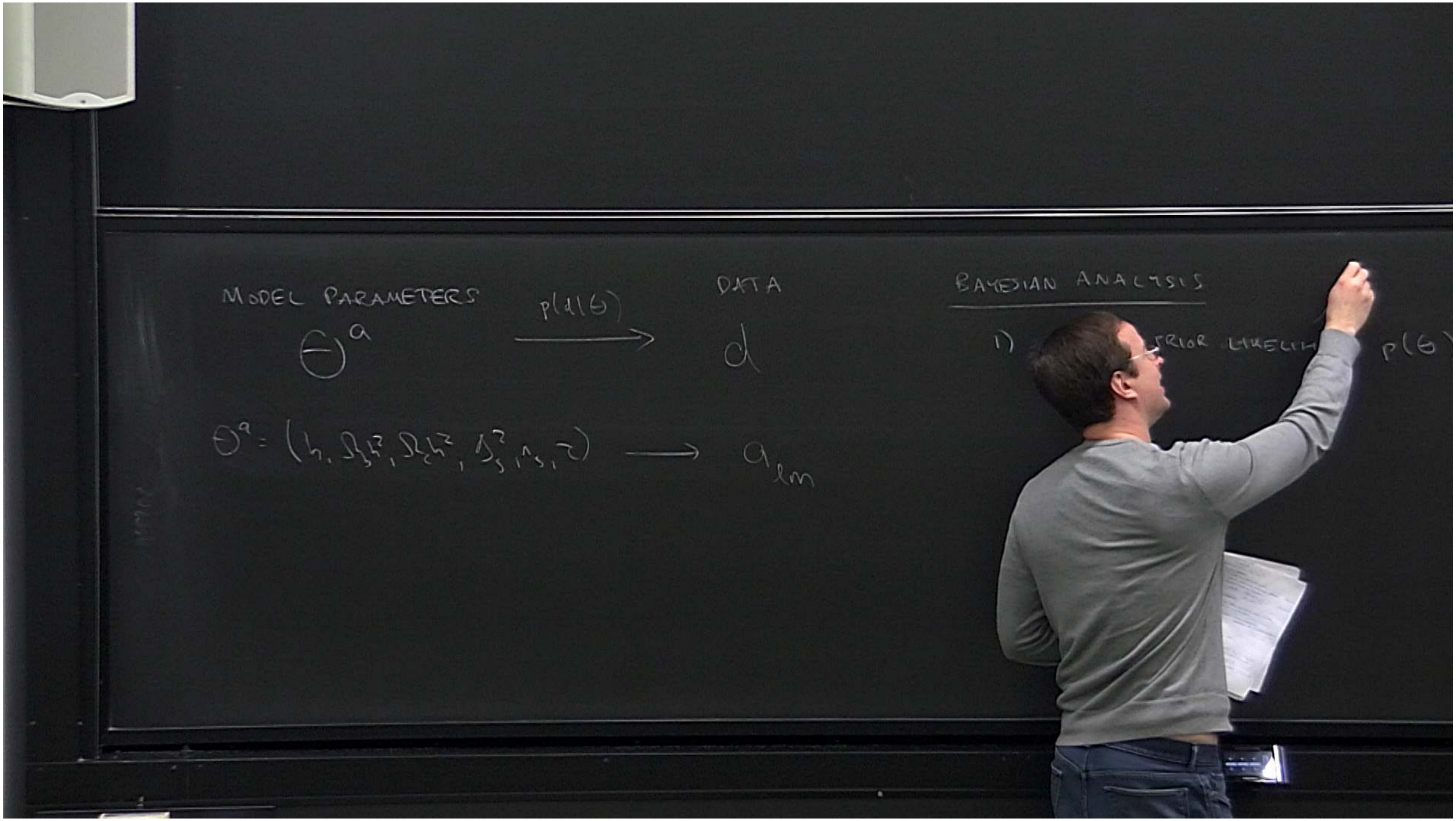
Title: PSI 2018/2019 - Explorations in Cosmology - Lecture 7

Speakers: Kendrick Smith

Collection: PSI 2018/2019 - Explorations in Cosmology (Smith)

Date: April 24, 2019 - 11:30 AM

URL: http://pirsa.org/19040069


MODEL PARAMETERS
p(d|Θ)
DATA
Θ^a
d
Θ^a = (h, Ω_b h^2, Ω_c h^2, Δ_s^2, n_s, τ)
a_lm
BAYESIAN ANALYSIS
1)
PRIOR
p(Θ

POWER SPECTRUM

BAYESIAN ANALYSIS

"PROBABILITY DISTRIBUTION FUNCTION"

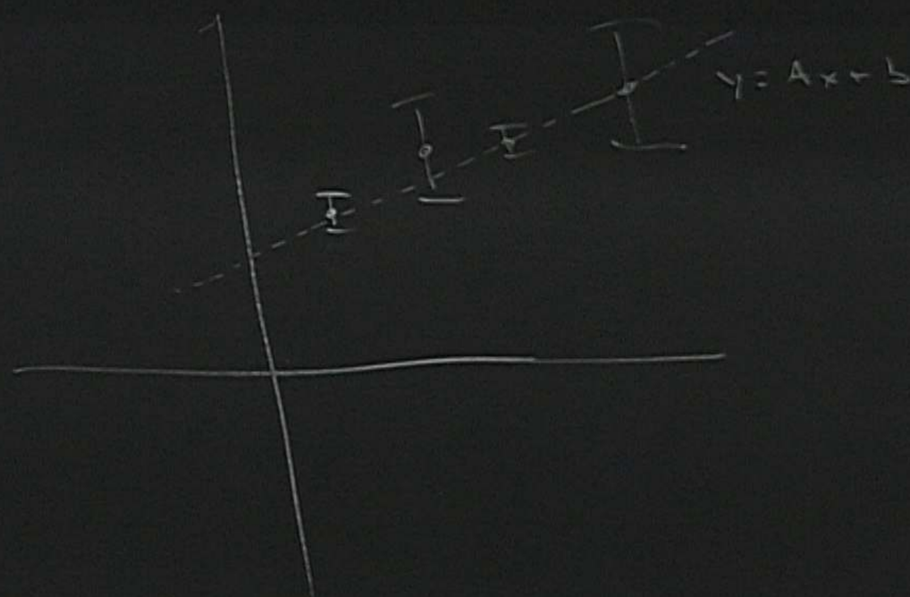
1) CHOOSE PRIOR LIKELIHOOD $p(\theta)$
2) COMPUTE CONDITIONAL LIKELIHOOD $p(d|\theta)$
3) COMPUTE POSTERIOR LIKELIHOOD

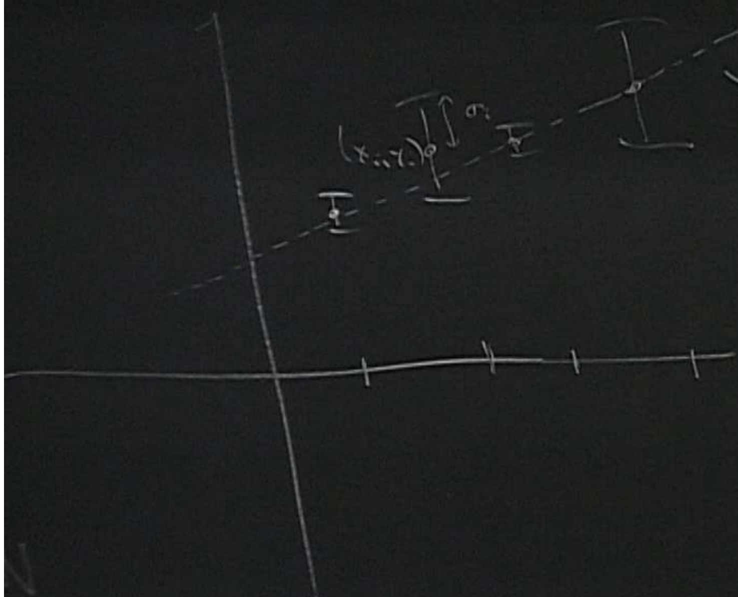
$$p(\theta|d) \propto p(d|\theta)\,p(\theta)$$

4) INTERPRET

CURVE FITTING

$y = Ax + b$

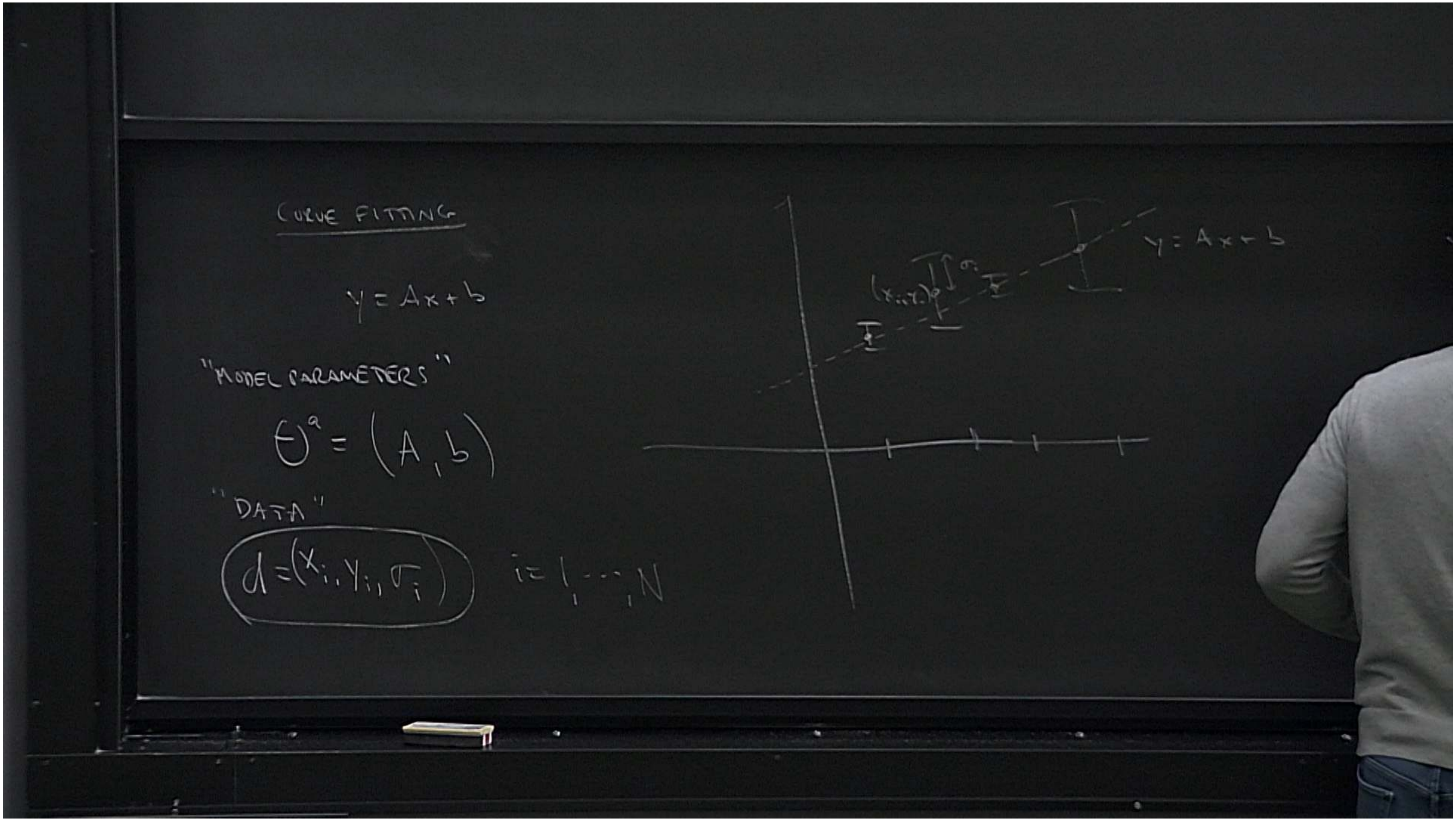


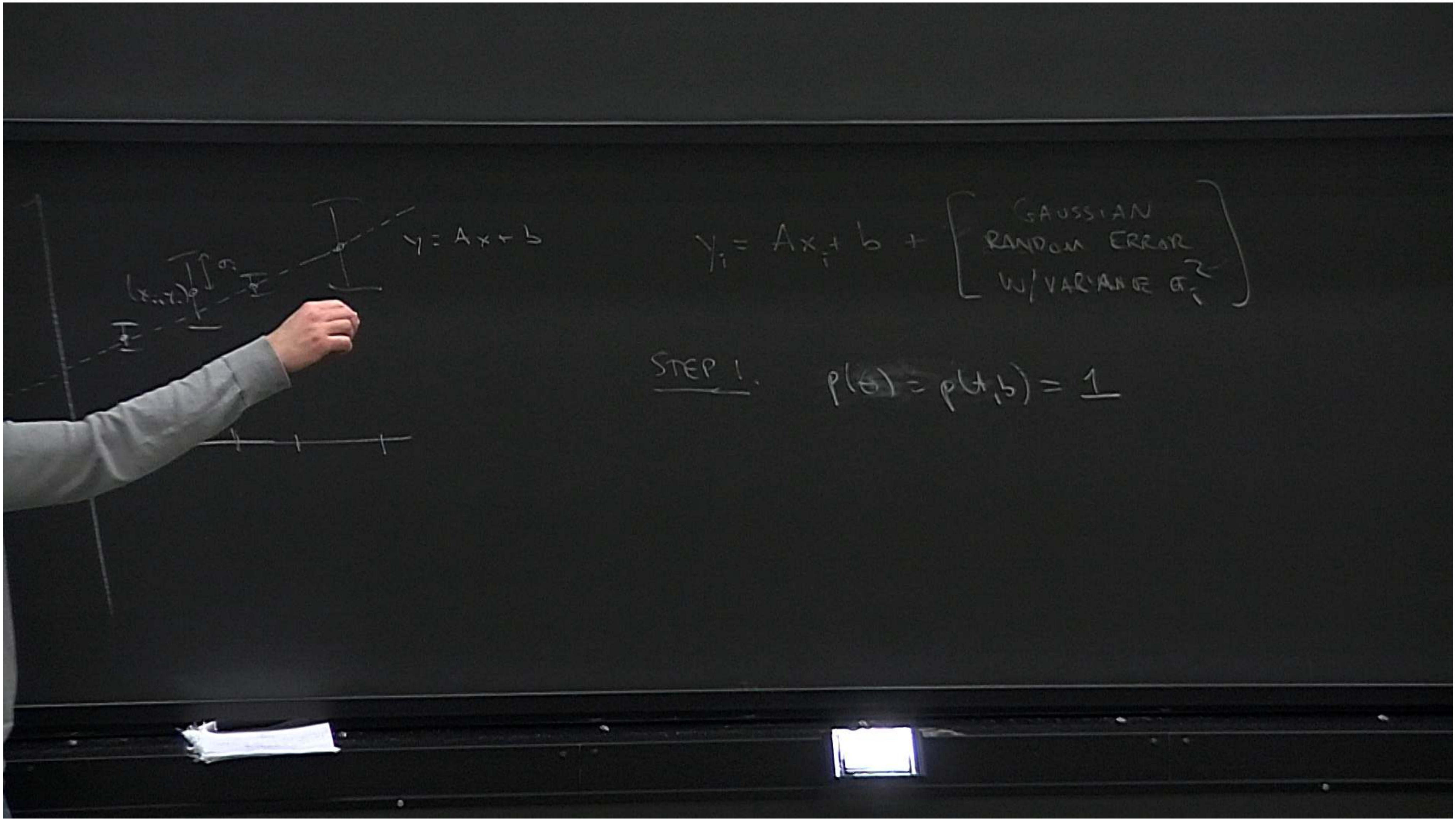


$y = Ax + b$

$$y_i = Ax_i + b + \left[\begin{array}{c}\text{GAUSSIAN}\\ \text{RANDOM ERROR}\\ \text{W/ VARIANCE } \sigma_i^2\end{array}\right]$$

STEP 1.


CURVE FITTING
y = Ax + b
"MODEL PARAMETERS"
θ^a = (A, b)
"DATA"
d = (x_i, y_i, σ_i)
i = 1, ..., N
(x_i, y_i)
σ_i
y = Ax + b


y = Ax + b
$y_i = Ax_i + b$ + GAUSSIAN RANDOM ERROR W/ VARIANCE σ_i^2
STEP 1. $p(\theta) = p(A,b) = 1$

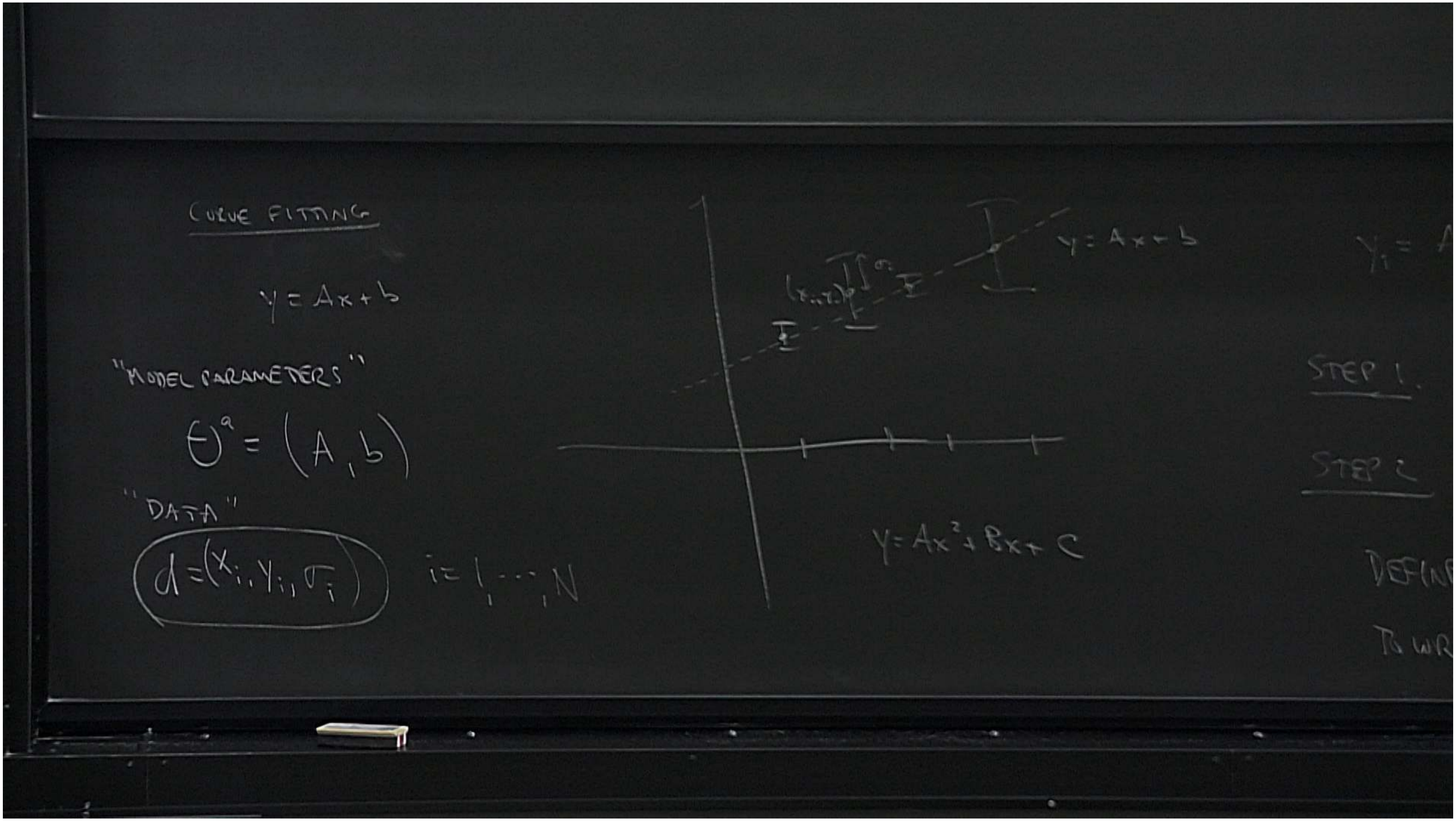
$$y_i = Ax_i + b + \left[\begin{array}{c}\text{GAUSSIAN}\\ \text{RANDOM ERROR}\\ \text{W/ VARIANCE } \sigma_i^2\end{array}\right]$$

STEP 1. $p(\theta) = p(A,b) = 1$

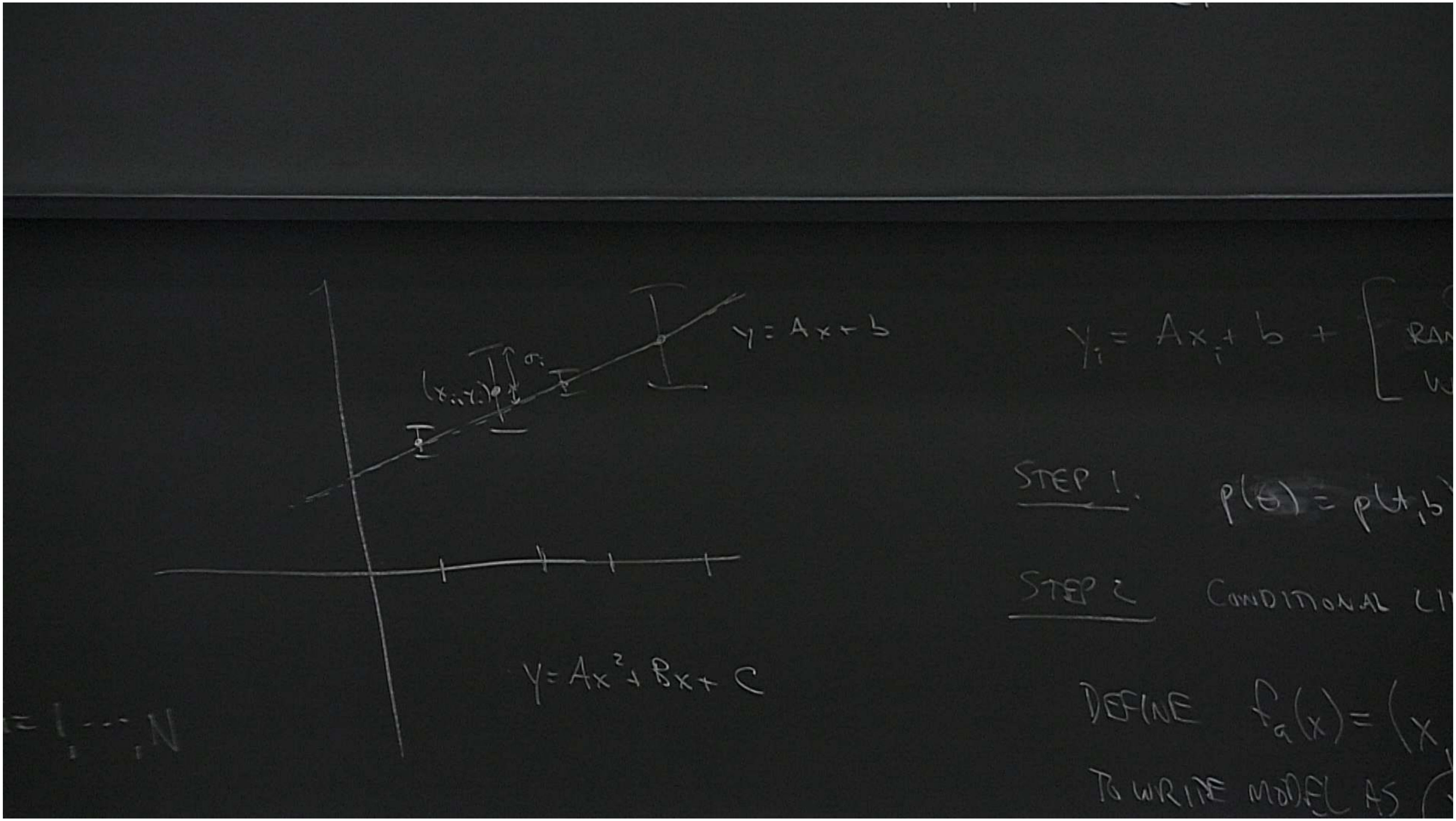
STEP 2 CONDITIONAL LIKELIHOOD $p(d|\theta)$

DEFINE $f_a(x) = (x, 1)$

TO WRITE MODEL AS $y_i = \theta^a f_a(x_i) + [\text{NOISE}]$


CURVE FITTING
y = Ax + b
"MODEL PARAMETERS"
θ^a = (A, b)
"DATA"
d = (x_i, y_i, σ_i)
i = 1, ..., N
y = Ax + b
y = Ax² + Bx + C
STEP 1.
STEP 2

$$p(d|\theta) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2\right]$$


$y = Ax + b$
(x_i, y_i)
σ_i
$y = Ax^2 + Bx + C$
$= 1, \ldots, N$
$y_i = Ax_i + b +$
STEP 1.
$p(\theta) = p(A, b$
STEP 2
CONDITIONAL
DEFINE $f_a(x) = (x$
TO WRITE MODEL AS

$y = Ax + b$

$$y_i = A x_i + b + \left[\begin{array}{c}\text{INDEPENDENT} \\ \text{GAUSSIAN} \\ \text{RANDOM ERROR} \\ \text{W/ VARIANCE } \sigma_i^2\end{array}\right]$$

STEP 1. $p(\theta) = p(A, b) = 1$

STEP 2 CONDITIONAL LIKELIHOOD $p(d|\theta)$

DEFINE $f_a(x) = (x_i, 1)$

TO WRITE MODEL AS $y_i = \theta^a f_a(x_i) + [\text{NOISE}]$

$$p(d|\theta) = \prod_{i=1}^{N} \cancel{\frac{1}{\sqrt{2\pi\sigma_i^2}}} \exp\left[-\frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2\right]$$

<u>STEP 3</u>: POSTERIOR $p(\theta|d) \propto \exp\left[\sum_{i=1}^{N}\left[-\frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2\right]\right]$

$$= \exp\left[-\sum_i \cancel{\frac{y_i^2}{2\sigma_i^2}} + \sum_{ia} \frac{\theta^a f_a(x_i)\, y_i}{\sigma_i^2} - \sum_{iab} \frac{\theta^a \theta^b f_a(x_i)\, f_b(x_i)}{2\sigma_i^2}\right]$$

$$p(d|\theta) = \prod_{i=1}^{N} \cancel{\frac{1}{\sqrt{2\pi\sigma_i^2}}} \exp\left[-\frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2 \right]$$

STEP 3: POSTERIOR

$$p(\theta|d) \propto \exp\left[\sum_{i=1}^{N} \left[-\frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2 \right] \right]$$

$$= \exp\left[\cancel{-\sum_i \frac{y_i^2}{2\sigma_i^2}} + \sum_{ia} \frac{\theta^a f_a(x_i)\, y_i}{\sigma_i^2} - \sum_{iab} \frac{\theta^a \theta^b f_a(x_i) f_b(x_i)}{2\sigma_i^2} \right]$$

$$\propto \exp\left[\theta^T V - \frac{1}{2}\theta^T F \theta \right]$$

WHERE $V_a = \sum_i \frac{f_a(x_i)\, y_i}{\sigma_i^2}$ AND $F_{ab} = \sum_i \frac{f_a(x_i) f_b(x_i)}{2\sigma_i^2}$

STEP 4: INTERPRET

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COMPARE w/ MULTIVARIATE GAUSSIAN

$$p(\theta) \propto \exp\left[-\frac{1}{2}(\theta-\bar{\theta})^T C^{-1}(\theta-\bar{\theta})\right]$$

OUR POSTERIOR IS

$$p(\theta|d) \propto \exp\left[-\frac{1}{2}(\theta-F^{-1}v)^T F(\theta-F^{-1}v)\right] \cancel{\exp\left[\frac{1}{2}v^T F^{-1}v\right]}$$

STEP 4: INTERPRET

COMPARE W/ MULTIVARIATE GAUSSIAN

$$p(\theta) \propto \exp\left[-\frac{1}{2}(\theta-\bar{\theta})^T C^{-1}(\theta-\bar{\theta})\right]$$

OUR POSTERIOR IS

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$\Rightarrow$ COVARIANCE F^{-1} AND MEAN $\bar{\theta} = F^{-1}v$

TERPRET

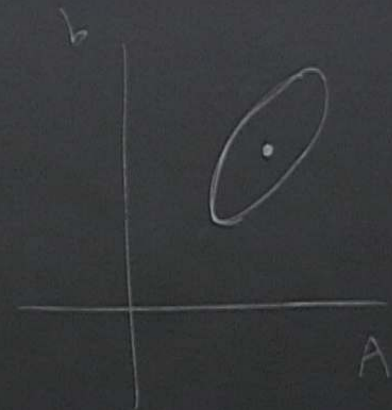
w/ MULTIVARIATE GAUSSIAN

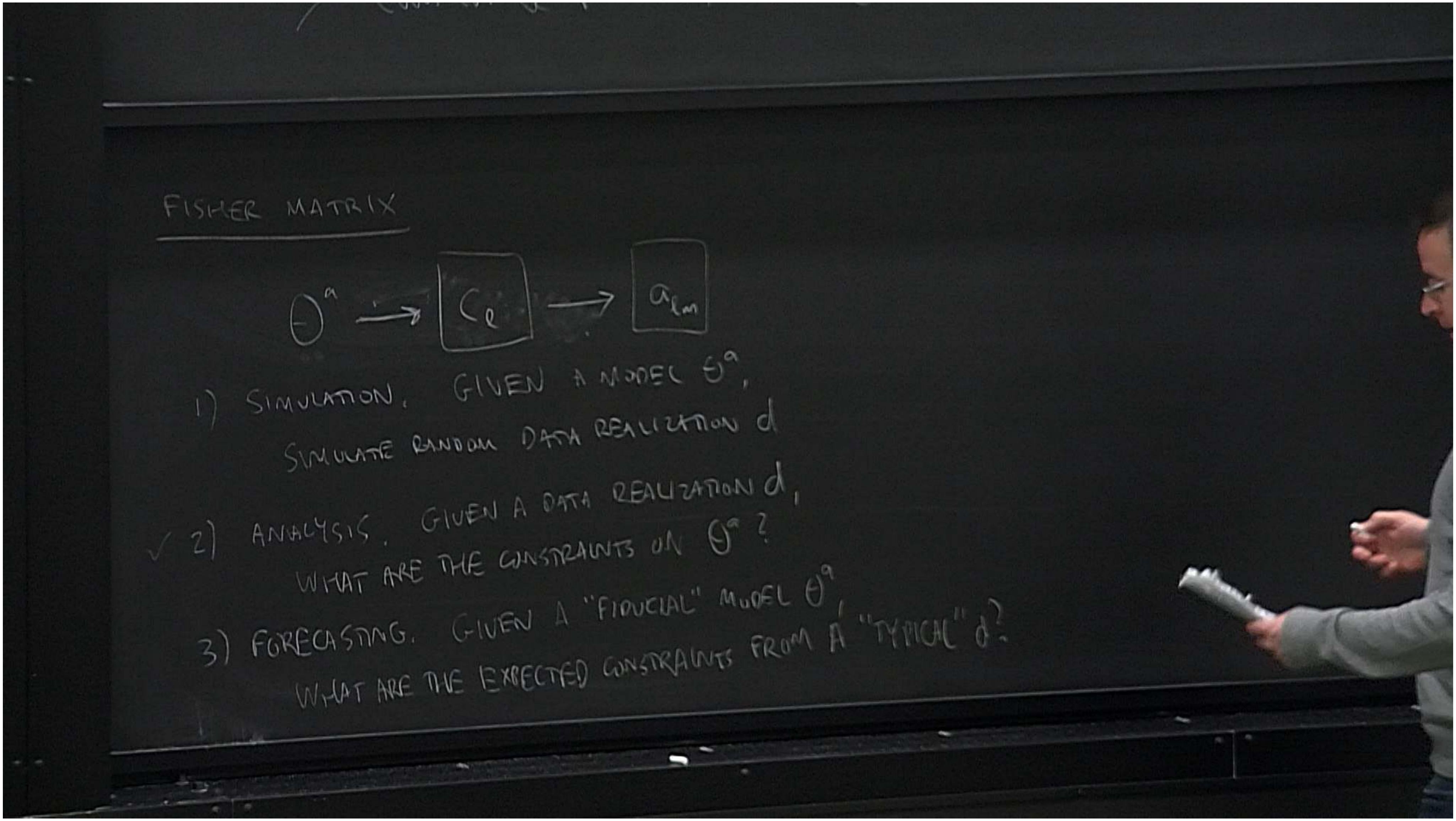
$$) \propto \exp\left[-\frac{1}{2}(\theta-\bar{\theta})^T C^{-1}(\theta-\bar{\theta})\right]$$

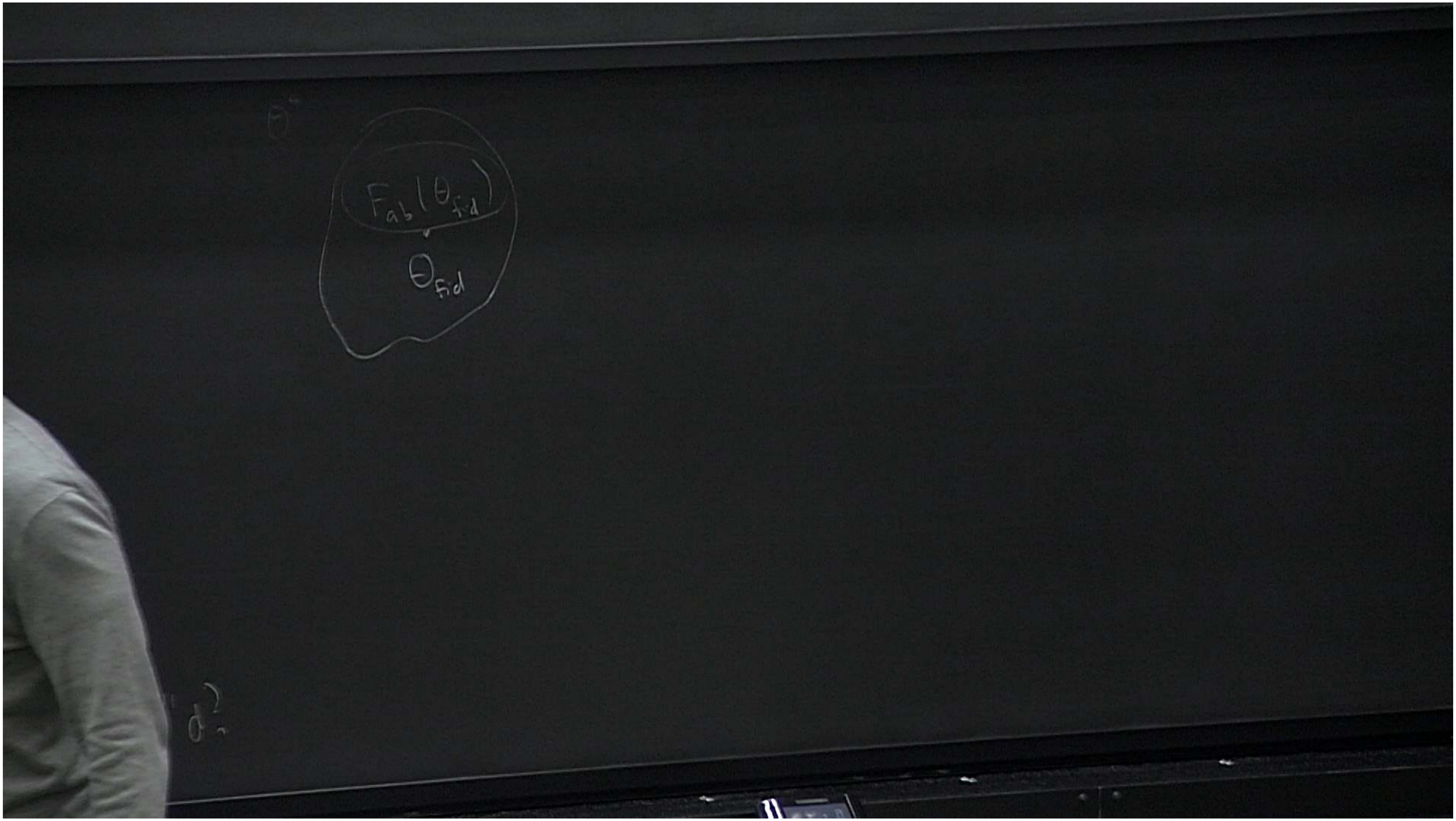
ERIOR IS

$$|d) \propto \exp\left[-\frac{1}{2}(\theta-F^{-1}v)^T F(\theta-F^{-1}v)\right] \cancel{\exp\left[\frac{1}{2}v^T F^{-1}v\right]}$$

VARIANCE F^{-1} AND MEAN $\bar{\theta} = F^{-1}v$




FISHER MATRIX
$\Theta^a \longrightarrow \boxed{C_\ell} \longrightarrow \boxed{a_{\ell m}}$
1) SIMULATION. GIVEN A MODEL Θ^a,
SIMULATE RANDOM DATA REALIZATION d
2) ANALYSIS. GIVEN A DATA REALIZATION d,
WHAT ARE THE CONSTRAINTS ON Θ^a?
3) FORECASTING. GIVEN A "FIDUCIAL" MODEL Θ^a,
WHAT ARE THE EXPECTED CONSTRAINTS FROM A "TYPICAL" d?


$F_{ab}(\theta_{fid})$
θ_{fid}

FISHER MATRIX

$$\Theta^a \longrightarrow \boxed{C_\ell} \longrightarrow \boxed{a_{\ell m}}$$

$p(d|\Theta)$

Θ^a

$F_{ab}|\Theta_{fid}$

Θ_{fid}

1) SIMULATION. GIVEN A MODEL Θ^a,
SIMULATE RANDOM DATA REALIZATION d

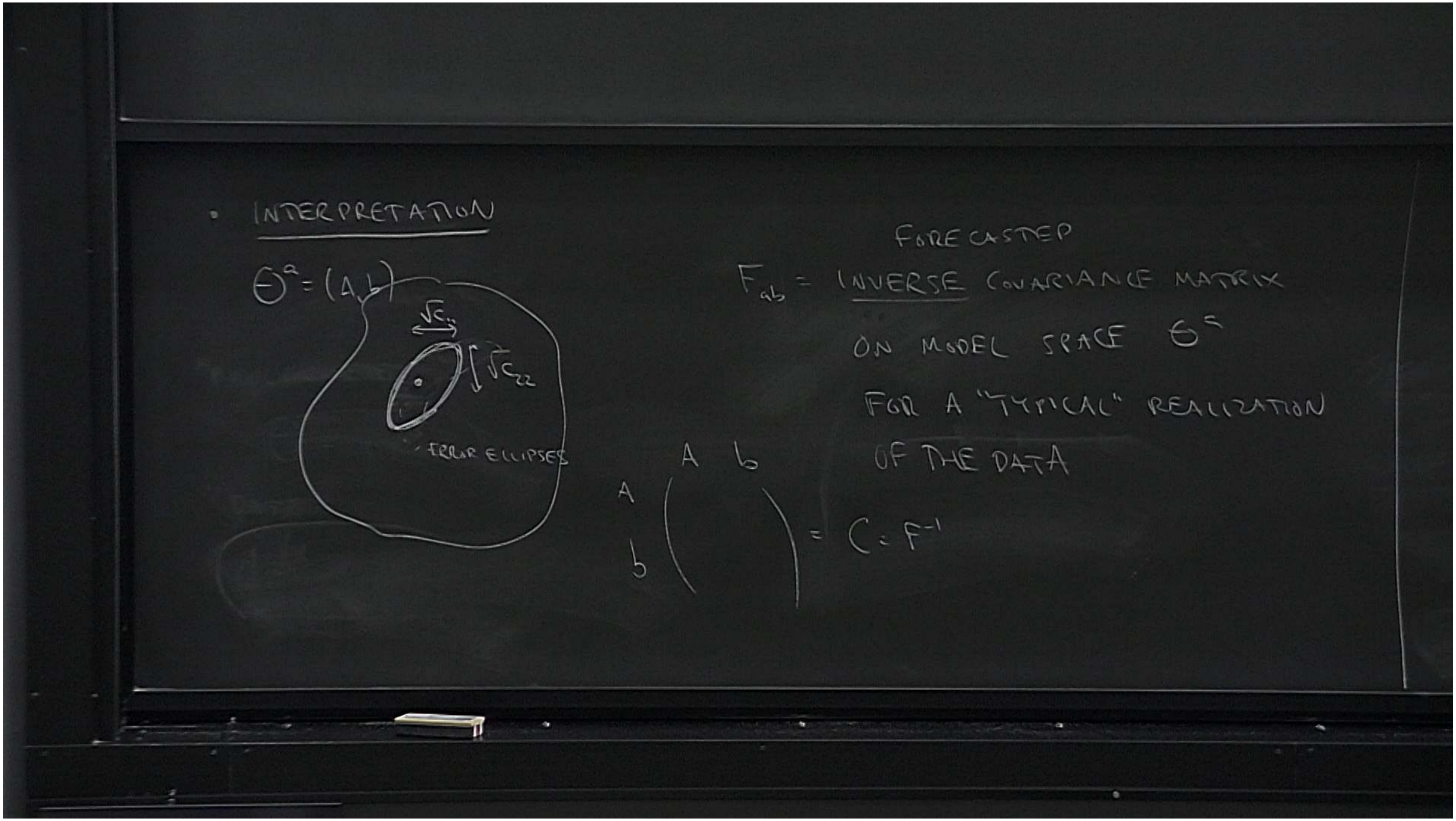
✓ 2) ANALYSIS. GIVEN A DATA REALIZATION d,
WHAT ARE THE CONSTRAINTS ON Θ^a?

3) FORECASTING. GIVEN A "FIDUCIAL" MODEL Θ^a,
WHAT ARE THE EXPECTED CONSTRAINTS FROM A "TYPICAL" d?

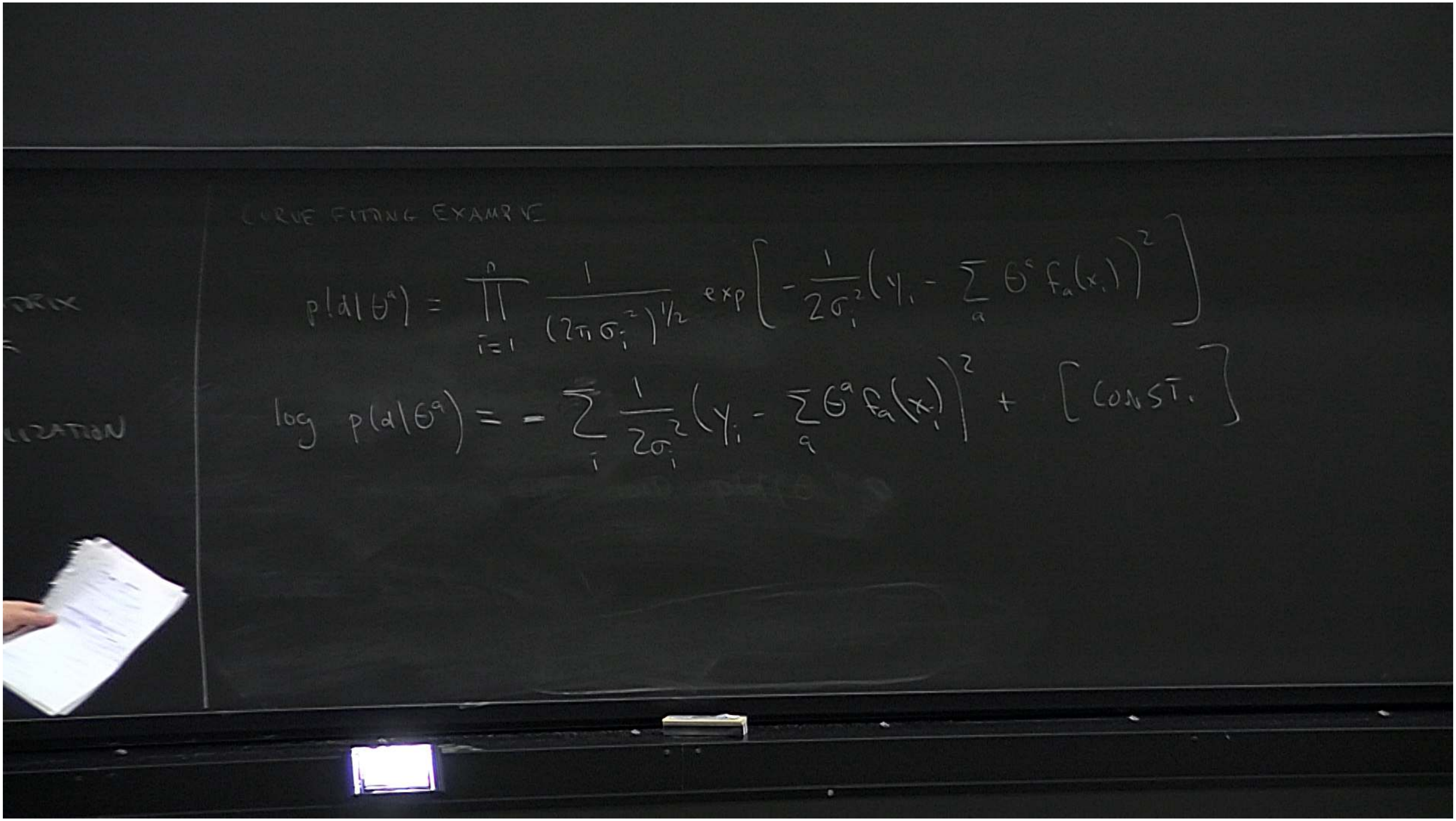
DEFINITION:

$$F_{ab}(\theta_{fid}) = \left\langle -\left(\frac{\partial^2 \log p(d|\theta)}{\partial\theta^a\,\partial\theta^b}\right)_{\theta_{fid}} \right\rangle_{d \sim \theta_{fid}}$$

$$= \int Dd\; p(d|\theta_{fid}) \left(-\frac{\partial^2 \log p(d|\theta)}{\partial\theta^a\,\partial\theta^b}\right)_{\theta = \theta_{fid}}$$



INTERPRETATION
$\Theta^a = (A,b)$
$\sqrt{C_{11}}$
$\sqrt{C_{22}}$
ERROR ELLIPSES
FORECASTED
F_{ab} = INVERSE COVARIANCE MATRIX
ON MODEL SPACE Θ^a
FOR A "TYPICAL" REALIZATION
OF THE DATA
A b
A
b
= $C = F^{-1}$


CURVE FITTING EXAMPLE
$p(d|\theta^a) = \prod_{i=1}^{n} \frac{1}{(2\pi\sigma_i^2)^{1/2}} \exp\left[-\frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2 \right]$
$\log p(d|\theta^a) = -\sum_i \frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2 + [\text{CONST.}]$

CURVE FITTING EXAMPLE

$$p(d|\theta^a) = \prod_{i=1}^{n} \frac{1}{(2\pi\sigma_i^2)^{1/2}} \exp\left[-\frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2 \right]$$

$$\log p(d|\theta^a) = -\sum_i \frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta^a f_a(x_i)\right)^2 + [\text{CONST.}]$$

$$F_{ab} = \left\langle -\frac{\partial^2 \log p(d|\theta)}{\partial\theta^a \partial\theta^b} \right\rangle_d$$

$$F_{ab} = \left\langle - \frac{\partial^2 \log p(d|\theta)}{\partial\theta^a \, \partial\theta^b} \right\rangle_d$$

$$= \left\langle - \sum_i - \frac{f_a(x_i)\, f_b(x_i)}{\sigma_i^2} \right\rangle_d$$

$$= \sum_i \frac{f_a(x_i)\, f_b(x_i)}{\sigma_i^2} \quad \checkmark$$