

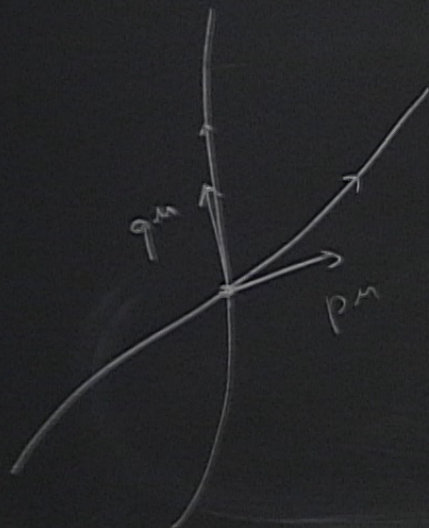
Title: PSI 2018/2019 - Explorations in Cosmology - Lecture 9

Speakers: Kendrick Smith

Collection: PSI 2018/2019 - Explorations in Cosmology (Smith)

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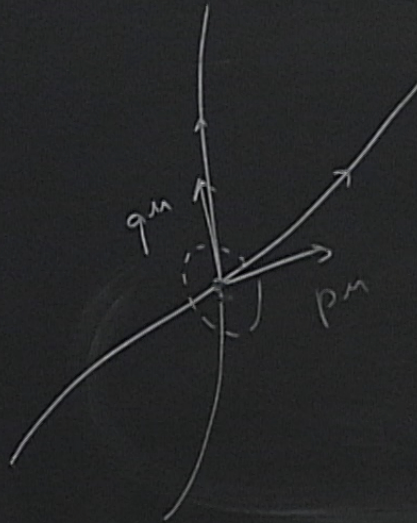


TWO PARTICLES OF MASS m
WHOSE WORLDLINES CROSS

$$p^\mu, q^\mu = \text{4-MOMENTA AT CROSSING}$$

E_{12} = "ENERGY OF PARTICLE 1, AS OBSERVED
IN REST FRAME OF PARTICLE 2"

$$= -\frac{1}{m} g_{\mu\nu} p^\mu q^\nu$$



TWO PARTICLES OF MASS m
WHOSE WORLDLINES CROSS

$$p^\mu, q^\mu = \text{4-MOMENTUM}$$

E_{12} = "ENERGY OF PARTICLE 1,
IN REST FRAME OF PARTICLE 2"

$$= -\frac{1}{m} g_{\mu\nu} p^\mu q^\nu$$

"BRINED" COORDS = REST-FRAME COORDS FOR PARTICLE 2

$$g'_{\mu\nu} = \begin{pmatrix} -1 & \\ & \delta_{ij} \end{pmatrix}$$

$$q'^{\mu} = \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$p'^{\mu} = \begin{pmatrix} E_{12} \\ p_{12} \end{pmatrix}$$

THEN $E_{12} = p'^0$ [NON-SCALAR]

$$= -\frac{1}{m} g'_{\mu\nu} p'^{\mu} q'^{\nu} \quad [\text{SCALAR}]$$

WORDS FOR PARTICLE 2

$$\begin{aligned}\text{THEN } E_{12} &= p^{10} && [\text{NON-SCALAR}] \\ &= -\frac{1}{m} g'_{\mu\nu} p'^{\mu} q'^{\nu} && [\text{SCALAR}]\end{aligned}$$

"SPECIAL COORDINATE SYSTEM ARGUMENT"

CMB FISHER MATRIX

$$\{\theta^a\} \longrightarrow \{C_\ell\} \longrightarrow \{a_{\ell m}\}$$

COSMOLOGICAL
PARAMETERS

DATA

ASSUME $C_\ell(\theta_{fid})$ AND $\frac{\partial C_\ell}{\partial \theta^a} \approx \frac{C_\ell(\theta_{fid}^a + \Delta\theta^a) - C_\ell(\theta_{fid}^a - \Delta\theta^a)}{2\Delta\theta^a}$

CAN BE CALCULATED NUMERICALLY

$$F_{ab} = \left\langle - \frac{\partial^2}{\partial \theta^a \partial \theta^b} \log p(a_m | \theta) \right\rangle_{a_m}$$

$$= \left\langle - \frac{\partial^2}{\partial \theta^a \partial \theta^b} \sum_l \left(- \frac{2l+1}{2} \log C_l - \sum_{m=-l}^l \frac{|a_{lm}|^2}{2C_l} \right) \right\rangle_{a_m}$$

$$= - \left\langle \sum_l \left(- \frac{2l+1}{2} \frac{1}{C_l} \frac{\partial^2 C_l}{\partial \theta^a \partial \theta^b} + \frac{2l+1}{2} \frac{1}{C_l^2} \frac{\partial C_l}{\partial \theta^a} \frac{\partial C_l}{\partial \theta^b} + \sum_{m=-l}^l \frac{|a_{lm}|^2}{2C_l^2} \frac{\partial^2 C_l}{\partial \theta^a \partial \theta^b} - \sum_{m=-l}^l \frac{|a_{lm}|^2}{C_l^3} \frac{\partial C_l}{\partial \theta^a} \frac{\partial C_l}{\partial \theta^b} \right) \right\rangle_{a_m}$$

$$= \sum_e \left(-\frac{2e-1}{2} \frac{1}{C_e^2} \frac{\partial^2 C_e}{\partial \theta^a \partial \theta^b} + \frac{2e+1}{2} \frac{1}{C_e^2} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \right. \\ \left. + (2e+1) \frac{C_e}{2 C_e^2} \frac{\partial^2 C_e}{\partial \theta^a \partial \theta^b} - (2e+1) \frac{C_e}{C_e^3} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \right)$$

$$= \sum_e \frac{2e+1}{2} \frac{1}{C_e^2} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b}$$

$$= \sum_e \left(-\frac{2\ell+1}{2} \frac{1}{C_e^2} \frac{\partial^2 C_e}{\partial \theta^a \partial \theta^b} + \frac{2\ell+1}{2} \frac{1}{C_e^2} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \right. \\ \left. + (2\ell+1) \frac{C_e}{2 C_e^2} \frac{\partial C_e}{\partial \theta^a \partial \theta^b} - (2\ell+1) \frac{C_e}{C_e^3} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \right)$$

$$= \sum_e \frac{2\ell+1}{2} \frac{1}{C_e^2} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \quad \left[\text{COSMIC VARIANCE LIMITED} \right]$$

$$\rightarrow \sum_e \frac{2\ell+1}{2} \frac{1}{(C_e + N_e)^2} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \quad \left[\text{W/NOISE} \right]$$

$$\text{"INDEPENDENT"} \Leftrightarrow p(d_1, d_2 | \theta) = p(d_1 | \theta) p(d_2 | \theta)$$

$$F_{ab} = \left\langle - \frac{\partial^2 \log p(d_1, d_2 | \theta)}{\partial \theta^a \partial \theta^b} \right\rangle$$

$$= \left\langle - \frac{\partial^2 \log p(d_1 | \theta)}{\partial \theta^a \partial \theta^b} - \frac{\partial^2 \log p(d_2 | \theta)}{\partial \theta^a \partial \theta^b} \right\rangle$$

$$= F_{ab}^{(1)} + F_{ab}^{(2)}$$

$$= - \sum_e \left(- \frac{2e+1}{2} \frac{1}{C_e^2} \frac{\partial^2 C_e}{\partial \theta^a \partial \theta^b} + \frac{2e+1}{2} \frac{1}{C_e^2} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} + (2e+1) \frac{C_e}{2 C_e^2} \frac{\partial C_e}{\partial \theta^a \partial \theta^b} - (2e+1) \frac{C_e}{C_e^3} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \right)$$

$$= \sum_e \frac{2e+1}{2} \frac{1}{C_e^2} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \quad \left[\text{COSMIC VARIANCE LIMITED} \right]$$

$$\rightarrow \sum_e \frac{2e+1}{2} \frac{1}{(C_e + N_e)^2} \frac{\partial C_e}{\partial \theta^a} \frac{\partial C_e}{\partial \theta^b} \quad \left[\text{W/NOISE} \right]$$

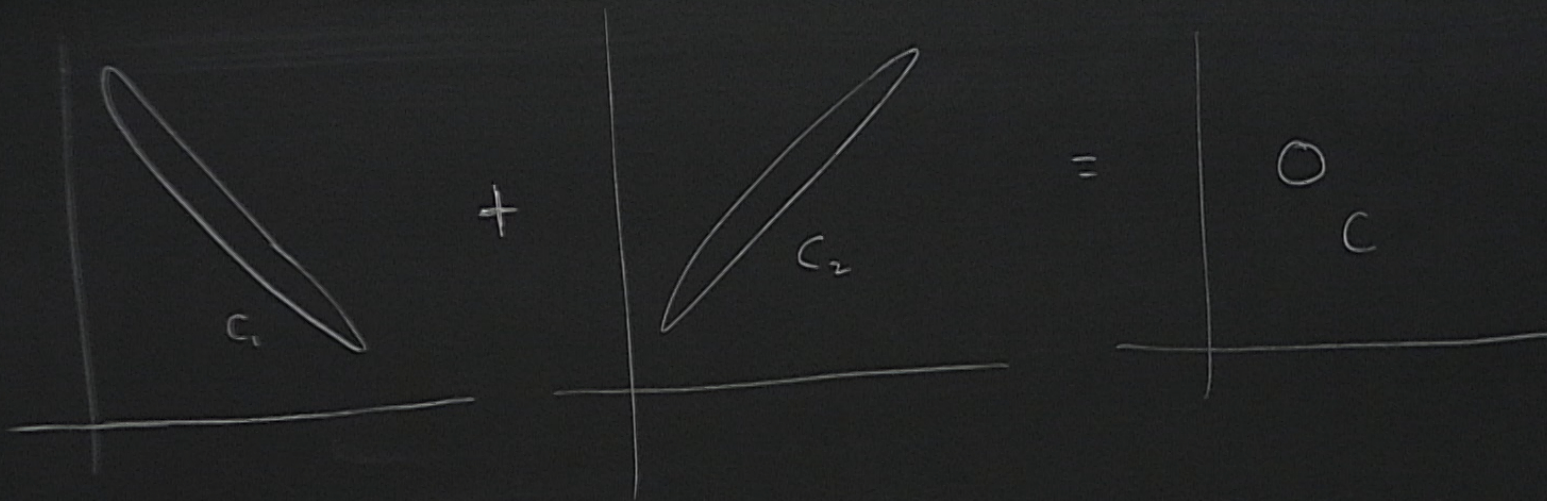
$$r^* \Leftrightarrow p(d_1, d_2 | \theta) = p(d_1 | \theta) p(d_2 | \theta)$$

"FISHER INFORMATION MATRIX"

$$\left\langle \frac{\partial^2 \log p(d_1, d_2 | \theta)}{\partial \theta^a \partial \theta^b} \right\rangle$$

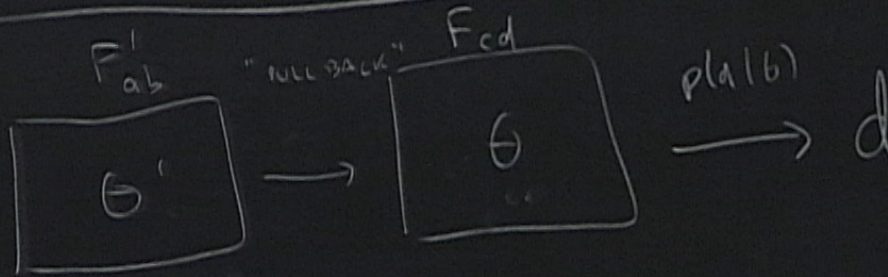
$$\left\langle \frac{\partial^2 \log p(d_1 | \theta)}{\partial \theta^a \partial \theta^b} + \frac{\partial^2 \log p(d_2 | \theta)}{\partial \theta^a \partial \theta^b} \right\rangle$$

$$F_{ab}^{(1)} + F_{ab}^{(2)}$$



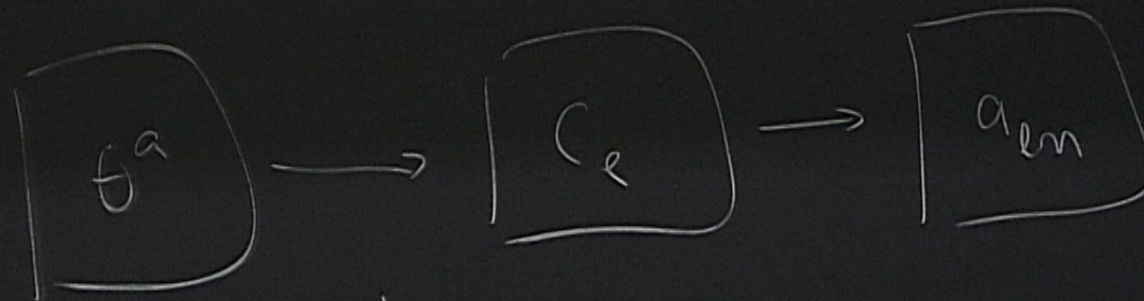
$$C^{-1} = C_1^{-1} + C_2^{-1}$$

TRANSFORMATION LAW



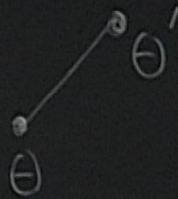
REPARAMETERIZATION
SUBMODEL

$$\Rightarrow F'_{ab} = \left(\frac{\partial \Theta^c}{\partial \Theta'^a} \right) \left(\frac{\partial \Theta^d}{\partial \Theta'^b} \right) F_{cd}$$



$$F_{ab} = \sum_e \frac{2\ell+1}{2} \frac{1}{c_e^2} \frac{\partial c_e}{\partial \theta^a} \frac{\partial c_e}{\partial \theta^b} \quad \bigg/ \quad F_{ee'} = \frac{2\ell+1}{2} \frac{1}{c_e^2} \delta_{ee'}$$

$$ds^2 = F_{ab} d\theta^a d\theta^b$$



EXAMPLE: GIVEN A "DERIVED" PARAMETER $\pi = \pi(\theta^1, \dots, \theta^N)$
WHAT IS THE "FULLY MARGINALIZED" ERROR $\sigma(\pi)$?

ANSWER:
$$\sigma(\pi) = \left[(F^{-1})^{ab} \frac{\partial \pi}{\partial \theta^a} \frac{\partial \pi}{\partial \theta^b} \right]^{1/2}$$

ARGUMENT: "PRIMED" COORDINATE SYSTEM $\theta'^1 = \pi$
AND F' IS DIAGONAL

$$\frac{\partial \pi}{\partial \theta^a} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$F'_{ab} = \begin{pmatrix} F'_{11} & & 0 \\ & \ddots & \\ 0 & & F'_{NN} \end{pmatrix}$$

$$\sigma(\pi) = \sqrt{\frac{1}{F'_{11}}}$$

$$\frac{\partial \pi}{\partial \theta^{ia}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad F'_{ab} = \begin{pmatrix} F'_{11} & 0 \\ & \ddots \\ 0 & F'_{NN} \end{pmatrix}$$

$$\sigma(\pi) = \sqrt{\frac{1}{F'_{11}}} \quad \text{[NON-SCALAR]}$$

$$= \left[\frac{\partial \pi}{\partial \theta^{ia}} \frac{\partial \pi}{\partial \theta^{ib}} (F^{-1})^{ab} \right]^{1/2} \quad \text{[SCALAR]}$$

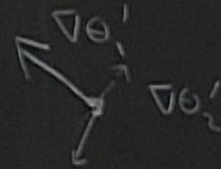
"DERIVED" PARAMETER $\pi = \pi(\theta^1, \dots, \theta^N)$
 THE "FULLY MARGINALIZED" ERROR $\sigma(\pi)$?

$$\frac{\partial \pi}{\partial \theta^{1a}} = \left(\begin{array}{c} \vdots \\ \vdots \end{array} \right)$$

$$\left[(F^{-1})^{ab} \frac{\partial \pi}{\partial \theta^a} \frac{\partial \pi}{\partial \theta^b} \right]^{1/2}$$

$$\theta_1' = \pi$$

$$\sigma(\pi) =$$



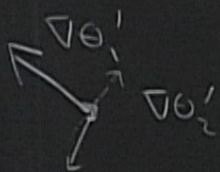
COORDINATE SYSTEM $\theta_1' = \pi$
IS DIAGONAL

ER $\pi = \pi(\theta^1, \dots, \theta^N)$
 "LIZED" ERROR $\sigma(\pi)$?

$$\frac{\partial \pi}{\partial \theta^{1a}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$F'_{ab} = \begin{pmatrix} F'_{11} & 0 \\ & \ddots \\ 0 & \ddots & F'_{NN} \end{pmatrix}$$

$$\theta^1_i = \pi$$



$$\sigma(\pi) = \sqrt{\frac{1}{F'_{11}}}$$

[NON-SCAL

$$= \left[\frac{\partial \pi}{\partial \theta^{1a}} \frac{\partial \pi}{\partial \theta^{1b}} (F^{-1})^{ab} \right]^{1/2}$$

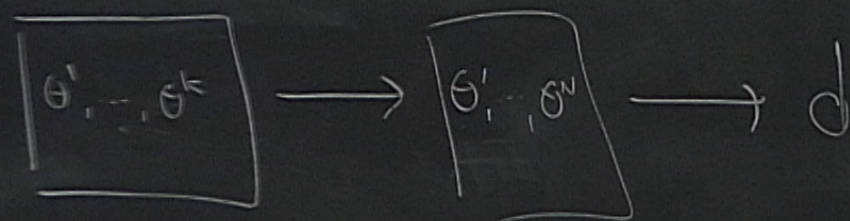
$$\sigma(\theta^a) = \sqrt{(F^{-1})^{aa}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

• EXAMPLE: GIVEN FISHER MATRIX IN COORDS $(\theta^1, \dots, \theta^N)$

WHAT'S THE STATISTICAL ERROR ON θ^1

ASSUMING $(\theta^2, \dots, \theta^k)$ ARE MARGINALIZED

& $(\theta^{k+1}, \dots, \theta^N)$ ARE FIXED TO FIDUCIAL VALUES?

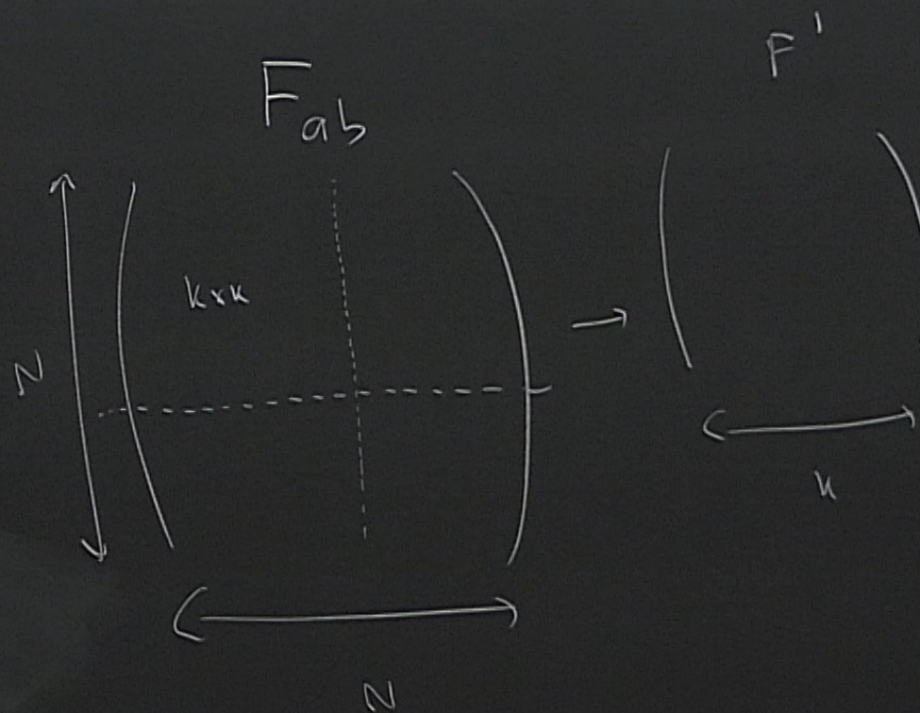


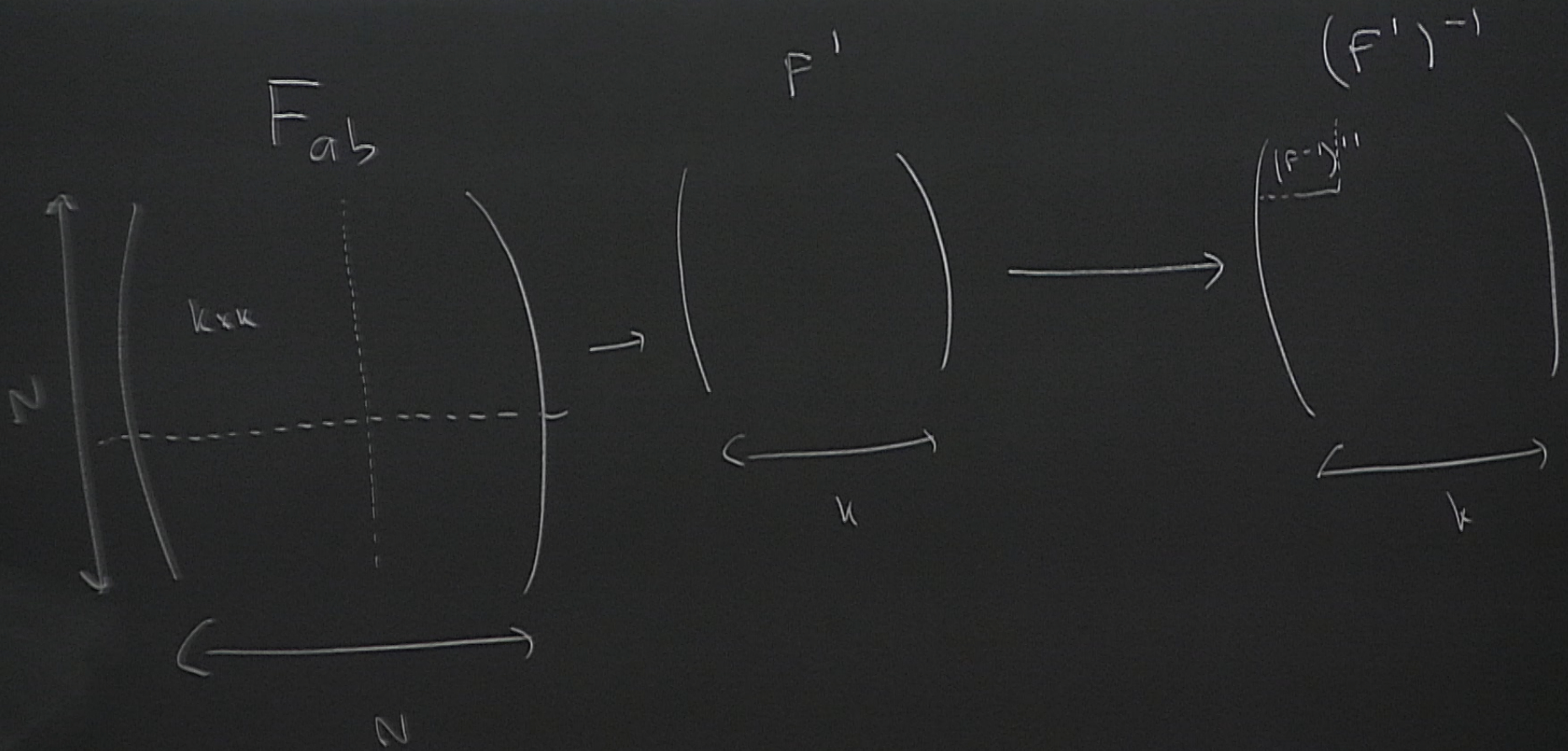
205 $(\theta^1, \dots, \theta^N)$

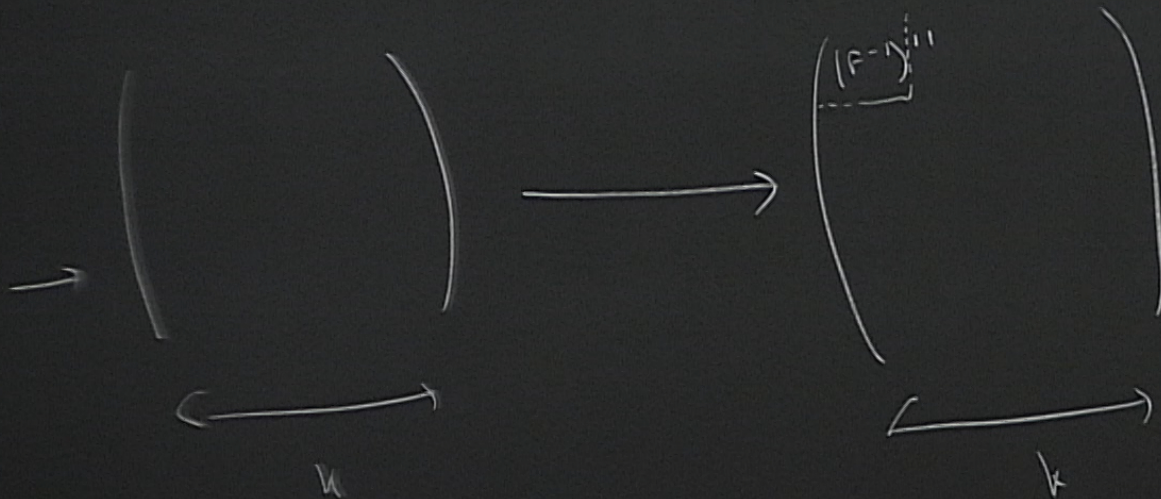
θ^1

VALUED

FIDUCIAL VALUES?





F' $(F')^{-1}$ 

$$\sigma(\theta') = \sqrt{(F')^{-1} 1L}$$

EXAMPLE 3

GIVEN FISHER MATRIX F_{ab} ,

HOW DO YOU MODIFY THE

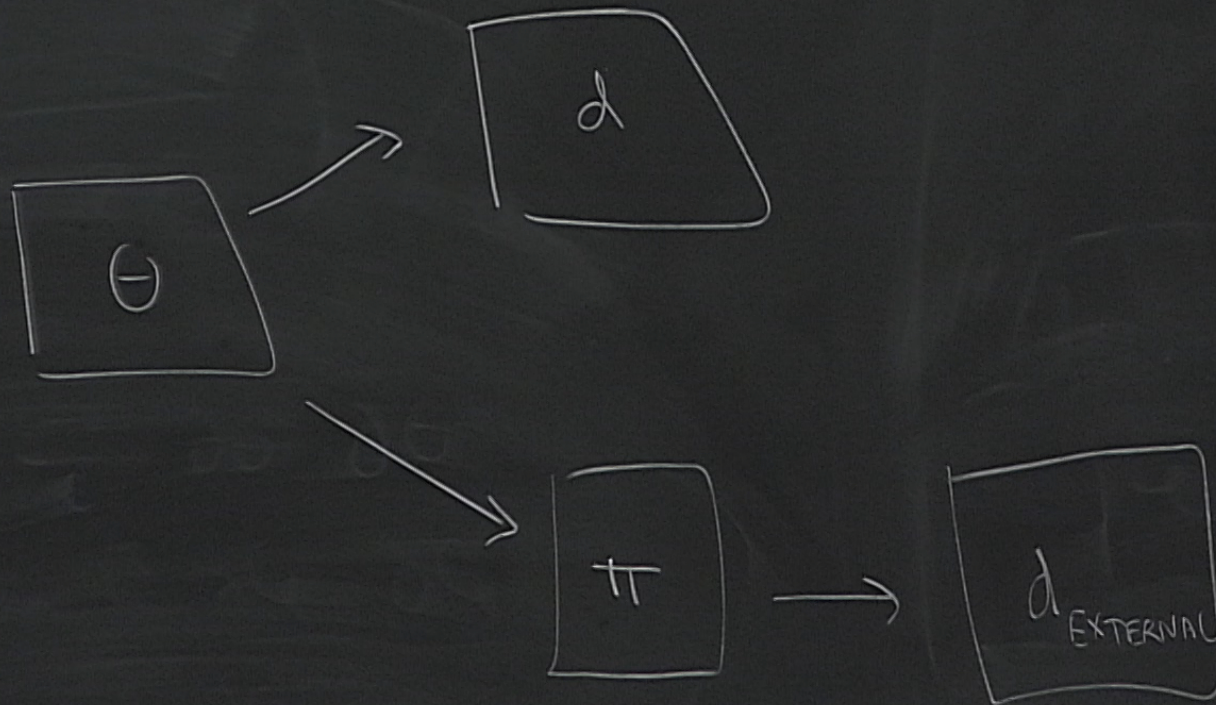
FORECAST TO INCLUDE AN

"EXTERNAL" MEASUREMENT

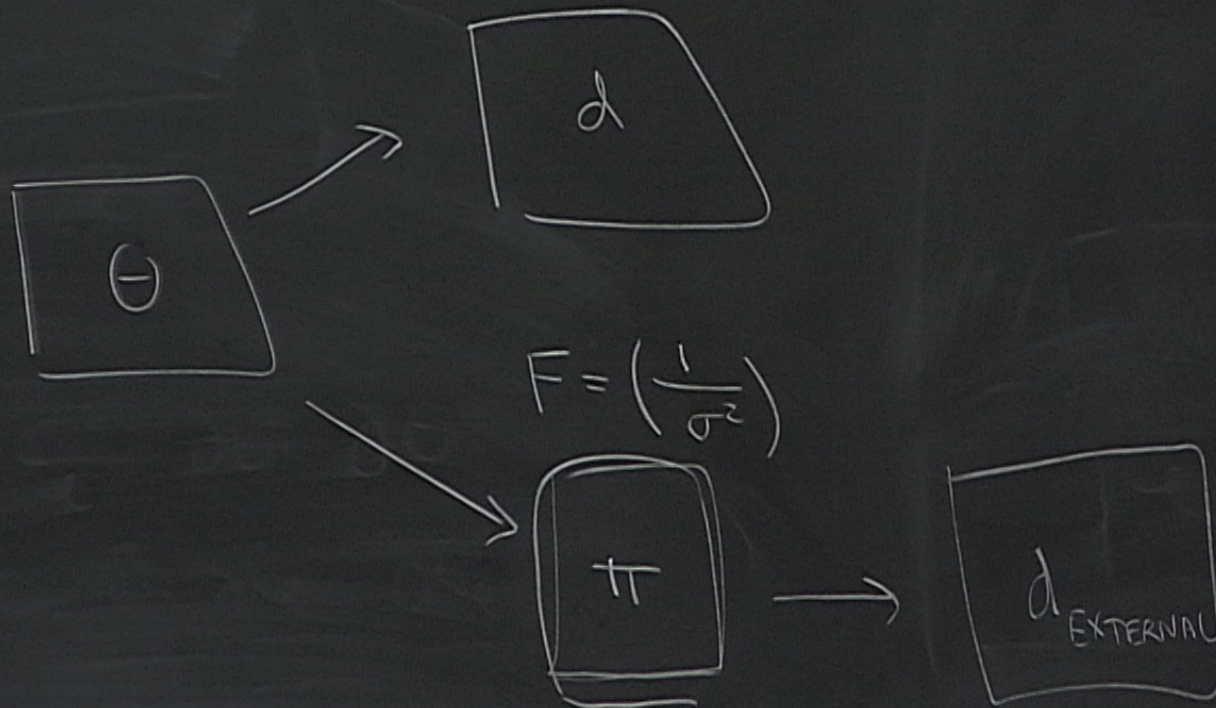
OF PARAMETER π WITH

VARIANCE σ^2 ?

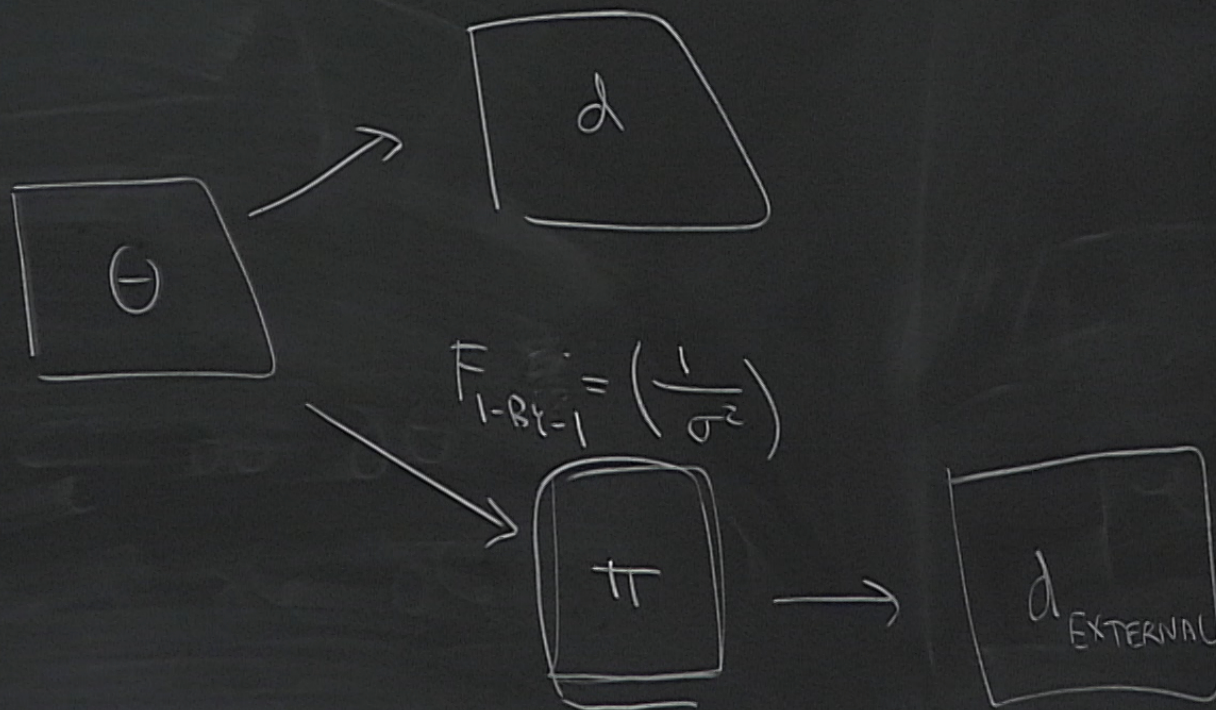
$$F_{ab} \rightarrow F_{ab} + \frac{1}{\sigma^2} \left(\frac{\partial \pi}{\partial \Theta^a} \right) \left(\frac{\partial \pi}{\partial \Theta^b} \right)$$



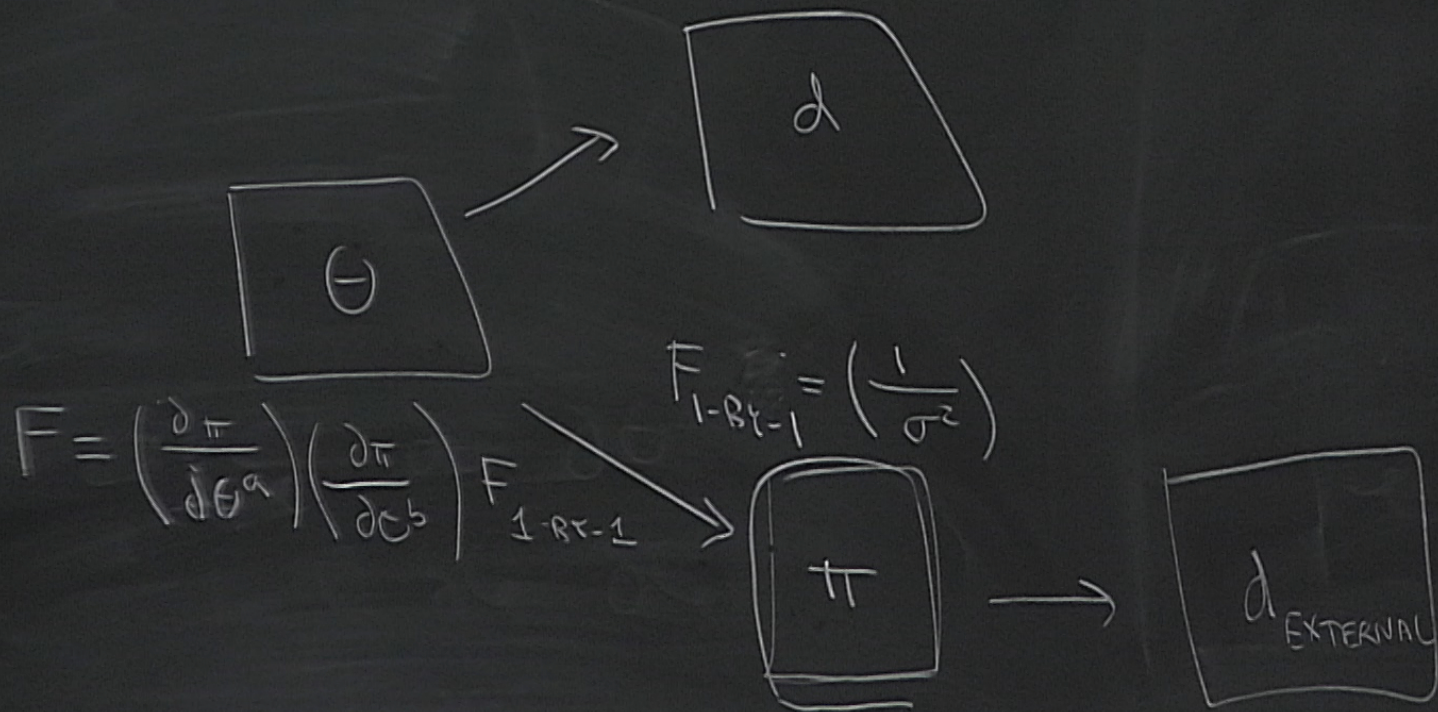
$$F_{ab} \rightarrow F_{ab} + \frac{1}{\sigma^2} \left(\frac{\partial \pi}{\partial \Theta^a} \right) \left(\frac{\partial \pi}{\partial \Theta^b} \right)$$



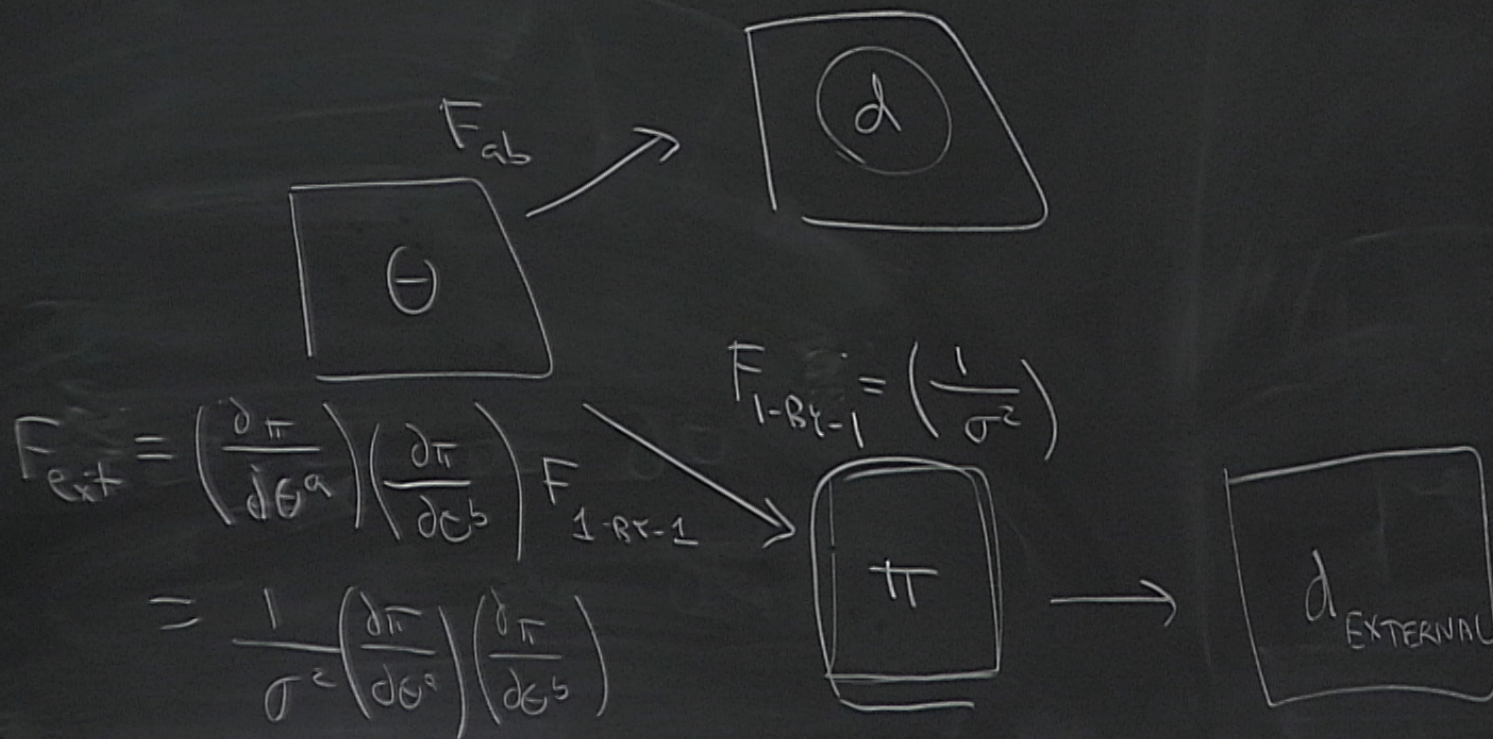
$$F_{ab} \rightarrow F_{ab} + \frac{1}{\sigma^2} \left(\frac{\partial \pi}{\partial \Theta^a} \right) \left(\frac{\partial \pi}{\partial \Theta^b} \right)$$



$$F_{ab} \rightarrow F_{ab} + \frac{1}{\sigma^2} \left(\frac{\partial \pi}{\partial \Theta^a} \right) \left(\frac{\partial \pi}{\partial \Theta^b} \right)$$



$$F_{ab} \rightarrow F_{ab} + \frac{1}{\sigma^2} \left(\frac{\partial \pi}{\partial \Theta^a} \right) \left(\frac{\partial \pi}{\partial \Theta^b} \right)$$



$$\rightarrow \boxed{C_e} \rightarrow \boxed{a_{em}}$$

$$F_{ii'} = \frac{u+1}{2} \cdot \frac{1}{C_e^2} \delta_{ii'}$$

$$ds^2 = (F_{ab}) d\theta^a d\theta^b$$

