Title: PSI 2018/2019 - Explorations in Cosmology - Lecture 8

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Collection: PSI 2018/2019 - Explorations in Cosmology (Smith)

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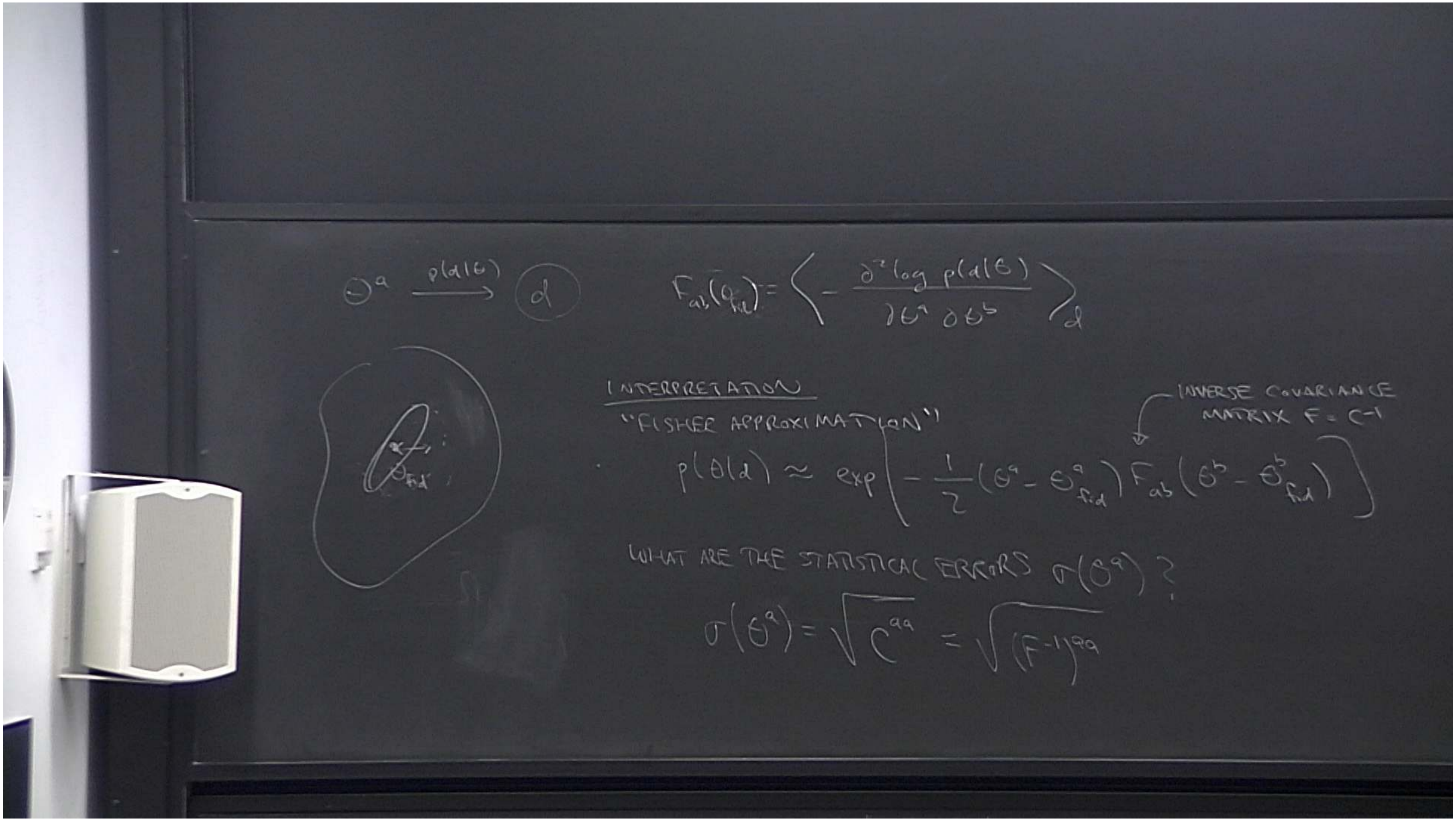
$$\theta \xrightarrow{p(d|\theta)} d$$

$$p(\theta) = 1$$

$$= \boxed{\sqrt{\mathrm{Det}\, F(\theta)}} \quad \text{"FISHER PRIOR"}$$

$$\int d^4x \sqrt{g}\,(\ldots)$$

$$= \int d^n\theta \sqrt{F}\,(\ldots)$$


$\Theta^a \xrightarrow{p(d|\theta)} d$
$F_{ab}(\theta_{fid}) = \left\langle - \frac{\partial^2 \log p(d|\theta)}{\partial\theta^a \partial\theta^b} \right\rangle_d$
INTERPRETATION
"FISHER APPROXIMATION"
INVERSE COVARIANCE MATRIX $F = C^{-1}$
$p(\theta|d) \approx \exp\left[-\frac{1}{2}(\theta^a - \theta^a_{fid}) F_{ab} (\theta^b - \theta^b_{fid}) \right]$
θ_{fid}
WHAT ARE THE STATISTICAL ERRORS $\sigma(\theta^a)$?
$\sigma(\theta^a) = \sqrt{C^{aa}} = \sqrt{(F^{-1})^{aa}}$

C_{12}

C_{22}

$\bar{\theta} = 0$

$] = \text{CONST.}$

$= 1$

$\cdot\ \theta^a C^{-1}_{ab} \theta^b = 1$

$\cdot$ FIX $\theta^2 = 0$: $\quad \theta^1 (C^{-1})_{11} \theta^1 = 1$

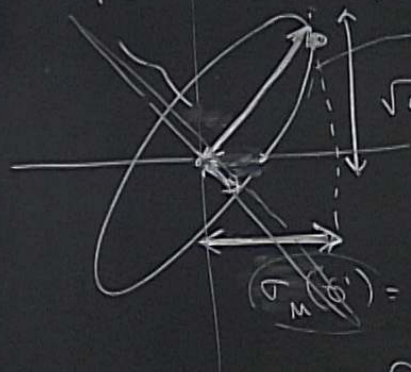
$$\Rightarrow \theta^1 = \frac{1}{\sqrt{(C^{-1})_{11}}}$$

$\cdot$ MARGINALIZE θ^2: SOLVE FOR $\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}$ W/ MAXIMIZES θ^1

S.T. CONSTRAINT $\theta^T C^{-1} \theta = 1$

$$\xRightarrow{\text{SOLUTION}} \theta^a = \begin{pmatrix} \sqrt{C^{11}} \\ C^{12}/\sqrt{C^{11}} \end{pmatrix}$$

$\sigma_F(\theta^1) = (C^{-1})_{11}^{-1/2}$ $\sqrt{\lambda_C} = (\lambda_F)^{-1/2}$



$\sqrt{C_{22}}$

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}$$

$\sigma_M(\theta^1) = (C_{11})^{1/2}$

$p(\theta^a) \propto \exp\left[-\theta^a C^{-1}_{ab}\theta^b\right]$ $\bar{\theta} = 0$

ERROR ELLIPSE $\exp\left[-\theta^a C^{-1}_{ab}\theta^b\right] = \text{CONST.}$

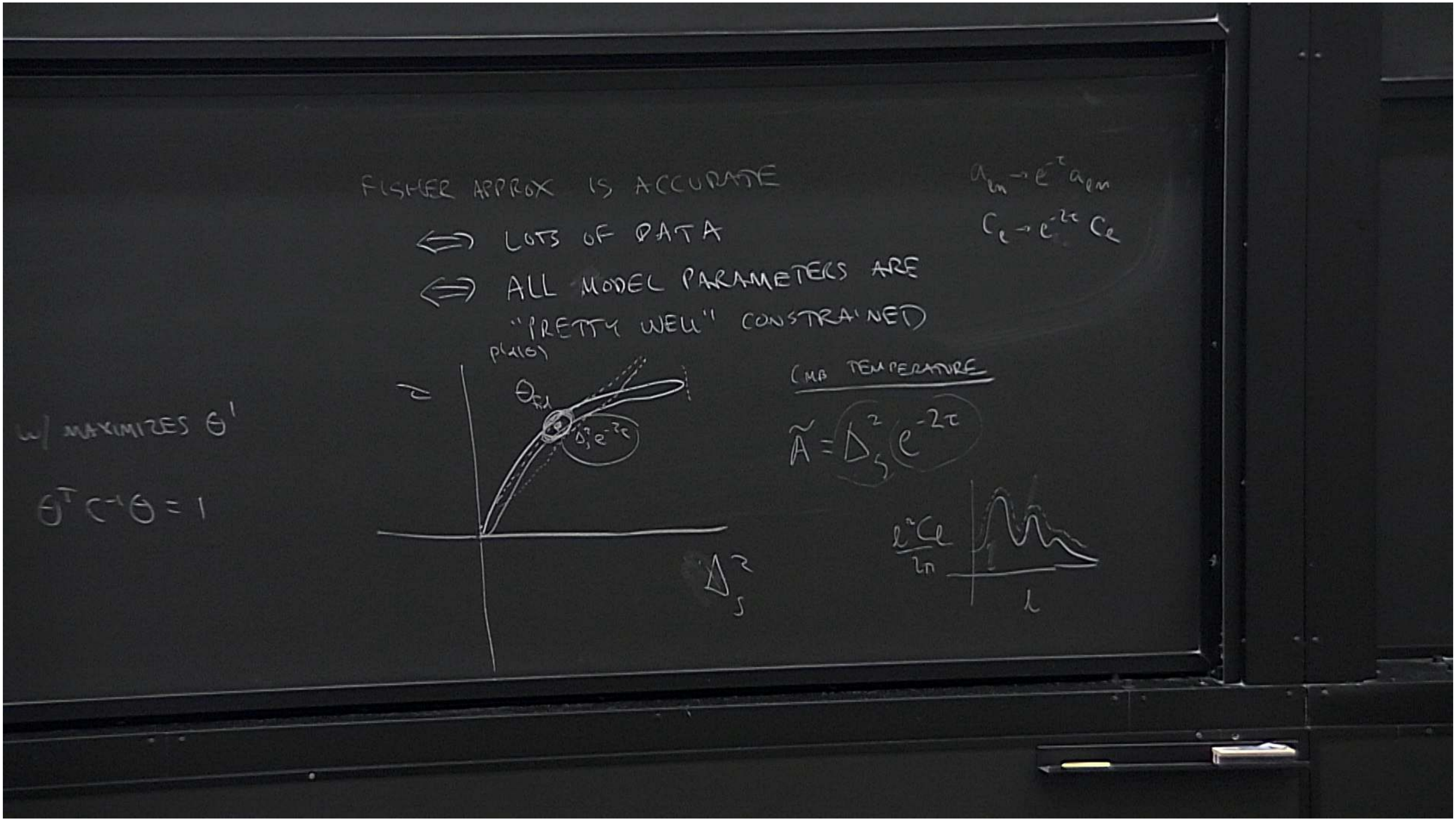
$\Leftrightarrow \boxed{\theta^a C^{-1}_{ab}\theta^b = 1}$

$\theta^a C^{-1}_{ab}\theta^b =$

- FIX $\theta^2 = 0$:

- MARGINALIZE θ^2

SOLUTION $\Longrightarrow$


FISHER APPROX IS ACCURATE
$\Leftrightarrow$ LOTS OF DATA
$\Leftrightarrow$ ALL MODEL PARAMETERS ARE
"PRETTY WELL" CONSTRAINED
$a_{\ell m} \to e^{-\tau} a_{\ell m}$
$C_\ell \to e^{-2\tau} C_\ell$
$p(d|\theta)$
τ
θ_{fid}
$\Delta_s^2 e^{-2\tau}$
Δ_s^2
CMB TEMPERATURE
$\tilde{A} = \Delta_s^2 e^{-2\tau}$
$\frac{\ell^2 C_\ell}{2\pi}$
ℓ
w/ MAXIMIZES θ'
$\theta^T C^{-1} \theta = 1$

CMB EXAMPLE

HOW WELL CAN THE CMB POWER SPECTRUM C_ℓ BE CONSTRAINED FROM A CMB MAP $a_{\ell m}$? [ASSUME MEASURED PERFECTLY, WITH NO NOISE]

MODEL PARAMS $\theta^a = \{ C_2, C_3, \dots \}$

DATA $d = \{ a_{\ell m} \}$

a_{00} = "MONOPOLE"

a_{1m} = "DIPOLE"

COMPUTE FISHER MATRIX $F_{\ell\ell'}$

CMB POWER
FROM A
EASURED

a_{00} = "MONOPOLE"

a_{1m} = "DIPOLE"

FIRST NEED TO WRITE DOWN $p(a_{\ell m} | C_\ell)$

RECALL:

- $a_{\ell 0}$ IS GAUSSIAN W/ VARIANCE C_ℓ
- FOR $m = 1, \ldots, \ell$, $\mathrm{Re}(a_{\ell m})$ & $\mathrm{Im}(a_{\ell m})$ ARE GAUSSIAN W/ VARIANCE $C_\ell / 2$
- FOR $m = -1, \ldots, -\ell$, $a_{\ell m} = (-1)^m a_{\ell,-m}^*$ IS NOT AN INDEPENDENT QUANTITY

DATA $\quad d = \{a_{\ell m}\}$

$a_{00} =$ "MONOPOLE"

$a_{1m} =$ "DIPOLE"

COMPUTE FISHER MATRIX $F_{\ell\ell'}$

• FOR $m = -1, \dots, -\ell$, $\quad a_{\ell m} = (-1)^m a^*_{\ell, -m}$

IS NOT AN INDEPENDENT QUANTITY

$$p(a_{\ell m} | C_\ell) = \prod_\ell \underbrace{\frac{1}{\sqrt{2\pi C_\ell}} \exp\left(-\frac{a_{\ell 0}^2}{2C_\ell}\right)}_{a_{\ell 0}} \prod_{m=1}^{\ell} \underbrace{\frac{1}{\sqrt{\pi C_\ell}} \exp\left(-\frac{(\mathrm{Re}\, a_{\ell m})^2}{C_\ell}\right)}_{\mathrm{Re}(a_{\ell m})} \underbrace{\frac{1}{\sqrt{\pi C_\ell}} \exp\left(-\frac{(\mathrm{Im}\, a_{\ell m})^2}{C_\ell}\right)}_{\mathrm{Im}(a_{\ell m})}$$

$$\propto \prod_\ell C_\ell^{-(2\ell+1)/2} \prod_{m=-\ell}^{\ell} \exp\left(-\frac{|a_{\ell m}|^2}{2C_\ell}\right)$$

$$F_{\ell\ell'} = \left\langle -\frac{\partial^2 \log p(a_{\ell m} | C_\ell)}{\partial C_\ell \, \partial C_{\ell'}} \right\rangle_{a_{\ell m}}$$

$$= \left\langle -\frac{\partial^2}{\partial C_\ell \partial C_{\ell'}} \sum_{\ell''} \left(-\frac{2\ell''+1}{2} \log C_{\ell''} - \sum_{m=-\ell''}^{\ell''} \frac{|a_{\ell'' m}|^2}{2 C_{\ell''}} \right) \right\rangle_{a_{\ell m}}$$

$$\left[\langle |a_{\ell m}|^2 \rangle = C_\ell \right]$$

$$= \left\langle -\left(\frac{2\ell+1}{2} \frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{C_\ell^3} \right) \delta_{\ell\ell'} \right\rangle_{a_{\ell m}}$$

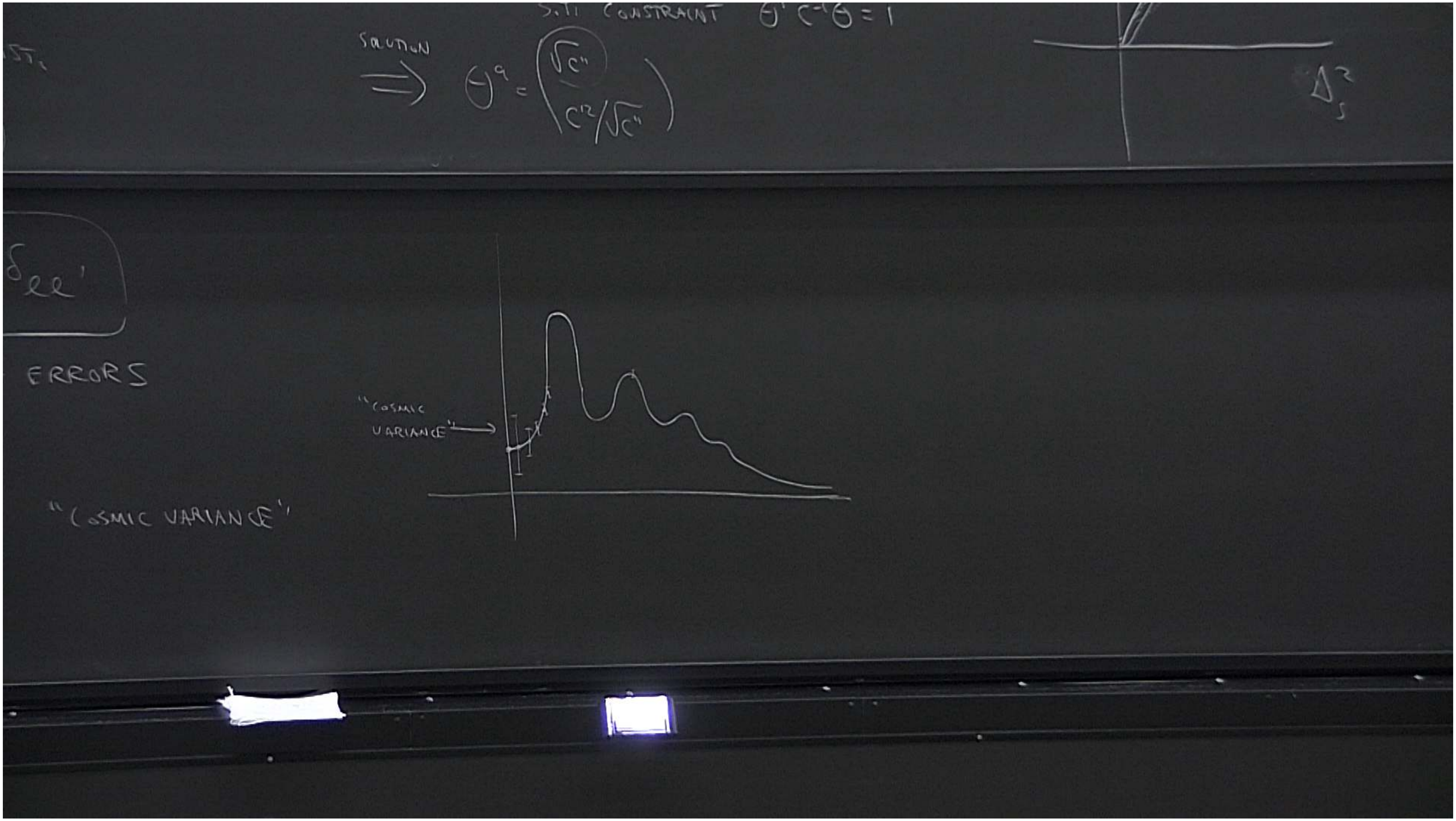
$$= -\left(\frac{2\ell+1}{2} \frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell} \frac{C_\ell}{C_\ell^3} \right) \delta_{\ell\ell'}$$

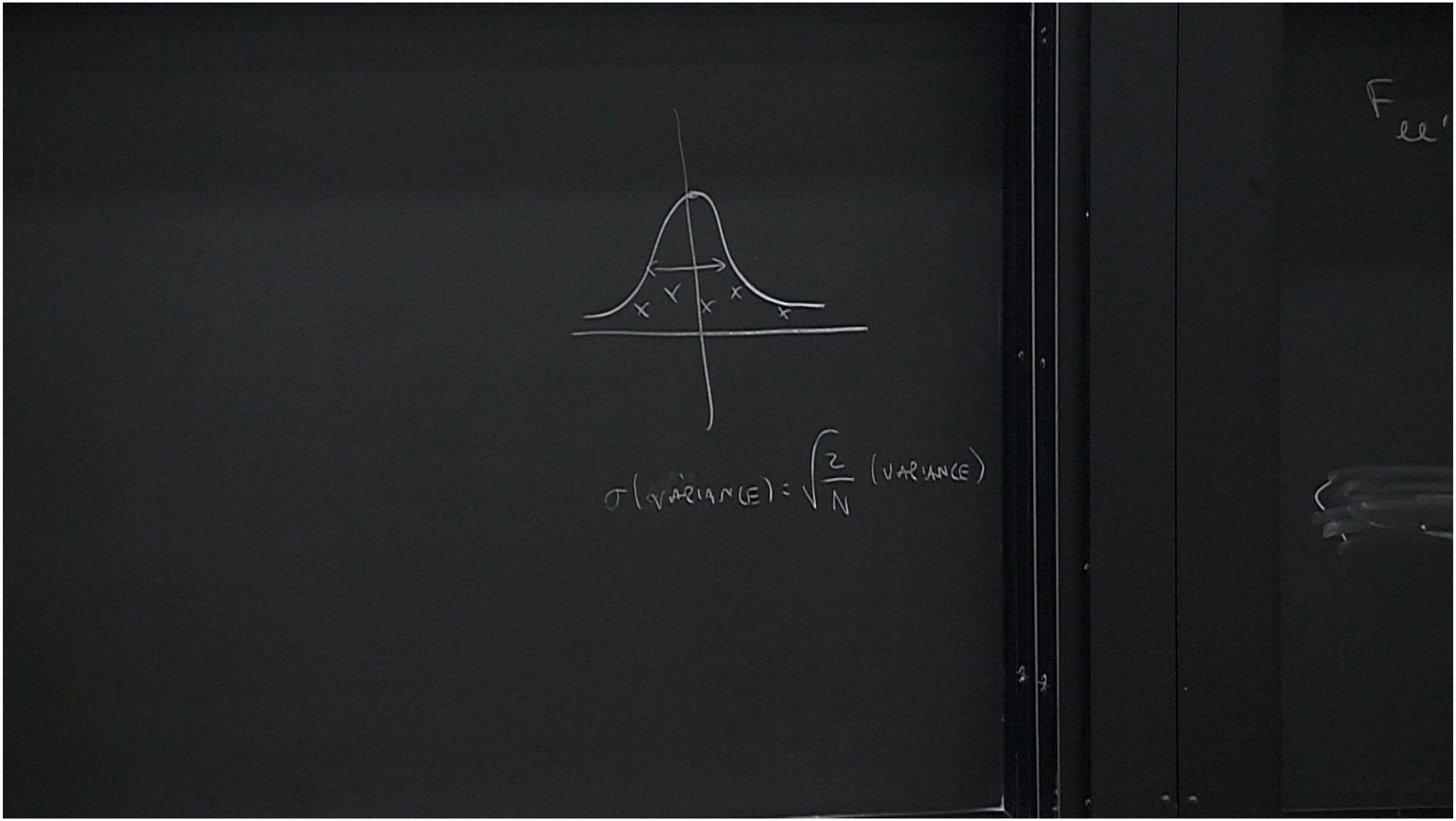
$$= \frac{2\ell+1}{C_\ell^2} \delta_{\ell\ell'}$$

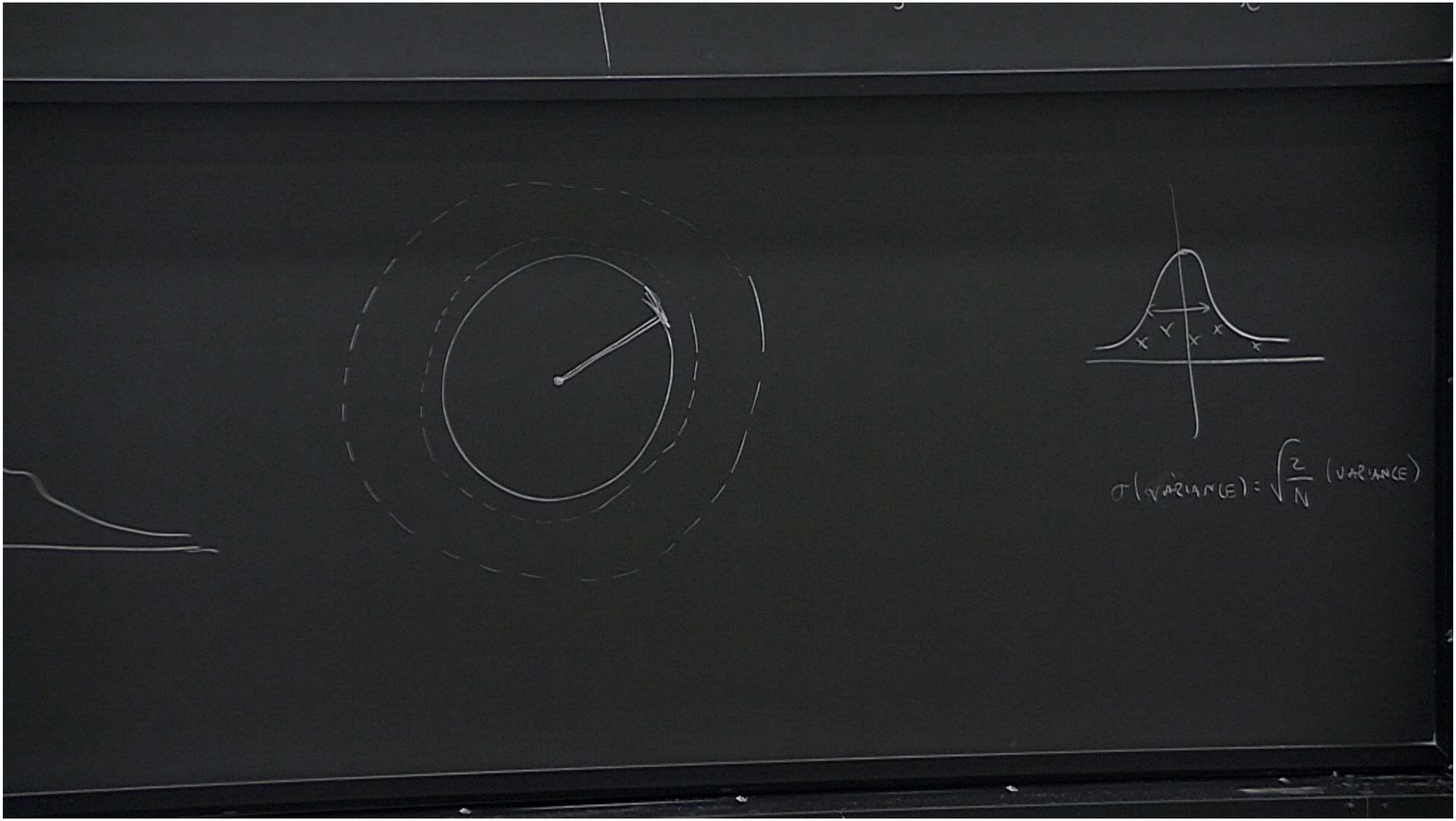
$$\boxed{F_{\ell\ell'} = \frac{2\ell+1}{2} \frac{1}{C_\ell^2} \delta_{\ell\ell'}}$$

- $C_\ell, C_{\ell'}$ HAVE INDEPENDENT ERRORS
- $\sigma(C_\ell) = \left(F^{-1}_{\ell\ell}\right)^{1/2}$

$$= \sqrt{\frac{2}{2\ell+1}}\, C_\ell \qquad \text{"COSMIC VARIANCE"}$$


S.T. CONSTRAINT
SOLUTION
ERRORS
"COSMIC VARIANCE"
"COSMIC VARIANCE"


$\sigma(\text{VARIANCE}) = \sqrt{\frac{2}{N}}\,(\text{VARIANCE})$


σ (VARIANCE) = √(2/N) (VARIANCE)

ELLIPSE $\exp\left[-\Theta^a C^{-1}_{ab}\Theta^b\right] = \text{CONST.}$

$$\Leftrightarrow \quad \boxed{\Theta^a C^{-1}_{ab}\Theta^b = 1}$$

SOLUTION

$$\Rightarrow \quad \Theta^a = \begin{pmatrix} \sqrt{C^{11}} \\ C^{12}/\sqrt{C^{11}} \end{pmatrix}$$

$$\boxed{F_{\ell\ell'} = \frac{2\ell+1}{2}\,\frac{1}{C_\ell^2}\,\delta_{\ell\ell'}}$$

- $C_\ell, C_{\ell'}$ HAVE INDEPENDENT ERRORS
- $\sigma(C_\ell) = \left(F^{-1}_{\ell\ell}\right)^{1/2}$

$$= \sqrt{\frac{2}{2\ell+1}}\,C_\ell \quad \text{"COSMIC VARIANCE"}$$

$$\rightarrow \sqrt{\frac{2}{2\ell+1}}\,(C_\ell + N_\ell) \quad \text{W/NOISE}$$

WHERE N_ℓ IS A "NOISE POWER SPECTRUM"