta^2 \dots d\theta^N \underbrace{\mathcal{L}(\theta^1, \theta^2, \dots, \theta^N)}_{10^5}$$







$$L(\theta^a)$$

METROPOLIS ALGORITHM #2

GIBBS SAMPLING

HAMILTONIAN SAMPLING

$$\rightarrow \theta^{(1)}, \theta^{(2)}, \dots$$

$\approx 10^5$ INDEPENDENT SAMPLES

$$\left(\frac{\text{\# LIKELIHOOD EVALUATIONS}}{\text{INDEPENDENT SAMPLE}} \right) \approx \Theta(N) \rightarrow \text{DIMENSION OF MODEL SPACE}$$

