Title: PSI 2018/2019 - Explorations in Cosmology - Lecture 5

Speakers: Kendrick Smith

Collection: PSI 2018/2019 - Explorations in Cosmology (Smith)

Date: April 22, 2019 - 11:30 AM

URL: http://pirsa.org/19040067

	N-DIMENSIONAL EUCLIDEAN	2-SPHERE
FOURIER TRANSFORM	$\phi(\vec{x}) = \int \frac{d^N k}{(2\pi)^N} \phi(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$	$\phi(\hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n})$
"REALITY" CONDITION (I.E. $\phi = \phi^*$ IN REAL SPACE)	$\phi(\vec{k})^* = \phi(-\vec{k})$	$a_{\ell m}^* = (-1)^m a_{\ell,-m}$
POWER SPECTRUM	$\langle \phi(\vec{k}) \phi(\vec{k}')^* \rangle = P(k) (2\pi)^N \delta^N(k-k')$	$\langle a_{\ell m} a_{\ell' m'}^* \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'}$

2-SPHERE

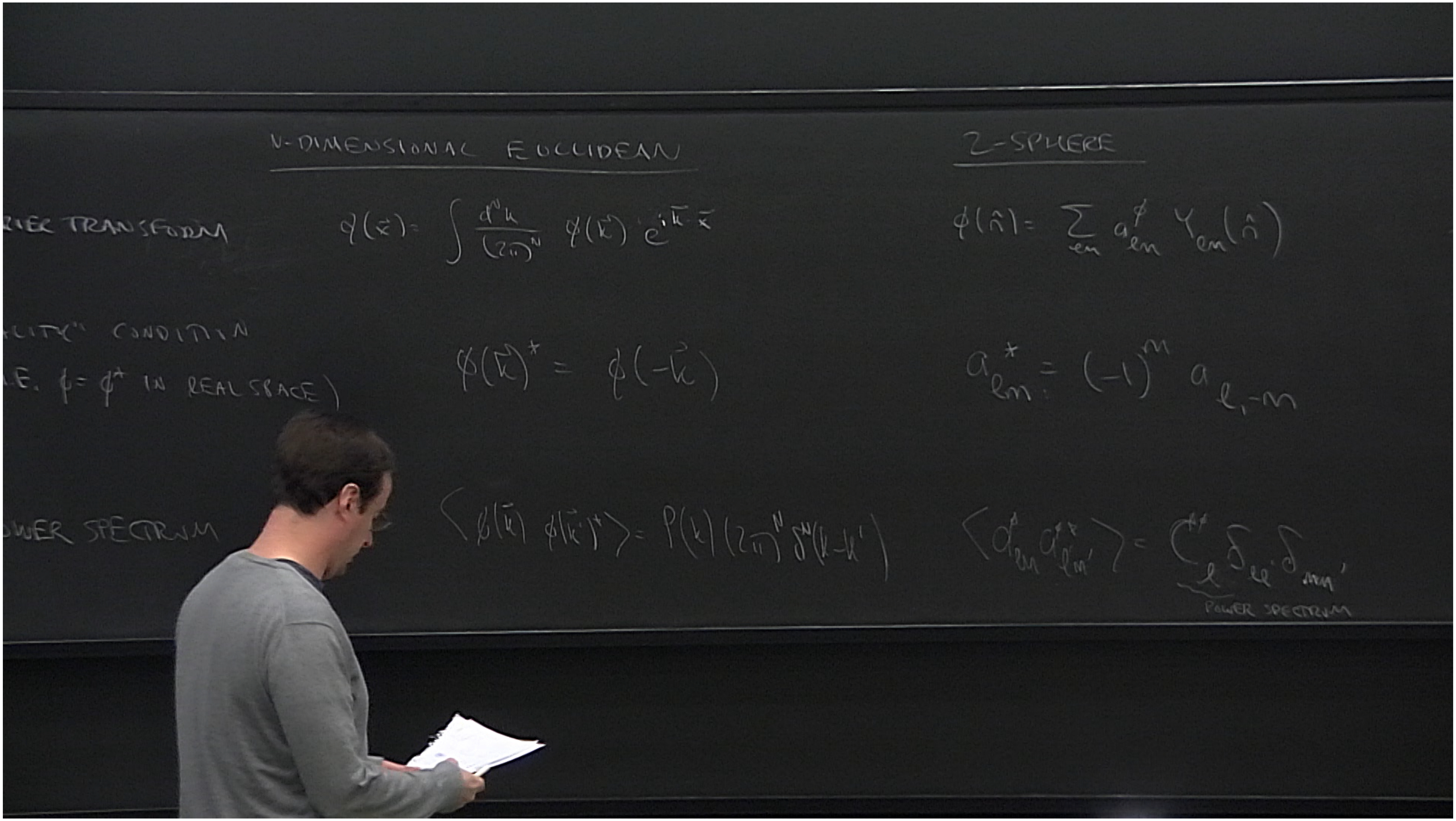
$$\phi(\hat{n}) = \sum_{\ell m} a^{\phi}_{\ell m} Y_{\ell m}(\hat{n})$$

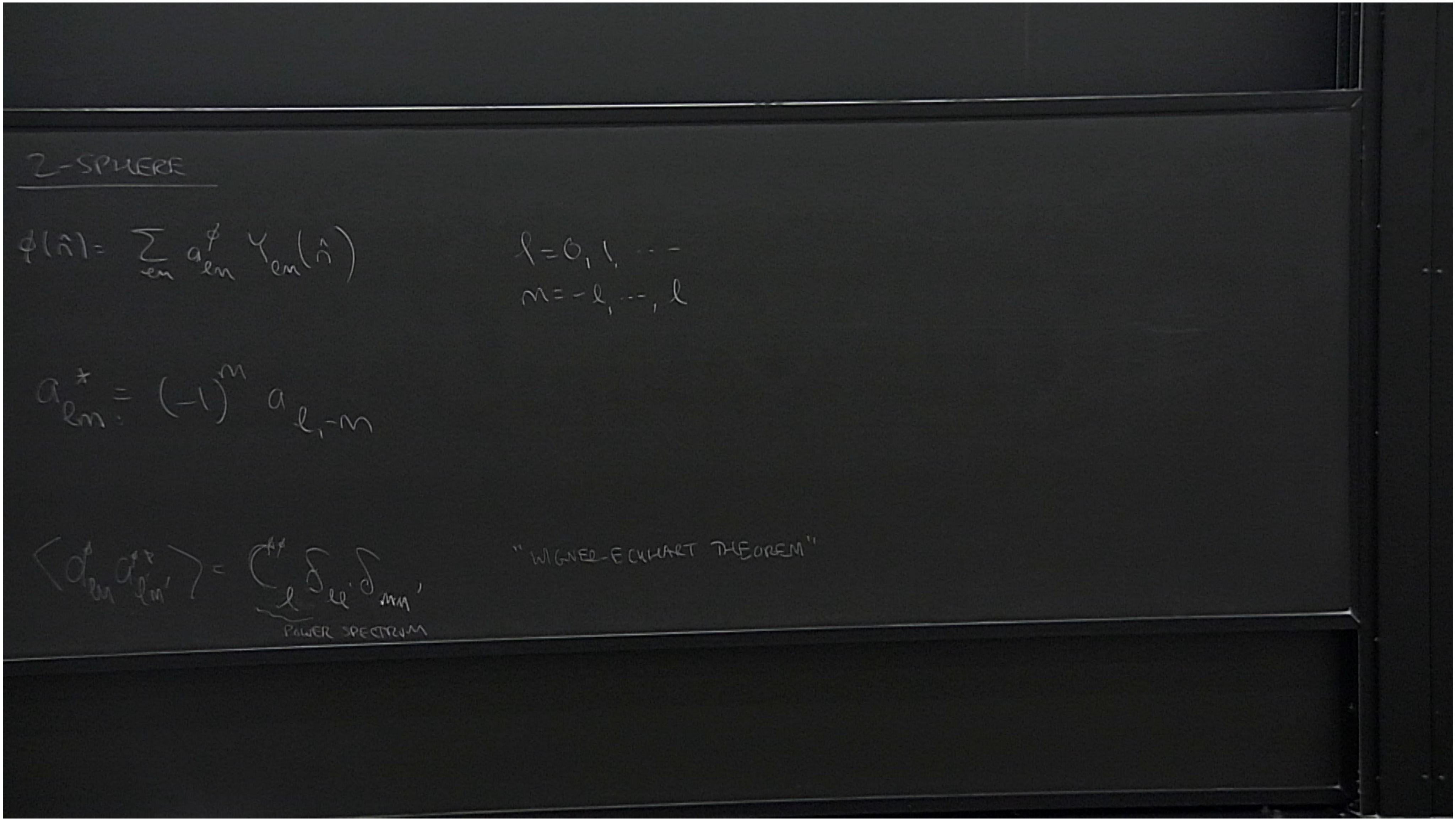
$\ell = 0, 1, \ldots$

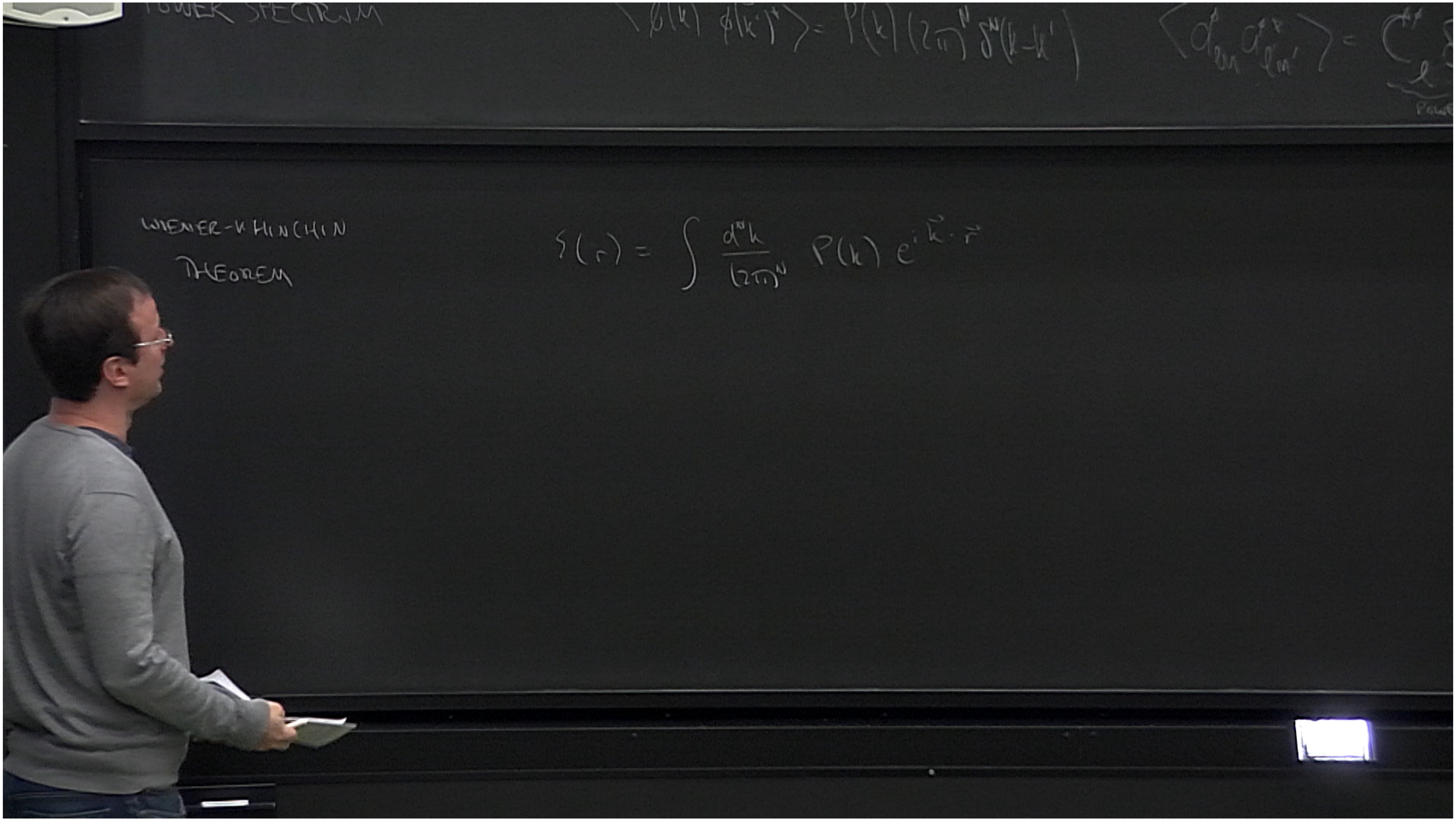
$m = -\ell, \ldots, \ell$

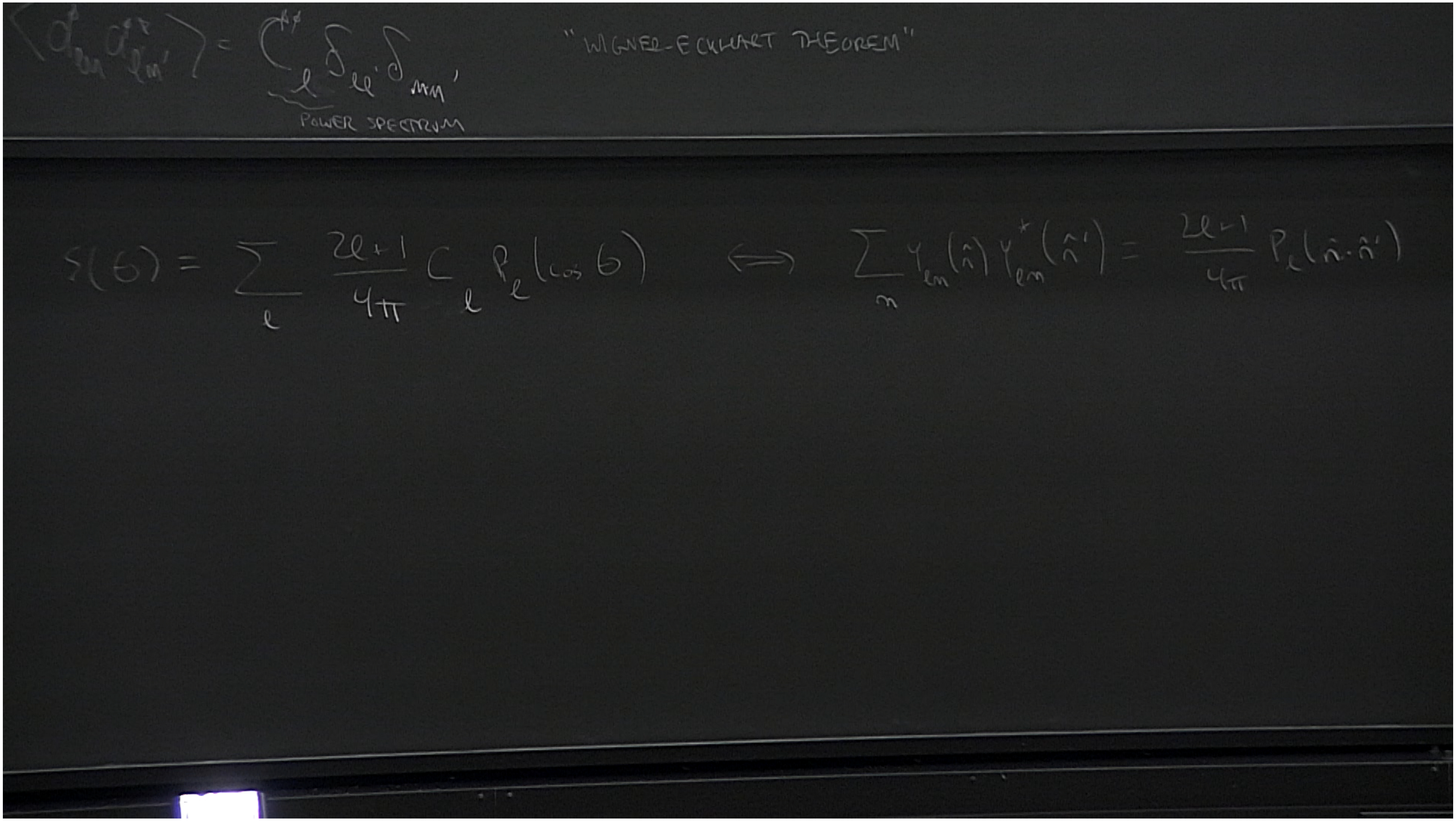
$$a^{*}_{\ell m} = (-1)^{m} a_{\ell, -m}$$

$$\langle a_{\ell m} a^{*}_{\ell' m'} \rangle = C_{\ell} \delta_{\ell \ell'} \delta_{m m'}$$

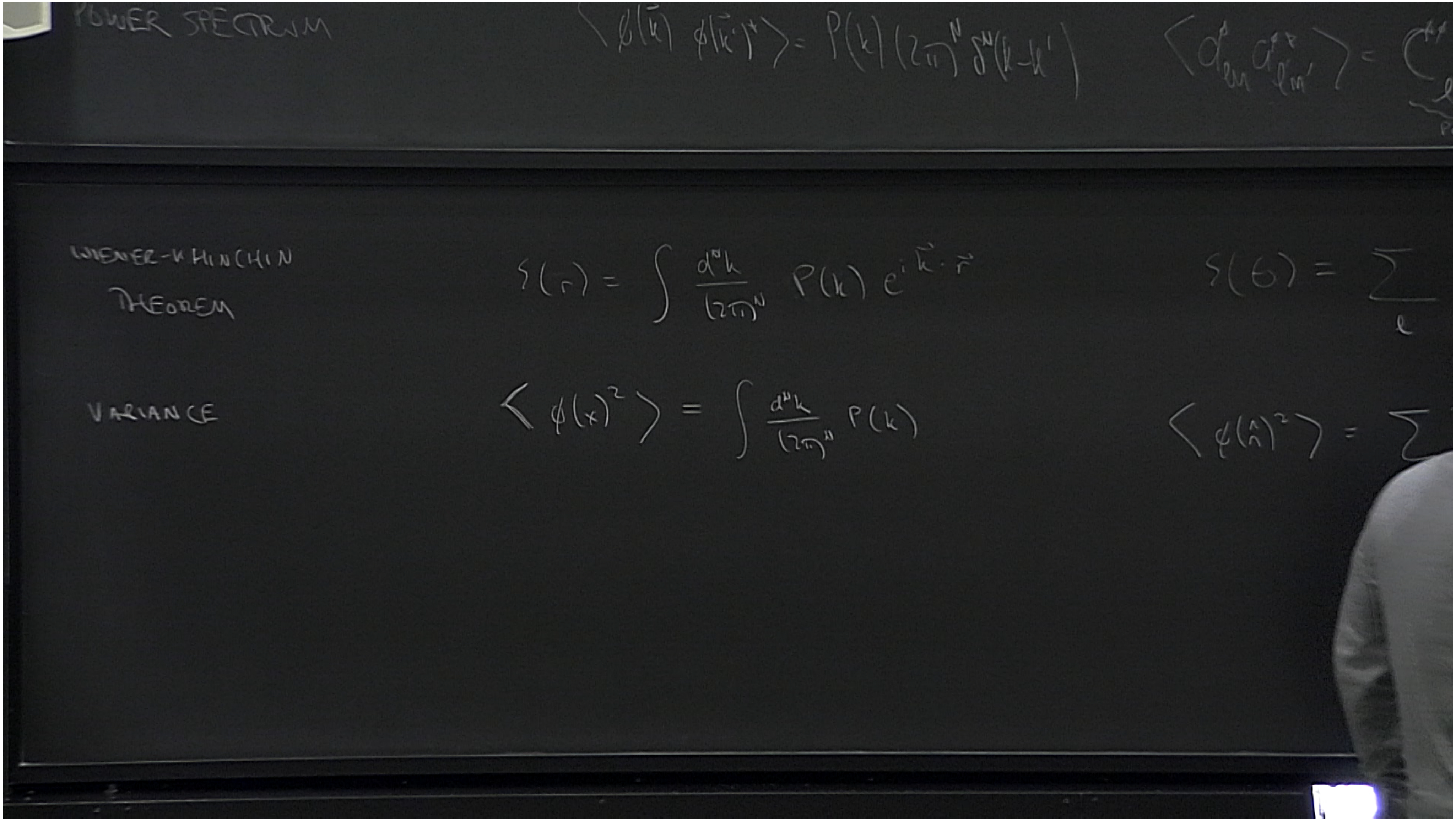

N-DIMENSIONAL EUCLIDEAN
2-SPHERE
RIER TRANSFORM
$\phi(\vec{x}) = \int \frac{d^N k}{(2\pi)^N} \phi(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$
$\phi(\hat{n}) = \sum_{\ell m} a^{\phi}_{\ell m} Y_{\ell m}(\hat{n})$
LITY" CONDITION
.E. $\phi = \phi^*$ IN REAL SPACE)
$\phi(\vec{k})^* = \phi(-\vec{k})$
$a^*_{\ell m} = (-1)^m a_{\ell,-m}$
WER SPECTRUM
$\langle \phi(\vec{k}) \phi(\vec{k}')^* \rangle = P(k) (2\pi)^N \delta^N(k-k')$
$\langle a^{\phi}_{\ell m} a^{\phi *}_{\ell' m'} \rangle = C^{\phi\phi}_{\ell} \delta_{\ell\ell'} \delta_{mm'}$
POWER SPECTRUM


2-SPHERE
$\phi(\hat{n}) = \sum_{lm} a^{\phi}_{lm} Y_{lm}(\hat{n})$
$l = 0, 1, \ldots$
$m = -l, \ldots, l$
$a^*_{lm} = (-1)^m a_{l,-m}$
$\langle a^{\phi}_{lm} a^{\phi *}_{l'm'} \rangle = C^{\phi\phi}_l \delta_{ll'} \delta_{mm'}$
"WIGNER-ECKHART THEOREM"
POWER SPECTRUM


POWER SPECTRUM
$\langle \phi(k)\, \phi(k')^* \rangle = P(k)\,(2\pi)^N\,\delta^N(k-k')$
WIENER-KHINCHIN
THEOREM
$\xi(r) = \int \frac{d^N k}{(2\pi)^N}\, P(k)\, e^{i\vec{k}\cdot\vec{r}}$



$\langle a_{\ell m}^{\phi} a_{\ell' m'}^{\phi *} \rangle = C_\ell^{\phi\phi} \delta_{\ell\ell'} \delta_{mm'}$
POWER SPECTRUM
"WIGNER-ECKHART THEOREM"
$S(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta) \iff \sum_m Y_{\ell m}(\hat{n}) Y_{\ell m}^*(\hat{n}') = \frac{2\ell+1}{4\pi} P_\ell(\hat{n}\cdot\hat{n}')$


POWER SPECTRUM
WIENER-KHINCHIN THEOREM
VARIANCE

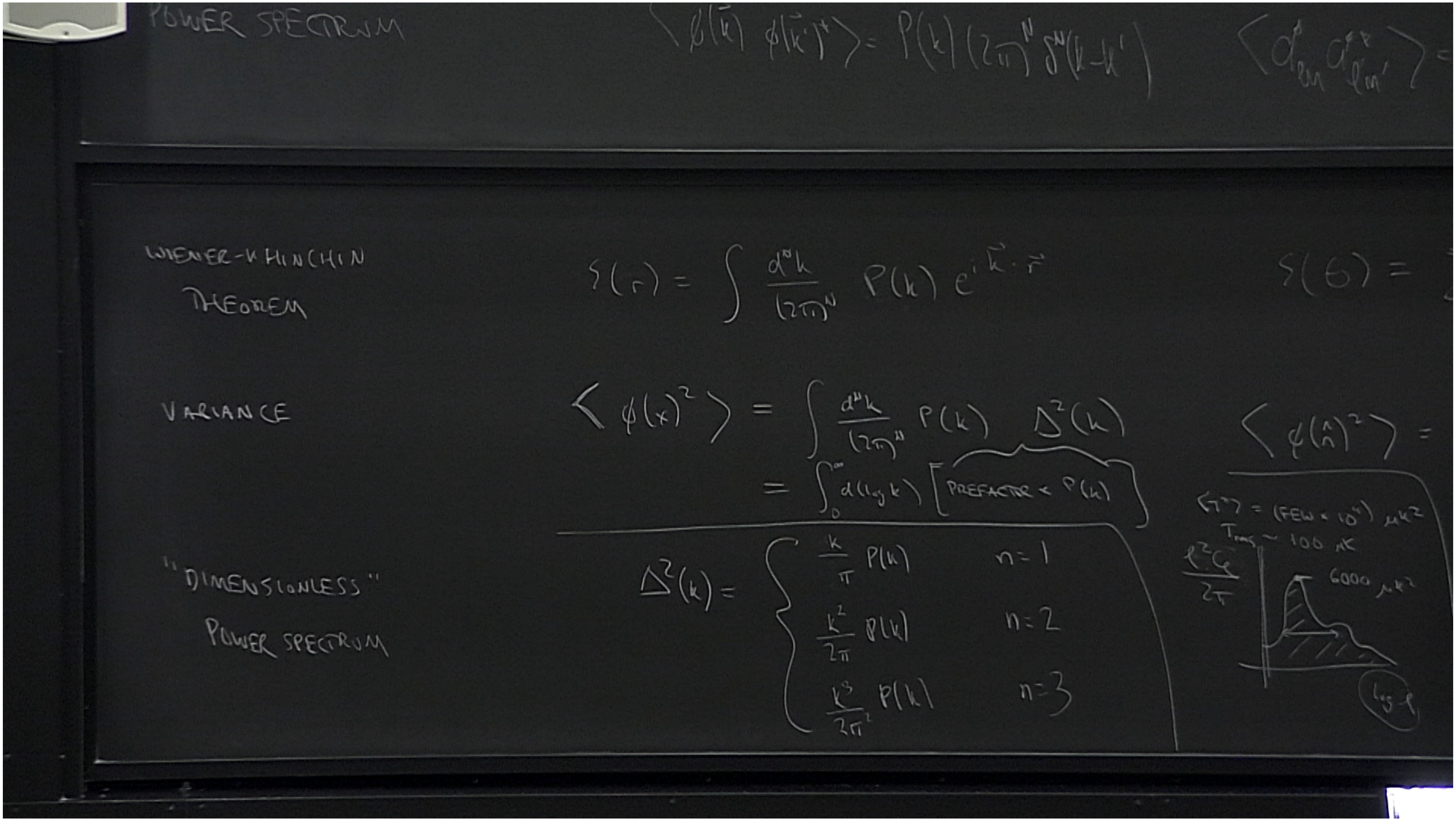
$$\langle a^{\phi}_{\ell m} a^{\phi *}_{\ell' m'} \rangle = C^{\phi\phi}_{\ell}\, \delta_{\ell\ell'}\, \delta_{mm'}$$

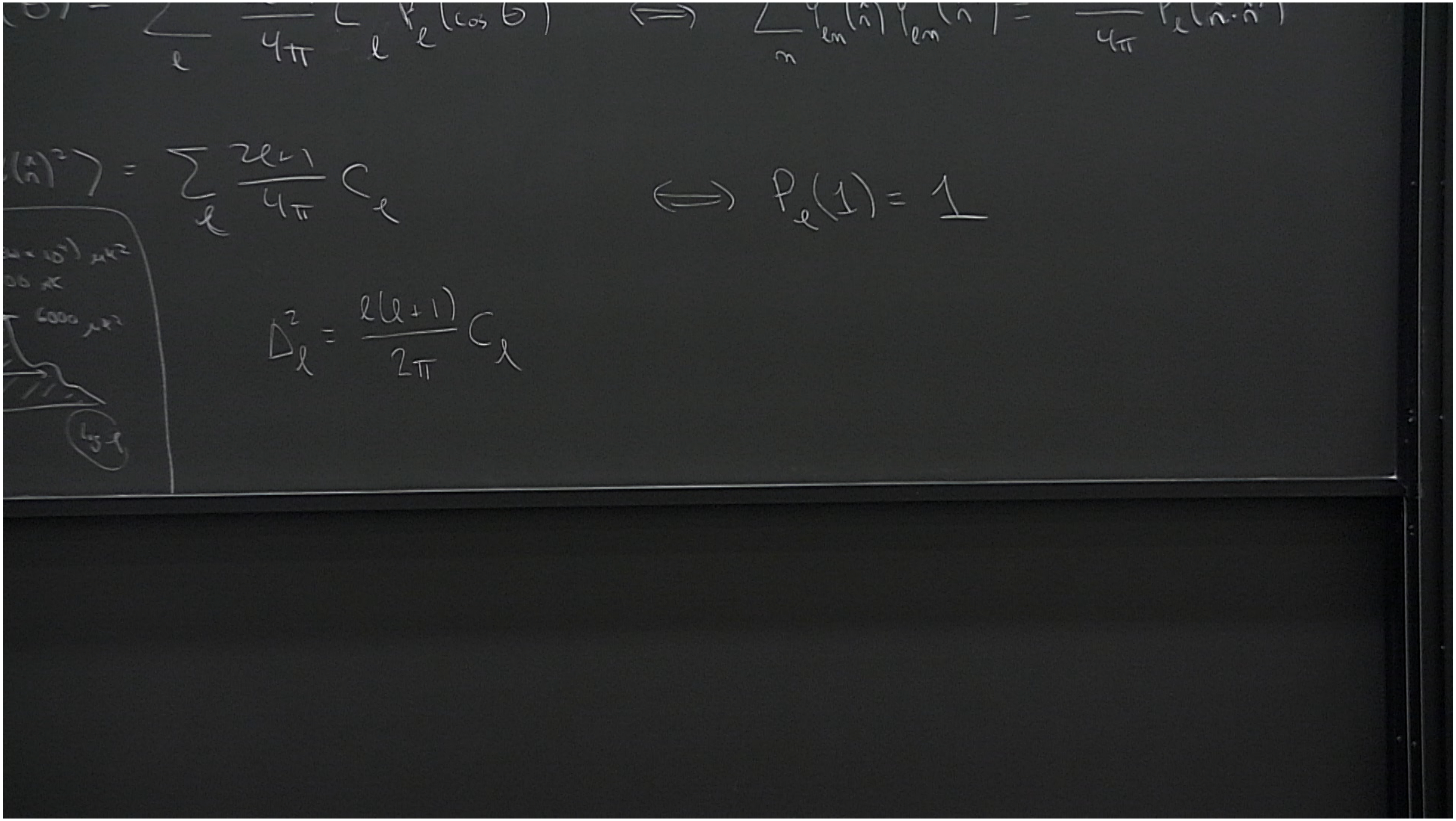
POWER SPECTRUM

"WIGNER-ECKHART THEOREM"

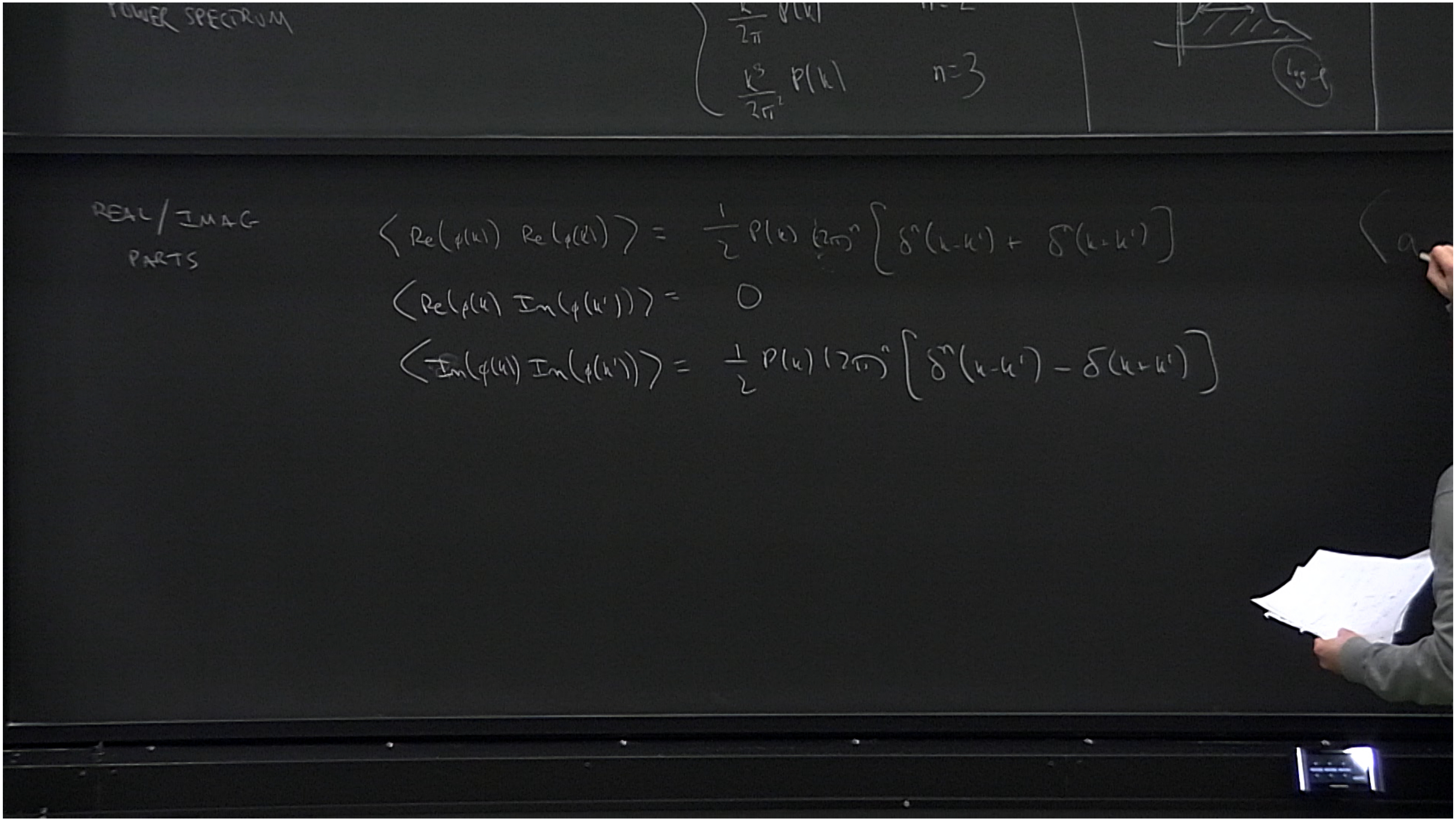
$$S(\theta) = \sum_{\ell} \frac{2\ell+1}{4\pi} C_{\ell} P_{\ell}(\cos\theta) \iff \sum_{m} Y_{\ell m}(\hat{n}) Y^{*}_{\ell m}(\hat{n}') = \frac{2\ell+1}{4\pi} P_{\ell}(\hat{n}\cdot\hat{n}')$$

$$\langle \phi(\hat{n})^2 \rangle = \sum_{\ell} \frac{2\ell+1}{4\pi} C_{\ell} \iff P_{\ell}(1) = 1$$


POWER SPECTRUM
⟨φ(k) φ(k')*⟩ = P(k) (2π)^N δ^N(k−k')
WIENER-KHINCHIN THEOREM
ξ(r) = ∫ d^N k/(2π)^N P(k) e^{i k·r}
VARIANCE
⟨φ(x)^2⟩ = ∫ d^N k/(2π)^N P(k)
= ∫_0^∞ d(log k) [PREFACTOR × P(k)]
Δ^2(k)
"DIMENSIONLESS" POWER SPECTRUM
Δ^2(k) = k/π P(k) n=1
k^2/2π P(k) n=2
k^3/2π^2 P(k) n=3
⟨φ(n̂)^2⟩ =
⟨T^2⟩ = (FEW × 10^3) μK^2
T_rms ~ 100 μK
6000 μK^2
l^2 C_l/2π
log l



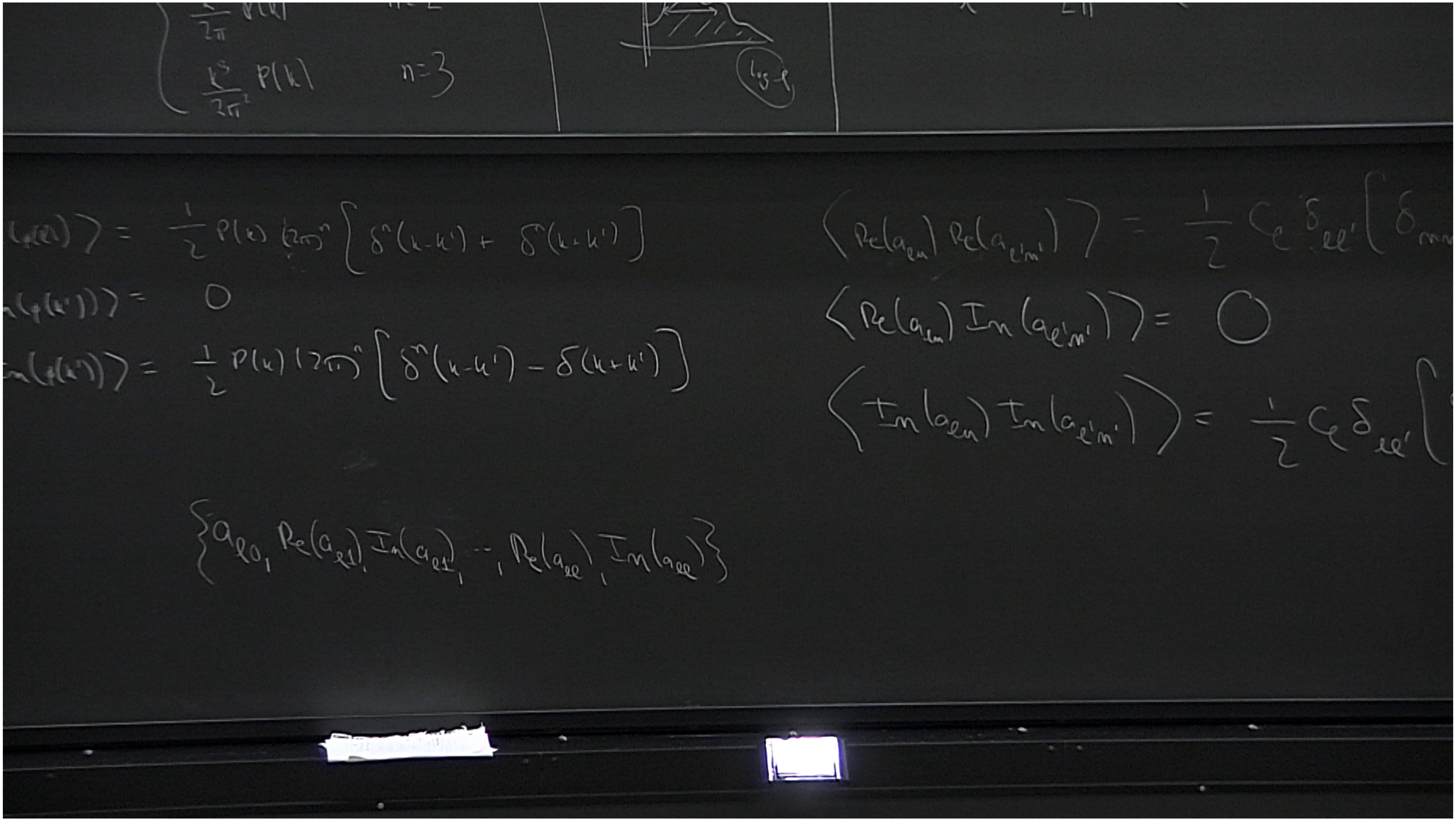
$= \sum_\ell \frac{2\ell+1}{4\pi} C_\ell$
$\Longleftrightarrow P_\ell(1) = 1$
6000 μK²
$\Delta_\ell^2 = \frac{\ell(\ell+1)}{2\pi} C_\ell$

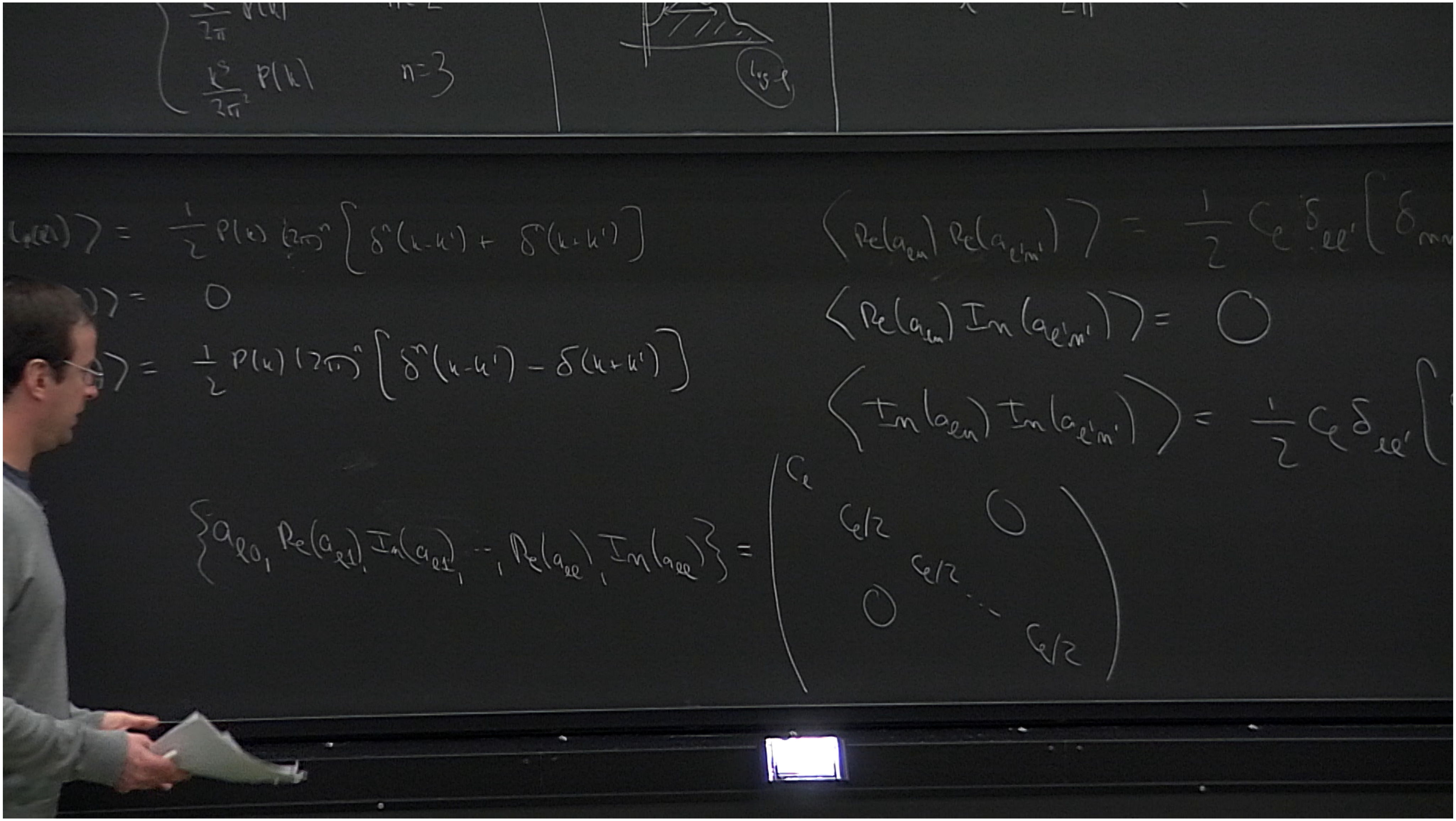

POWER SPECTRUM
$\frac{k^3}{2\pi^2} P(k)$ $n=3$
REAL / IMAG PARTS
$\langle Re(\phi(k))\, Re(\phi(k')) \rangle = \frac{1}{2} P(k) (2\pi)^n [\delta^n(k-k') + \delta^n(k+k')]$
$\langle Re(\phi(k))\, Im(\phi(k')) \rangle = 0$
$\langle Im(\phi(k))\, Im(\phi(k')) \rangle = \frac{1}{2} P(k) (2\pi)^n [\delta^n(k-k') - \delta(k+k')]$

$$\langle \mathrm{Re}(a_{\ell m})\, \mathrm{Re}(a_{\ell' m'}) \rangle = \frac{1}{2} C_\ell\, \delta_{\ell\ell'} \left[\delta_{mm'} + (-1)^m \delta_{m,-m'} \right]$$

$$\langle \mathrm{Re}(a_{\ell m})\, \mathrm{Im}(a_{\ell' m'}) \rangle = 0$$

$$\langle \mathrm{Im}(a_{\ell m})\, \mathrm{Im}(a_{\ell' m'}) \rangle = \frac{1}{2} C_\ell\, \delta_{\ell\ell'} \left[\delta_{mm'} - (-1)^m \delta_{m,-m'} \right]$$


n=3
⟨Re(a_{ℓm}) Re(a_{ℓ'm'})⟩ = 1/2 C_ℓ δ_{ℓℓ'}
⟨Re(a_{ℓm}) Im(a_{ℓ'm'})⟩ = 0
⟨Im(a_{ℓm}) Im(a_{ℓ'm'})⟩ = 1/2 C_ℓ δ_{ℓℓ'}
{a_{ℓ0}, Re(a_{ℓ1}), Im(a_{ℓ1}), ..., Re(a_{ℓℓ}), Im(a_{ℓℓ})}


n=3
⟨Re(a_{ℓm}) Re(a_{ℓ'm'})⟩ = 1/2 C_ℓ δ_{ℓℓ'}
⟨Re(a_{ℓm}) Im(a_{ℓ'm'})⟩ = 0
⟨Im(a_{ℓm}) Im(a_{ℓ'm'})⟩ = 1/2 C_ℓ δ_{ℓℓ'}
{a_{ℓ0}, Re(a_{ℓ1}), Im(a_{ℓ1}), ..., Re(a_{ℓℓ}), Im(a_{ℓℓ})} =
C_ℓ
C_ℓ/2
C_ℓ/2
C_ℓ/2

$$\langle \mathrm{Re}(a_{\ell m})\, \mathrm{Re}(a_{\ell' m'}) \rangle = \frac{1}{2} C_\ell\, \delta_{\ell\ell'} \left[\delta_{mm'} + (-1)^m \delta_{m,-m'} \right]$$

$$\langle \mathrm{Re}(a_{\ell m})\, \mathrm{Im}(a_{\ell' m'}) \rangle = 0$$

$$\langle \mathrm{Im}(a_{\ell m})\, \mathrm{Im}(a_{\ell' m'}) \rangle = \frac{1}{2} C_\ell\, \delta_{\ell\ell'} \left[\delta_{mm'} - (-1)^m \delta_{m,-m'} \right]$$

$$\ldots m(a_{\ell\ell})\} = \begin{pmatrix} C_\ell & & & & 0 \\ & C_\ell/2 & & & \\ & & C_\ell/2 & & \\ & & & \ddots & \\ 0 & & & & C_\ell/2 \end{pmatrix}$$

$$\boxed{a_{\ell m} \rightarrow e^{im\phi}\, a_{\ell m}}$$

Random field slideshow

Kendrick Smith
PSI 2019, Explorations in Cosmology

Reminder: the correlation function ζ(r) and the power spectrum P(k) contain the same information, and are alternate ways of parameterizing the two-point statistics of a random field.

$$\langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle = \zeta(|\mathbf{x}-\mathbf{y}|)$$ (definition of ζ(r))

$$\langle \phi(\mathbf{k})\,\phi(\mathbf{k}')^*\rangle = P(k)\,(2\pi)^n\delta^n(\mathbf{k}-\mathbf{k}')$$ (definition of P(k))

$$\zeta(r) = \int \frac{d^n\mathbf{k}}{(2\pi)^n} P(k) e^{i\mathbf{k}\cdot\mathbf{r}}$$ (Wiener-Khinchin theorem)

In these slides, we'll use the power spectrum P(k).

In cosmology,

- 1-d random fields are usually time series

  Denoted f(t) in real space, or f(ω) in Fourier space

- 2-d random fields are usually functions of angular coordinates

  Flat-sky: $f(\theta_x,\theta_y)$ in real space, $f(l_x,l_y)$ in Fourier space
  All-sky: f(n) in real space, f_{lm} in harmonic space
  Power spectrum is usually denoted C_l (not P(l))
  Dimensionless power spectrum is $\Delta_l^2 = \frac{l(l+1)}{2\pi} C_l$

- 3-d random fields are usually functions of spatial coordinates

  Denoted f(x) in real space, or f(k) in Fourier space
  Dimensionless power spectrum is $\Delta(k)^2 = \frac{k^3}{2\pi^2} P(k)$

Curved sky: The spherical harmonic $Y_{lm}(\theta,\phi)$ is a special function defined for integers $\ell = 0,1,2,\ldots$ and $m = -\ell, (-\ell+1), \ldots, \ell$.

Spherical analogue of a plane wave e^{ikx}. The wavenumber ℓ is quantized (an integer), and there are $(2\ell+1)$ harmonics for each ℓ.

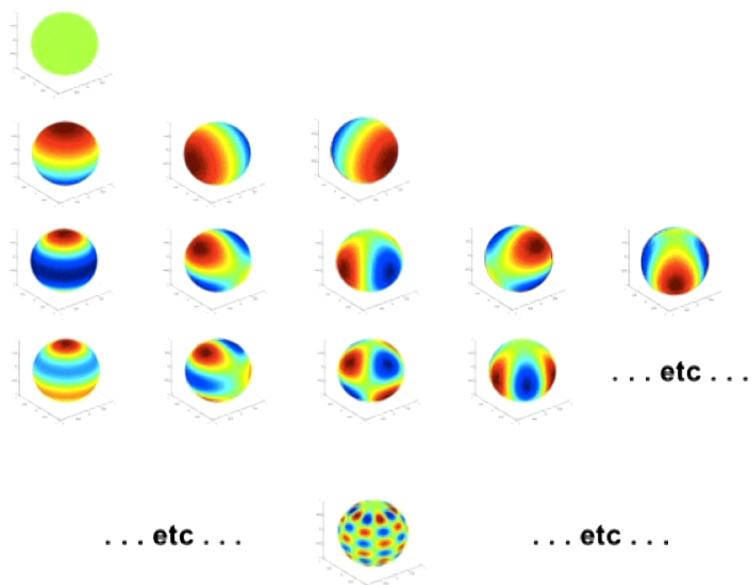
Any function $f(\theta,\phi)$ is representable as $f(\theta,\phi) = \sum_{lm} a_{lm} Y_{lm}(\theta,\phi)$

$\ell=0$ (monopole)

$\ell=1$ (dipole)

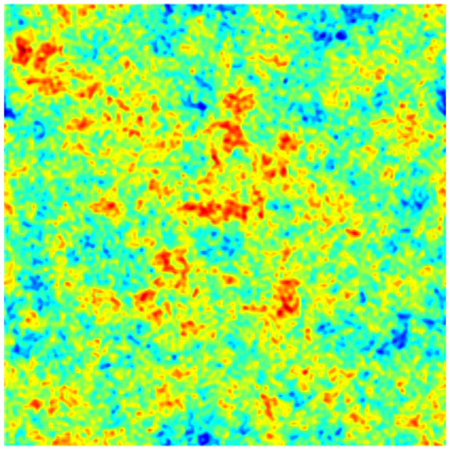
$\ell=2$ (quadrupole)

$\ell=3$ (octopole)



Reminder: A Euclidean Gaussian field $\phi(x)$ is easy to simulate.

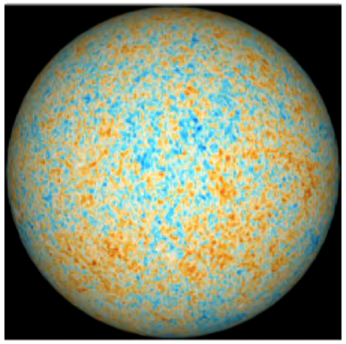
- For each Fourier mode k, let the real and imaginary parts of $\phi(k)$ be random Gaussian numbers with variance $P(k)/2$.
- Detail: don't forget $\phi(-k) = \phi(k)^*$
- If the real-space representation is desired (e.g. for plotting purposes), just take the Fourier transform $\phi(k) \to \phi(x)$.



A curved-sky Gaussian field $f(\theta,\phi)$ is simulated as follows:

- For each multipole ℓ,
  - Let $a_{\ell 0}$ be a real Gaussian number with variance C_l
  - For $m=1,\ldots,\ell$, let $a_{\ell m}$ be a complex number whose real and imaginary parts are Gaussians with variance $C_\ell/2$.
  - Fill in negative m-values using the identity $a_{\ell,-m} = (-1)^m (a_{\ell m})^*$
- If the real-space representation is desired (e.g. for plotting purposes), take the "spherical transform"

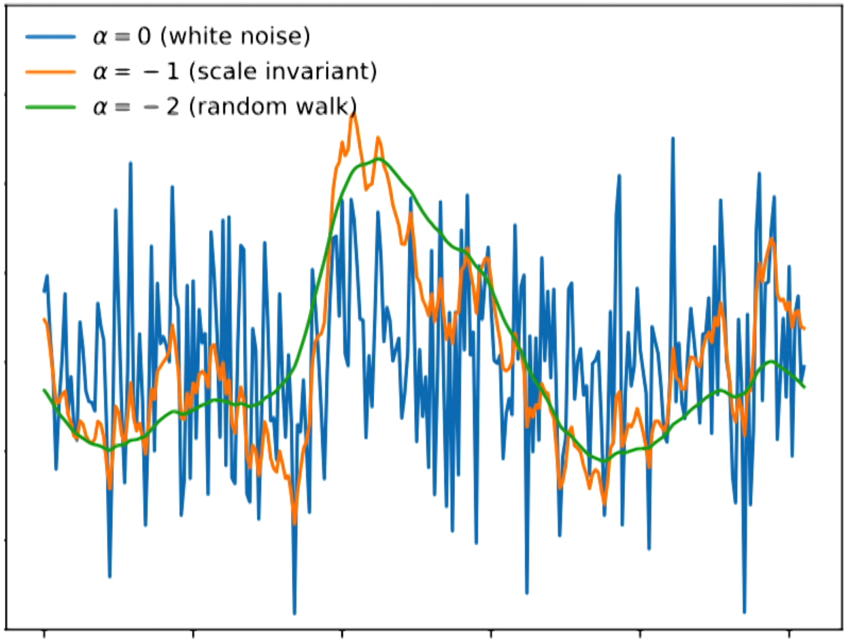
$$f(\theta,\phi) = \sum_{lm} a_{lm} Y_{lm}(\theta,\phi)$$



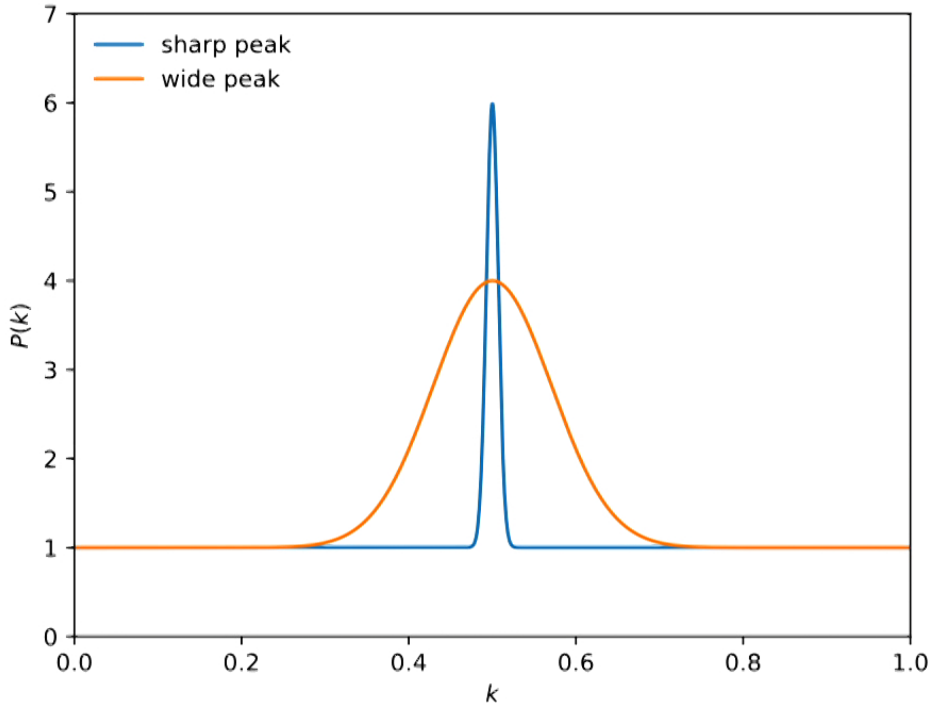
One-dimensional examples. First, Gaussian random fields with power-law spectra $P(\omega) \propto \omega^{\alpha}$.

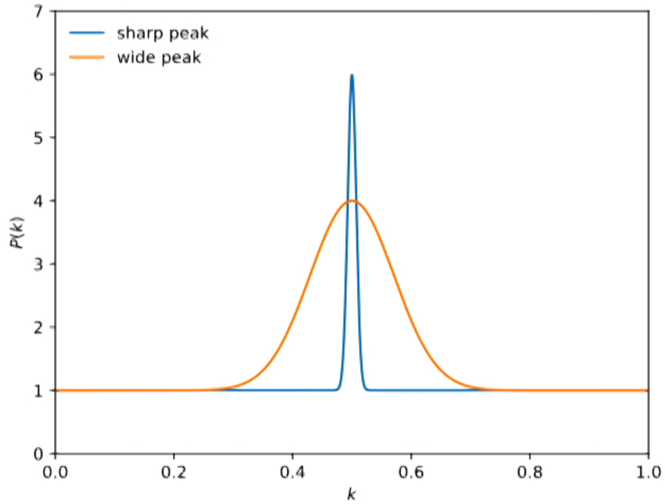
$\alpha = 0$: White noise

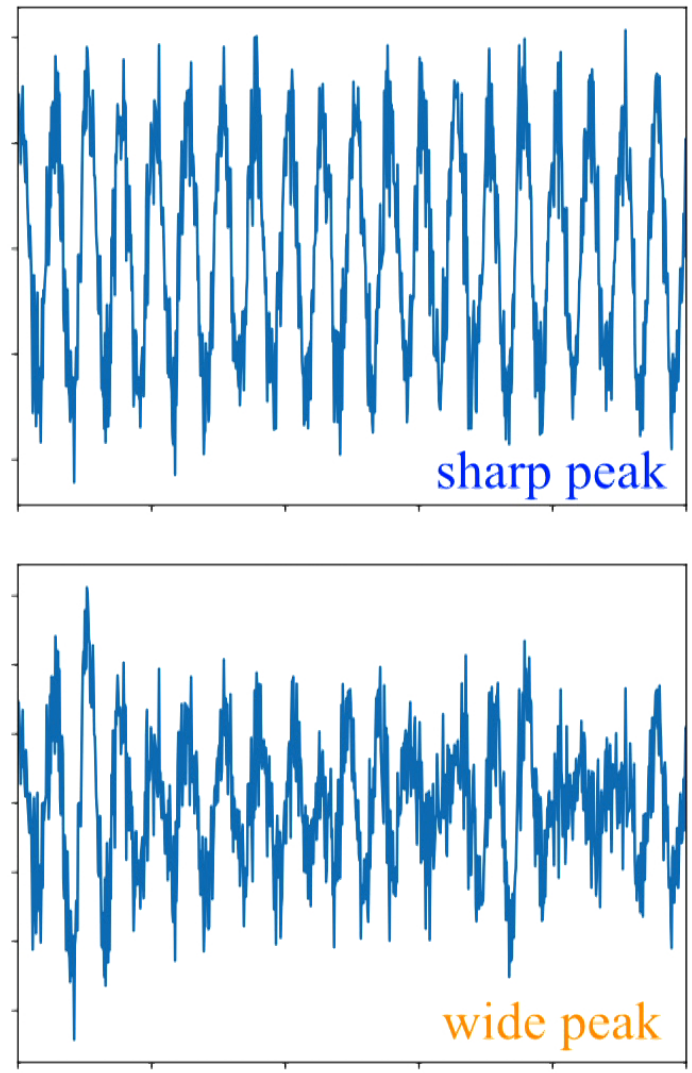
$\alpha < 0$: "Red" noise



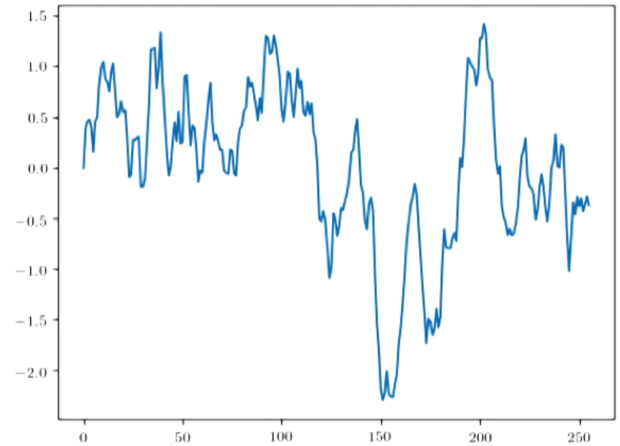
Next example: one-dimensional Gaussian random field, whose power spectrum is a constant (white noise), plus a peak which can be either "sharp" or "wide" in k.




sharp peak
wide peak
7
6
5
4
3
2
1
0
P(k)
0.0
0.2
0.4
0.6
0.8
1.0
k

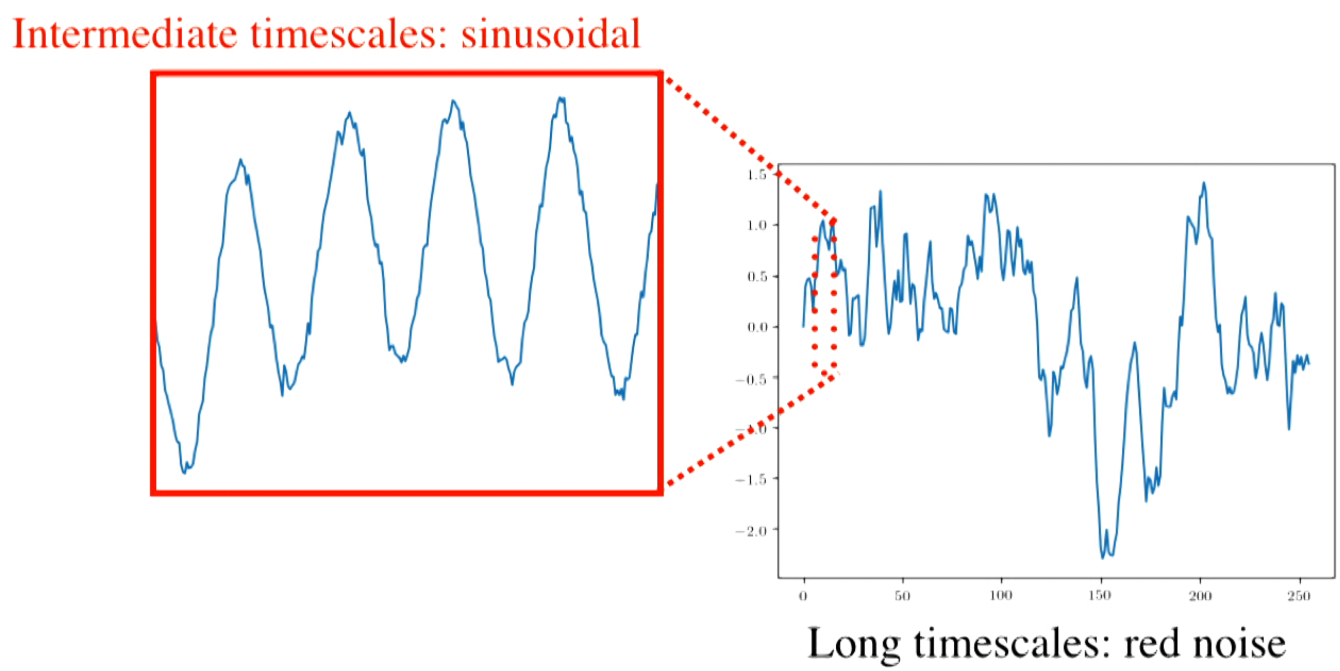

sharp peak
wide peak

A realistic detector noise timestream might look like this:

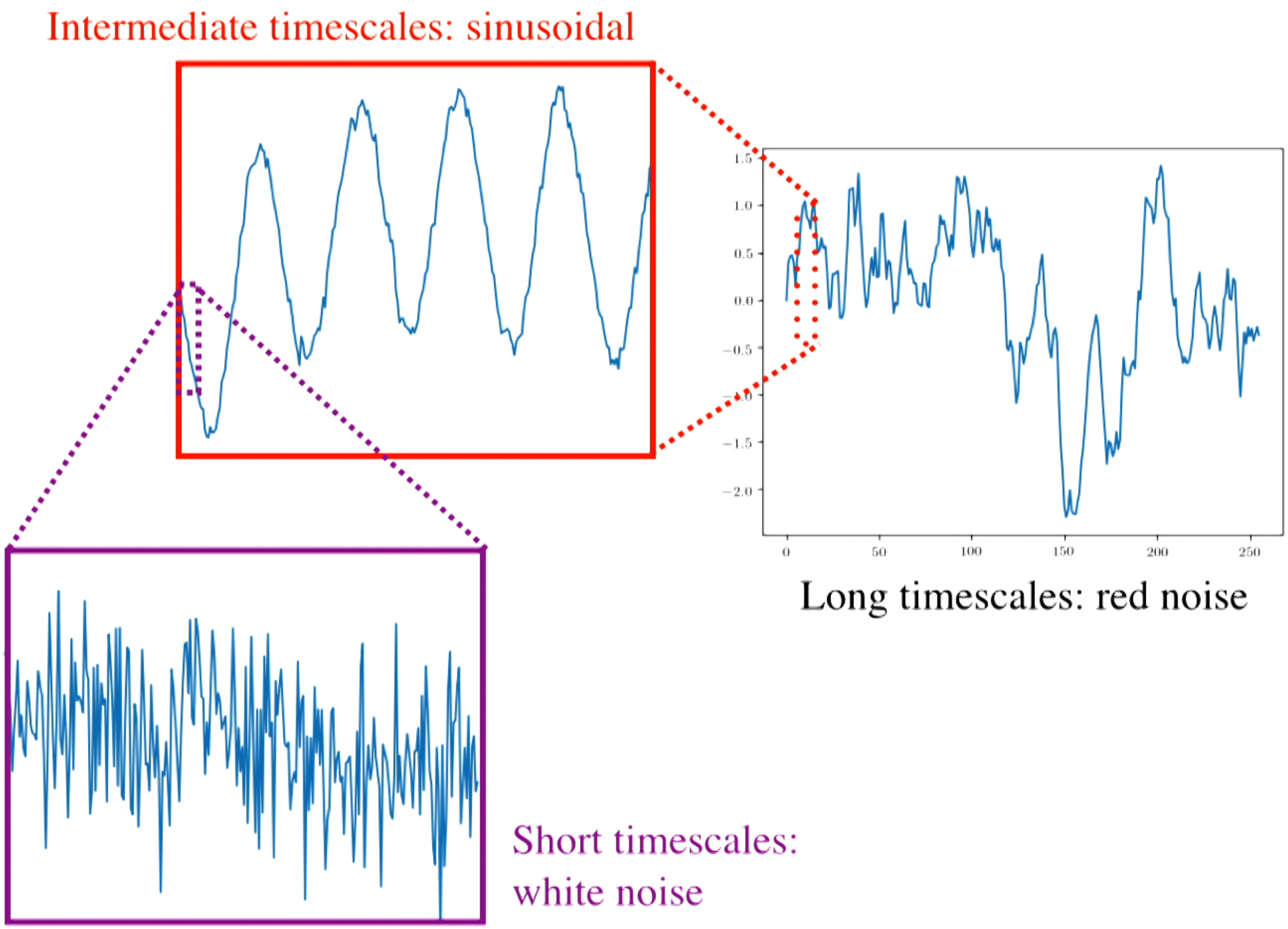


Long timescales: red noise

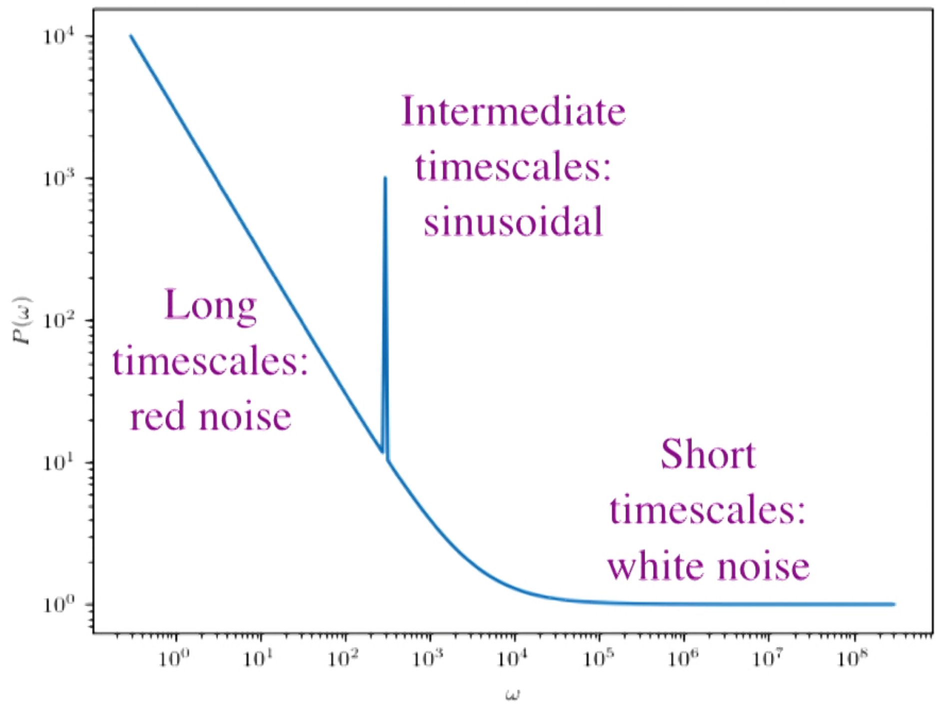
A realistic detector noise timestream might look like this:



A realistic detector noise timestream might look like this:

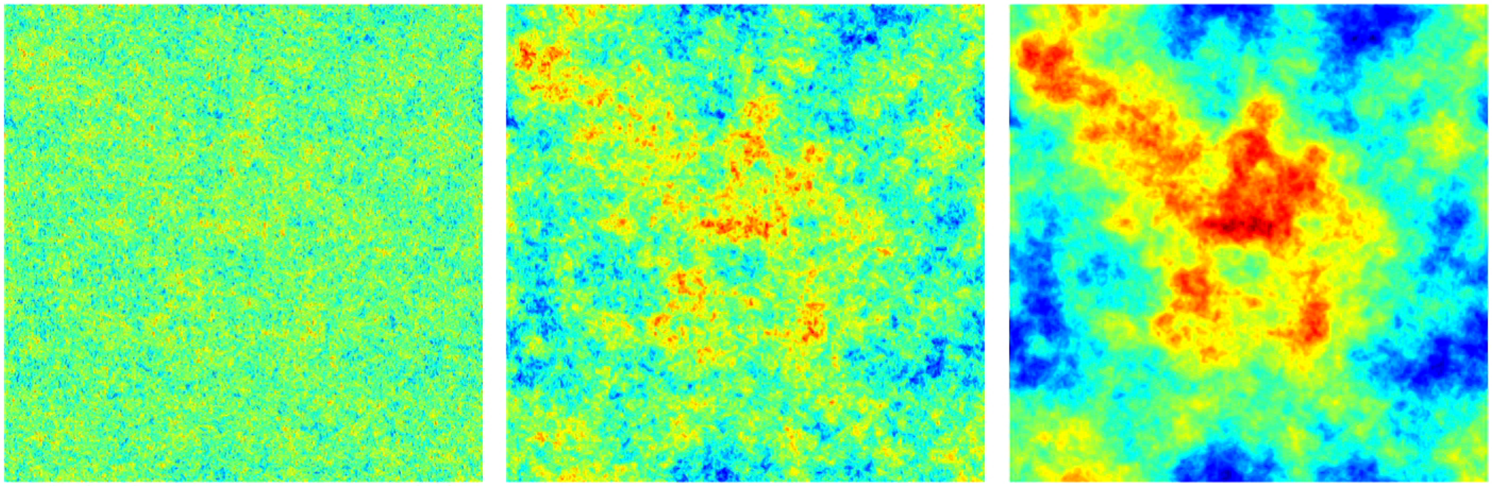


Associated noise power spectrum:



(Plots on the previous slide were generated by simulating a 1-d Gaussian random field with this power spectrum.)

Two-dimensional Gaussian random fields with power-law spectra $P(l) \propto l^{\alpha}$

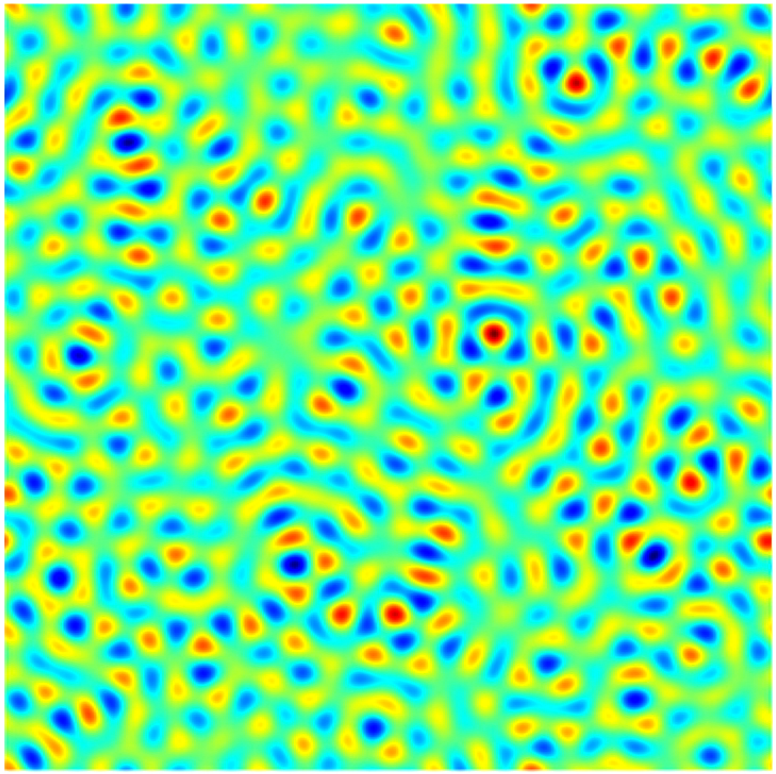


$\alpha = -1$
"blue" spectrum

$\alpha = -2$
scale invariant

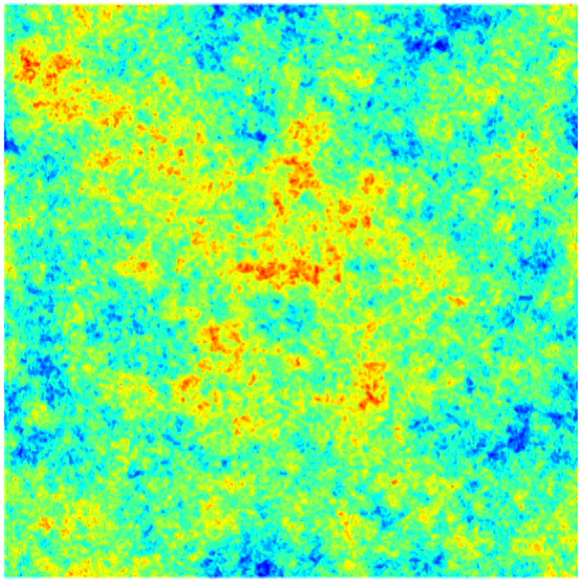
$\alpha = -3$
"red" spectrum

Delta function power spectrum: $P(l) \propto \delta(l-l_0)$

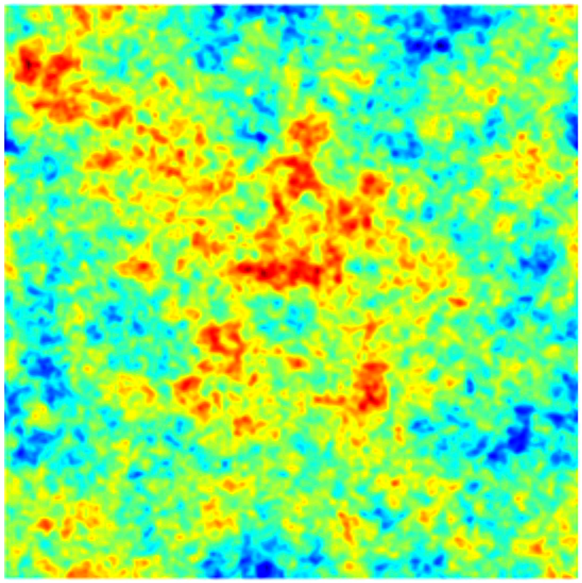


Well-defined characteristic scale

Putting a “cutoff” in the power spectrum removes power on small scales.

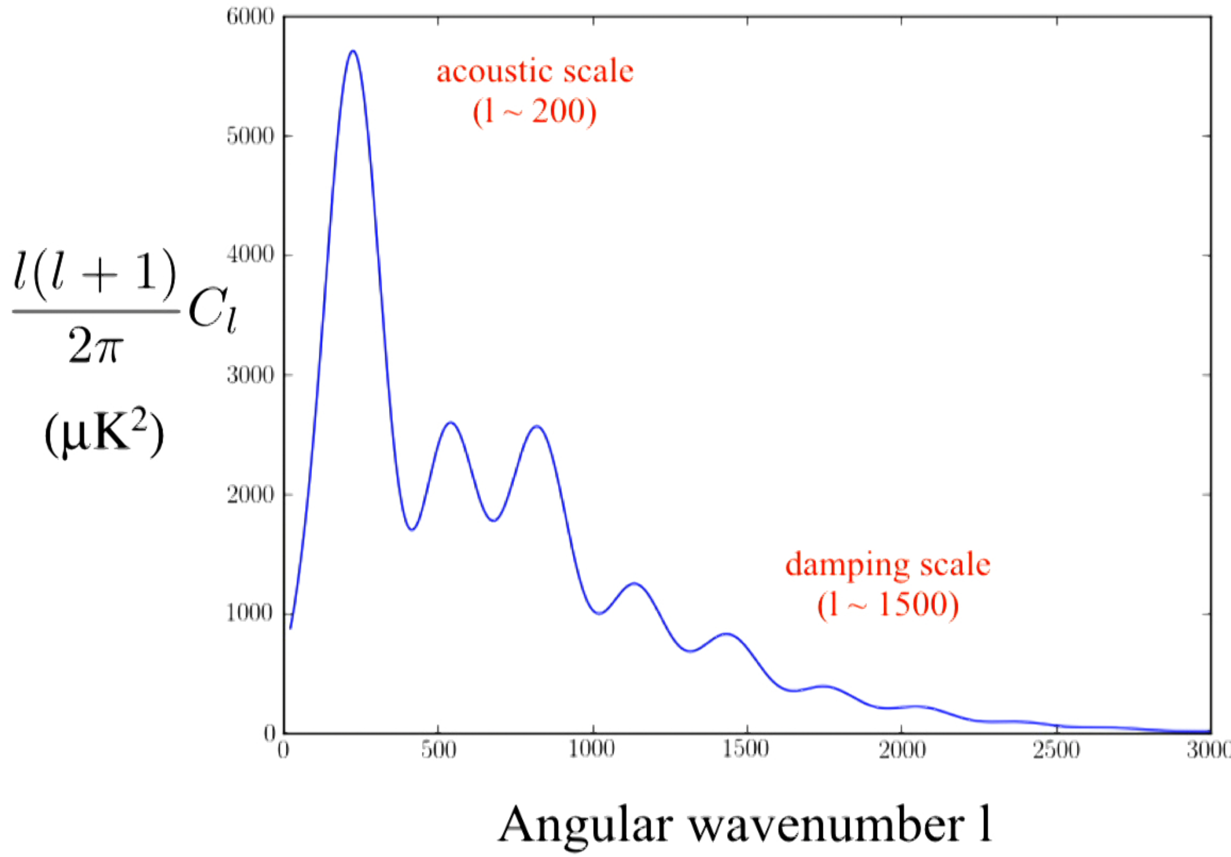


Scale invariant power spectrum: $P(l) \propto l^{-2}$



Scale invariant power spectrum with cutoff: $P(l) \propto l^{-2} \exp(-(l/l_0)^2)$

CMB power spectrum

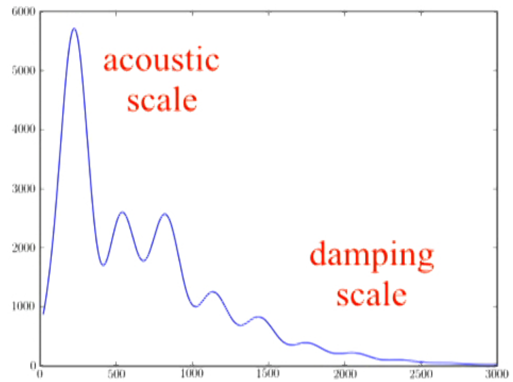


Calculated numerically!

A hand-waving explanation of the physics which gives rise to these two scales.

- Acoustic scale: arises because the universe is a plasma between the big bang (a=0) and CMB formation ($a=10^{-3}$).

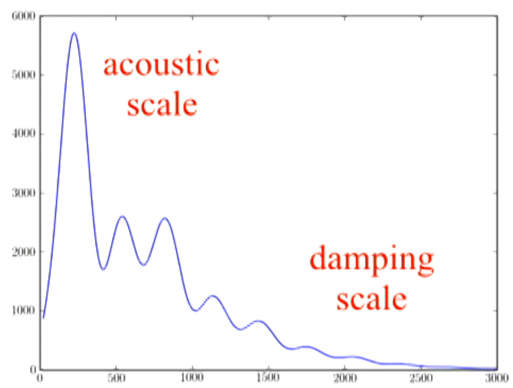
  In a plasma, the perturbations satisfy equations of motion which are similar to the "wave equation" example from lecture. As in that example, the effect is to imprint oscillatory features on the power spectrum.



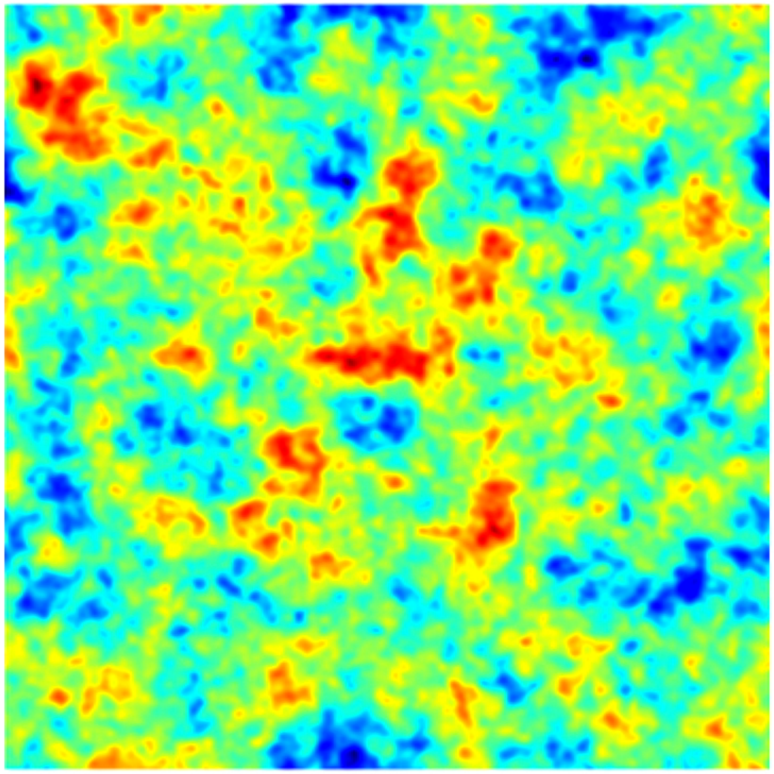
A hand-waving explanation of the physics which gives rise to these two scales.

- Damping scale: arises because photons have a nonzero mean free path (plasma is not "tightly coupled").

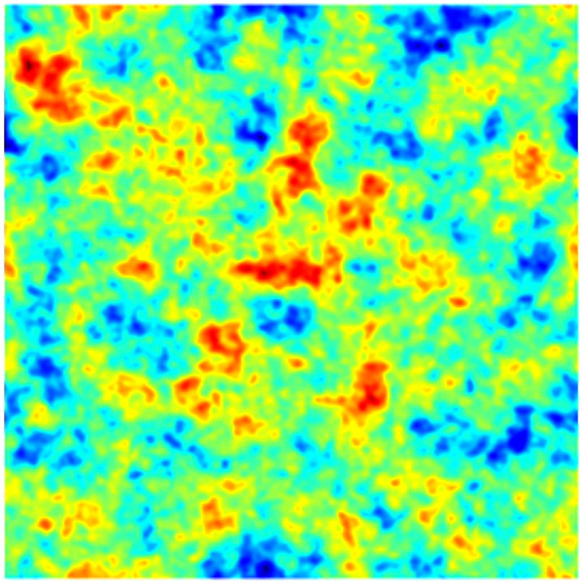
  This acts as a source of diffusion, "damping" or reducing fluctuations on small scales. Similar to the "boxcar filter" example from lecture!



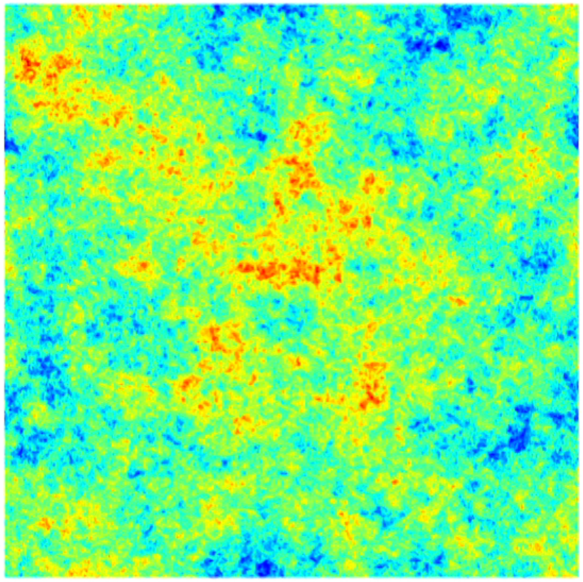
A Gaussian map with the CMB power spectrum.



Since the CMB is a Gaussian field (as far as we know), this is a perfect simulation of the true CMB!

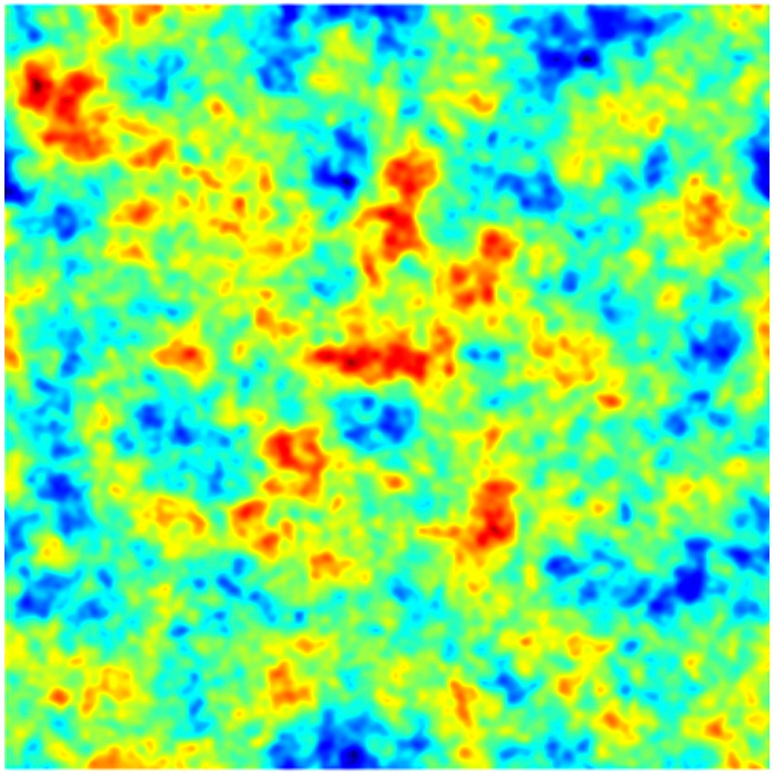


CMB power spectrum



Scale invariant power spectrum

A Gaussian map with the CMB power spectrum.

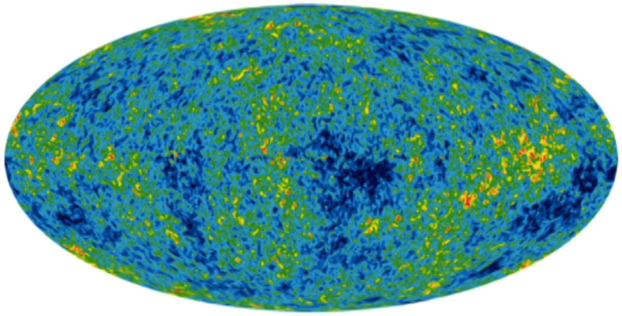


Since the CMB is a Gaussian field (as far as we know), this is a perfect simulation of the true CMB!

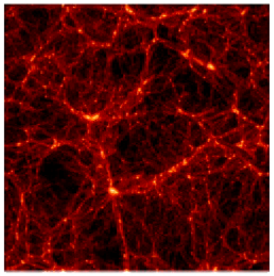
Slow-roll single-field inflation predicts that initial perturbations are small (~10^{-5}) and Gaussian (at ~10^{-7} level).

At early times, the perturbations are small enough that linear perturbation theory is a good approximation. This implies that perturbations are Gaussian at early times.

Perturbations grow via gravitational instability, and are order-one at late times. The perturbations start to evolve nonlinearly, and perturbations are non-Gaussian at late times.



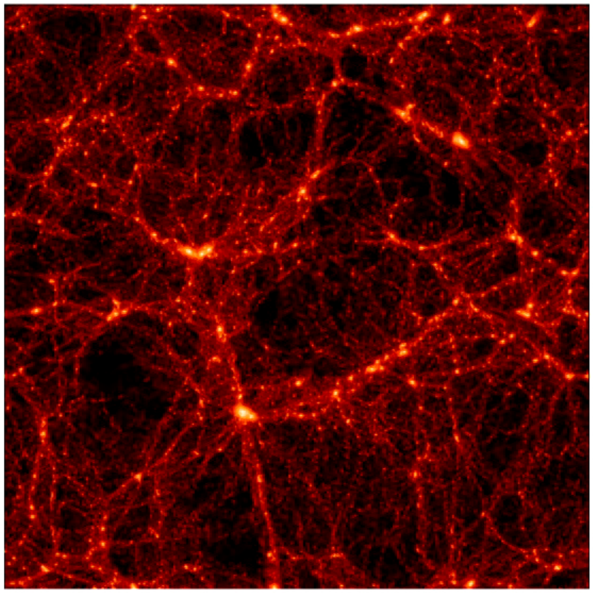
$z=10^3$ (Gaussian)



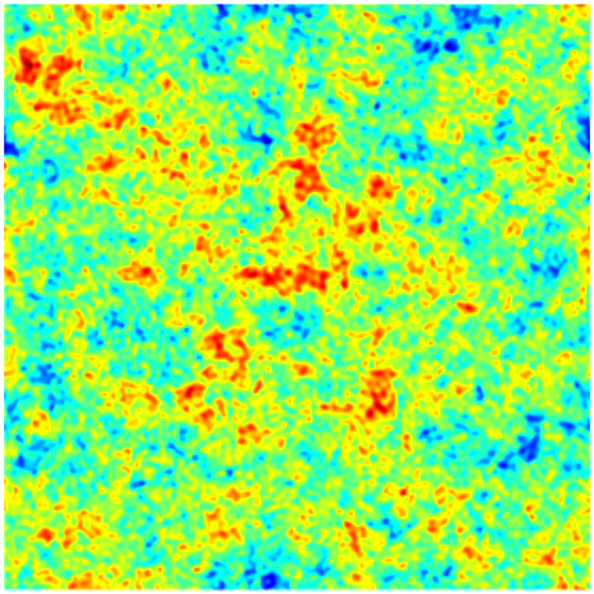
z=0 (non-Gaussian)

Reminder: for a Gaussian field, the statistics of the field are completely determined by the power spectrum P(k).

For a non-Gaussian field, this is not true!



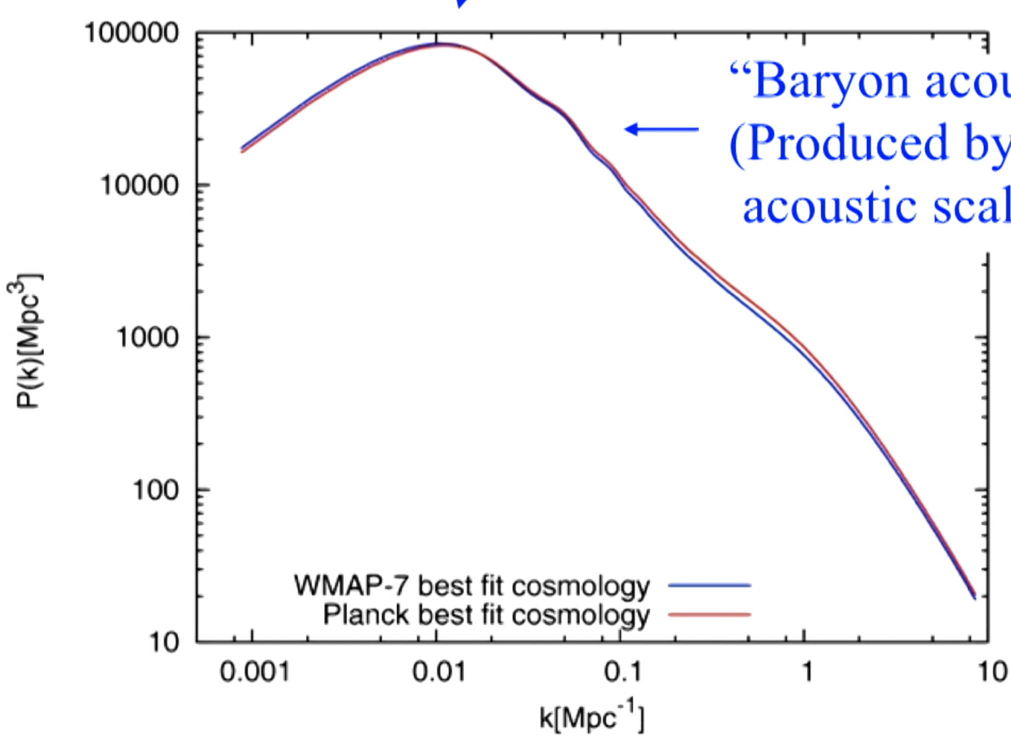
cosmological density
field at z=0



Gaussian field with
same power spectrum

3-d power spectrum P(k) of the density field at z=0

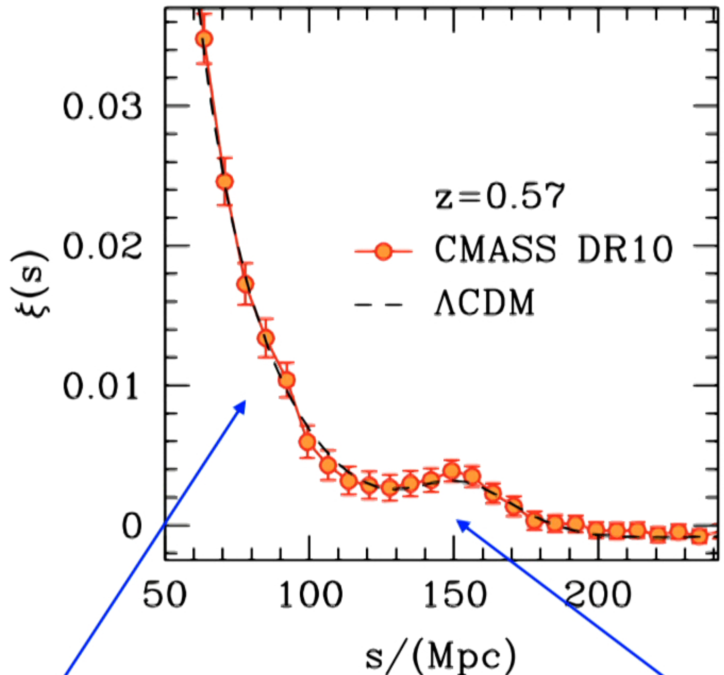
“Scale of matter-radiation equality”
k_{eq} = a H(a), evaluated at a=(3 x 10^{-4})
when $\rho_m(a) = \rho_{rad}(a)$.



“Baryon acoustic oscillations”
(Produced by same physics as
acoustic scale in the CMB).

Associated correlation function ζ(r)

(Taken from a random paper, ignore data points!)



Matter-radiation scale $k_{eq} \sim 0.02$ Mpc^{-1} shows up as "correlation length" $(k_{eq})^{-1} \sim 50$ Mpc in ζ(r)

Baryon acoustic oscillations show up as a single peak in ζ(r).