Title: PSI 2018/2019 - Explorations in Cosmology - Lecture 4

Speakers: Kendrick Smith

Collection: PSI 2018/2019 - Explorations in Cosmology (Smith)

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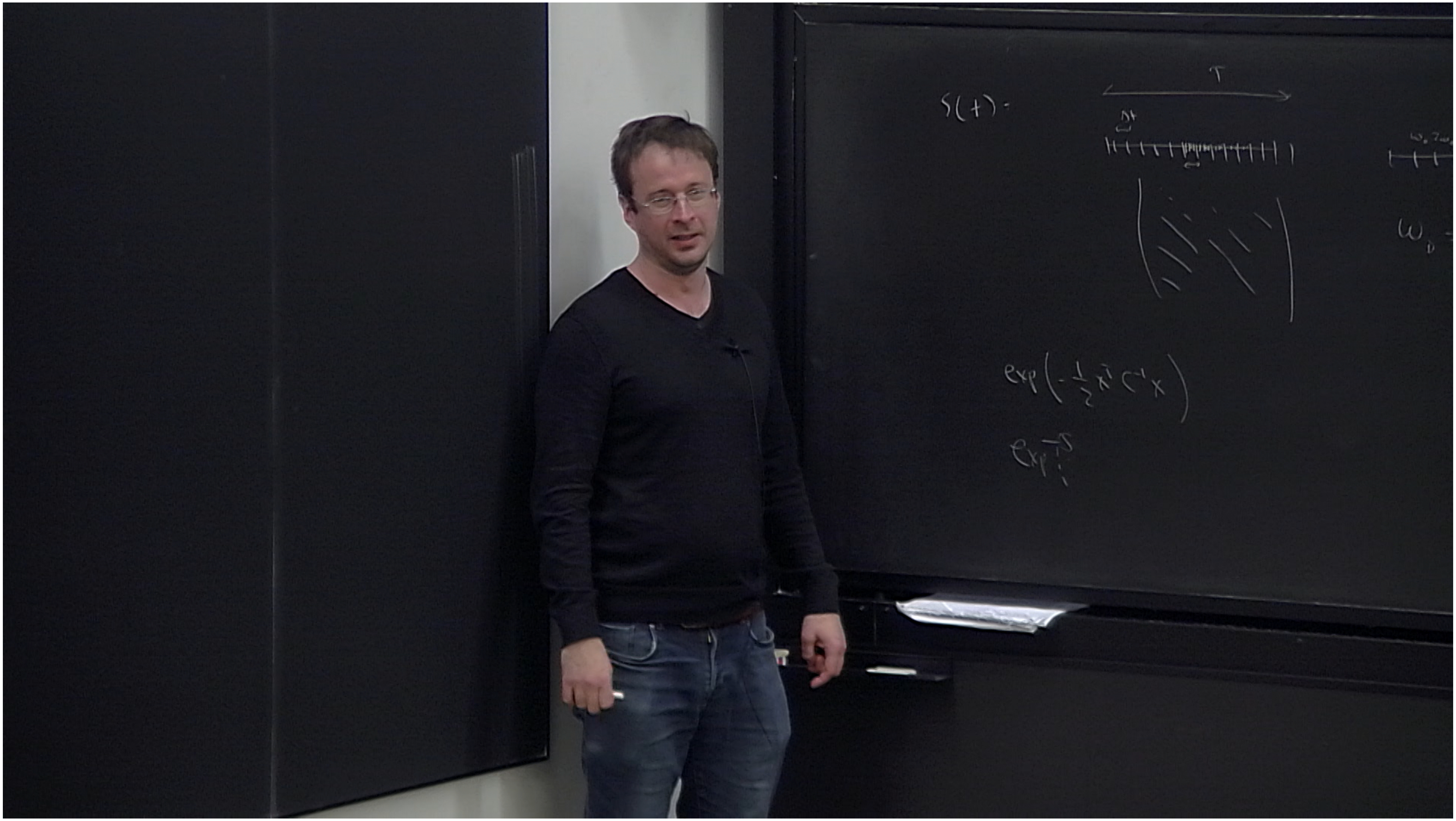
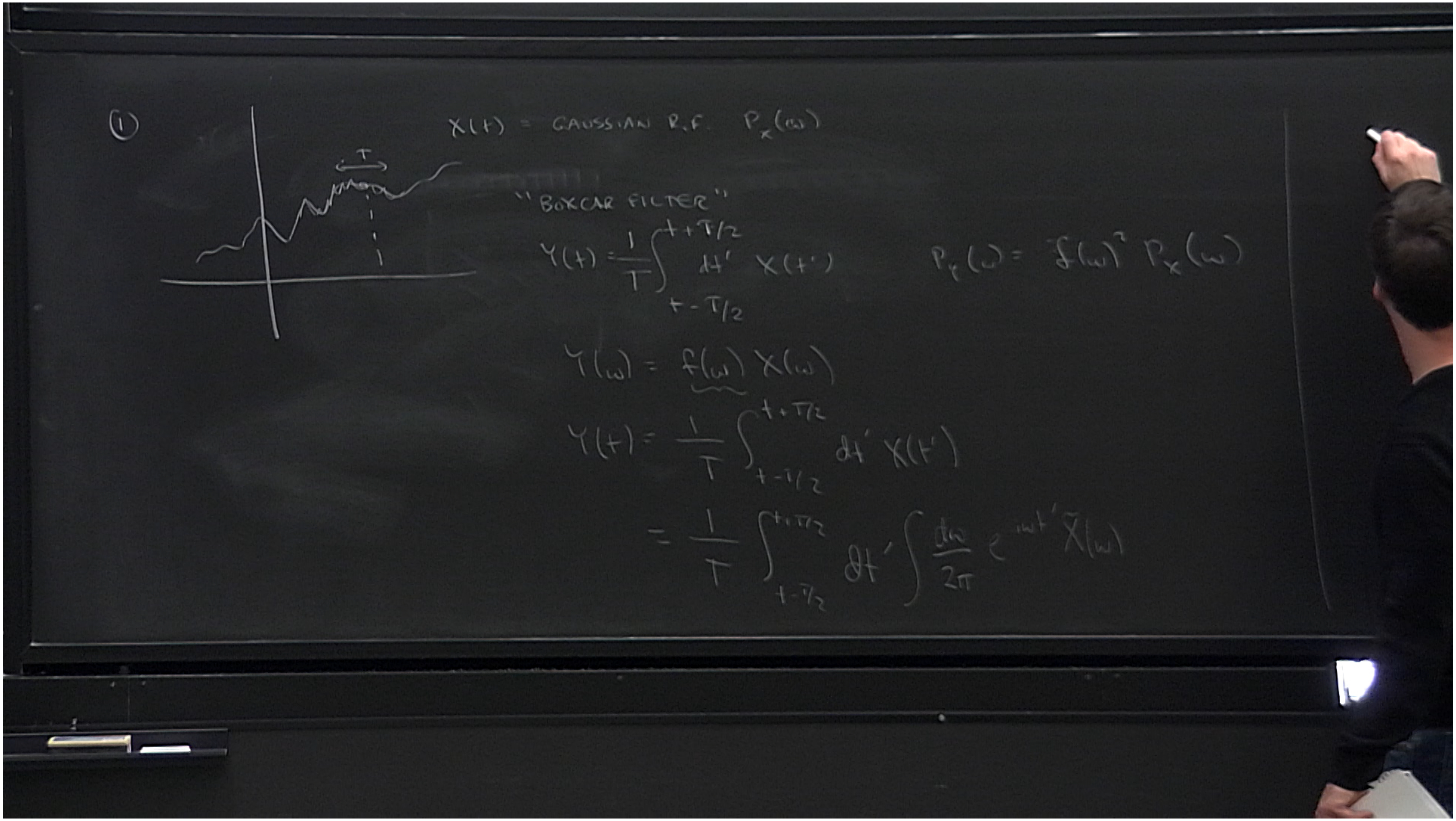
URL: http://pirsa.org/19040066


S(t) =
T
Δt
exp(- 1/2 x^T C^-1 x)


1
$X(t)$ = GAUSSIAN R.F. $P_x(\omega)$
T
"BOXCAR FILTER"
$Y(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} dt'\, X(t')$
$P_Y(\omega) = f(\omega)^2 P_x(\omega)$
$Y(\omega) = f(\omega)\, X(\omega)$
$Y(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} dt'\, X(t')$
$= \frac{1}{T}\int_{t-T/2}^{t+T/2} dt' \int \frac{d\omega}{2\pi} e^{-i\omega t'} \tilde{X}(\omega)$

$P_X(\omega)$

$$Y(t) = \frac{1}{T}\int \frac{d\omega}{2\pi} \, \frac{e^{i\omega(t+T/2)} - e^{-i\omega(t-T/2)}}{i\omega} \, X(\omega)$$

$$= \int \frac{d\omega}{2\pi} \, \underbrace{\frac{\sin(\omega T/2)}{\omega T/2} \, X(\omega)}_{= Y(\omega)} \, e^{i\omega t}$$

$$Y(\omega) = \left(\frac{\sin(\omega T/2)}{\omega T/2}\right) X(\omega)$$

$$P_Y(\omega) = \left[\frac{\sin(\omega T/2)}{\omega T/2}\right]^2 P_X(\omega)$$

(1+1)-DIM WAVE EQ:

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial x^2}\right) \phi(x,t) = 0$$

c_s = "SOUND SPEED"

WITH THE FOLLOWING INITIAL CONDITIONS.

$$\phi(x,0) = \phi_I(x)$$ GAUSSIAN RANDOM FIELD w/ P.S. $P_I(k)$

$$\partial_t (x,t)\Big|_{t=0} = 0$$

EVOLVE TO TIME T AND TAKE A "SNAPSHOT": $\phi_F(x) = \phi(x,T)$

WHAT IS THE POWER SPECTRUM $P_F(k)$?

$$\tilde{\phi}_t(k) = \int dx\, \phi(t,x)\, e^{-ikx}$$

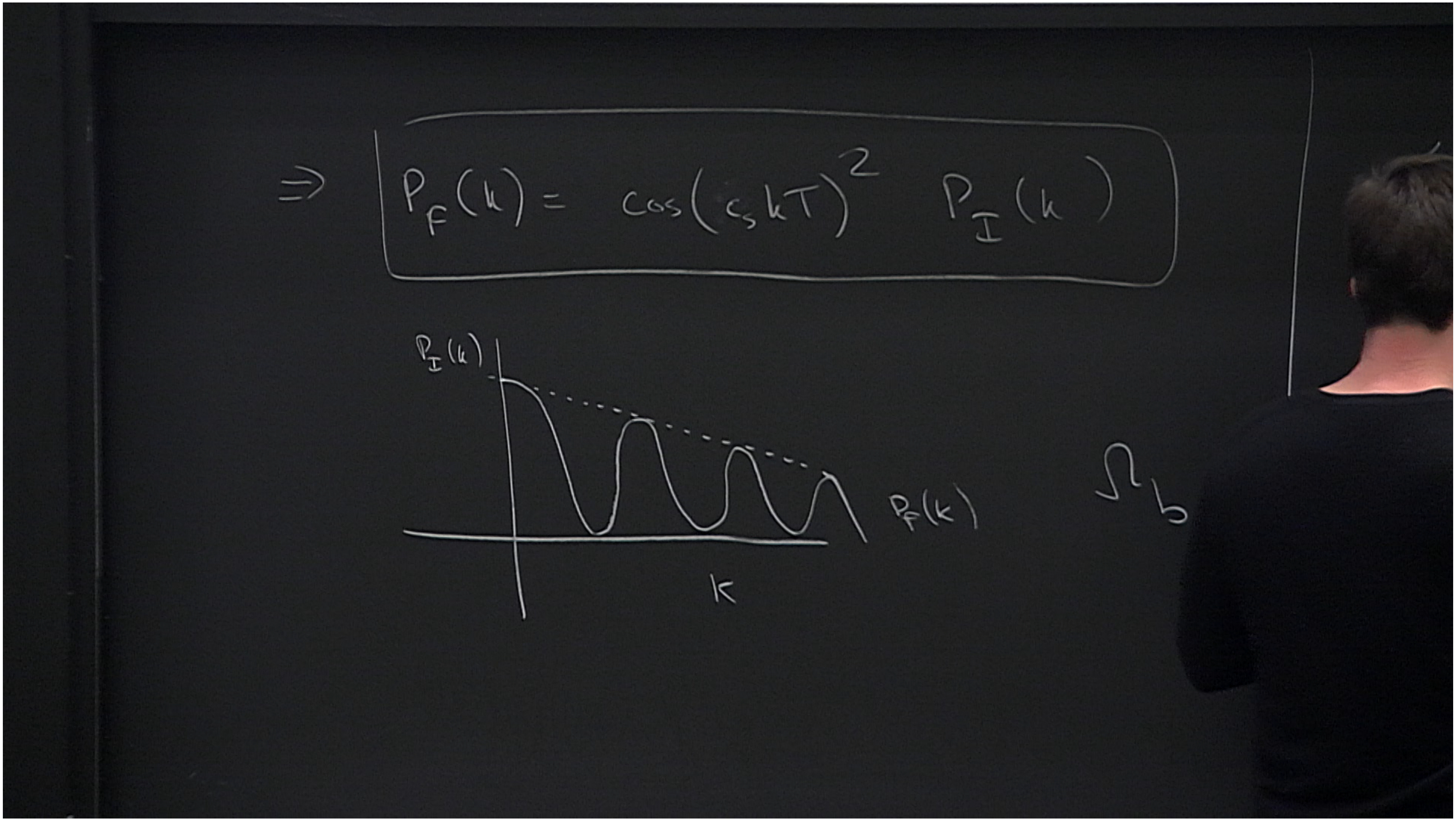
WAVE EQUATION BECOMES:

$$\left(\frac{\partial^2}{\partial t^2} + c_s^2 k^2\right)\phi_t(k) = 0$$

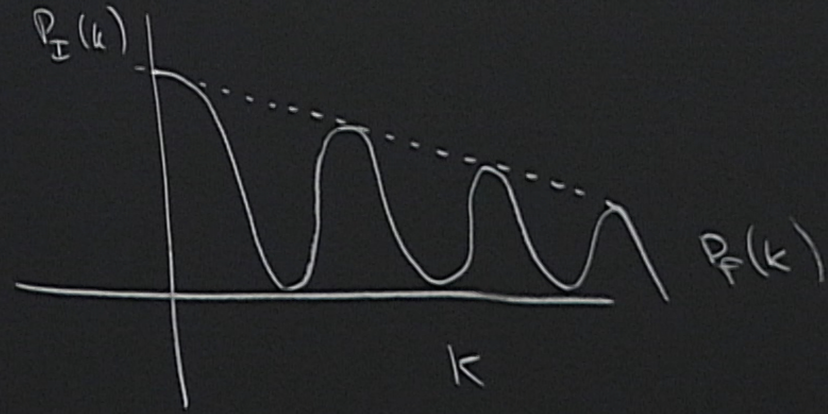
W/ INITIAL CONDITIONS: $\phi_0(k) = \phi_I(k)$ [GAUSSIAN RANDOM NUMBER]

$$\partial_t \phi_t(k)\big|_{t=0} = 0$$

$$\Rightarrow \boxed{\phi_t(k) = \phi_I(k)\cos(c_s k t)}$$


$\Rightarrow \boxed{P_F(k) = \cos(c_s k T)^2 \; P_I(k)}$
$P_I(k)$
$P_F(k)$
k
Ω_b

$$\Rightarrow \quad \boxed{P_F(k) = \cos(c_s k T)^2 \; P_I(k)}$$



Ω_b

$$\Rightarrow \boxed{\phi_t(k) = \phi_I(k)\cos(c_s k t)}$$

$e^{i\omega c_s t}$ $e^{-i\omega c_s t}$

$$\langle X(\omega)\, X(\omega')^* \rangle = P(\omega)\,(2\pi)\,\delta(\omega-\omega')$$

$$\Updownarrow$$

$$\langle X(\omega)\, X(\omega') \rangle = P(\omega)\,(2\pi)\,\delta(\omega+\omega')$$

$X(t)^* = X(t)$

$\tilde{X}(\omega)^* = \tilde{X}(-\omega)$

$$\langle \mathrm{Re}\,X(\omega)\ \mathrm{Re}\,X(\omega') \rangle = \left\langle \left(\frac{X(\omega) + X(\omega')^*}{2}\right)\left(\frac{X(\omega) + X(\omega')^*}{2}\right)\right\rangle = \frac{1}{2} P(\omega)\left[\delta(\omega-\omega') + \delta(\omega+\omega')\right]$$

$$\langle \mathrm{Re}\,X(\omega)\ \mathrm{Im}\,X(\omega') \rangle = 0$$

$$\langle \mathrm{Im}\,X(\omega)\ \mathrm{Im}\,X(\omega') \rangle = \frac{1}{2} P(\omega)\left[\delta(\omega-\omega') - \delta(\omega+\omega')\right]$$

$X(\omega_0)\ X(2\omega_0)\ \cdots$

$\mathrm{Re}(\omega_0),\ \mathrm{Im}(\omega_0),\ \mathrm{Re}(2\omega_0),\ \mathrm{Im}(2\omega_0),\ \cdots$

$$\begin{pmatrix} \frac{P(\omega_0)}{2} & 0 & & & \\ 0 & \frac{P(\omega_0)}{2} & & & \\ & & \frac{P(2\omega_0)}{2} & & \\ & & & \frac{P(2\omega_0)}{2} & \\ & & & & \ddots \end{pmatrix}$$

n-DIMENSIONAL EUCLIDEAN | 2-SPHERE

"FULL SYMMETRY"

TRANSLATIONS [n]

ROTATIONS [n(n-1)/2]

ROTATIONS [3]

REAL-SPACE FIELD $\phi(\vec{x})$ | $\phi(\hat{n})$ $\hat{n} = (n_x, n_y, n_z)$

CORRELATION FUNCTION $\langle \phi(\vec{x})\,\phi(\vec{x}')\rangle = \xi(|\vec{x}-\vec{x}'|)$ | $\langle \phi(\hat{n})\,\phi(\hat{n}')\rangle = \xi(\theta_{nn'})$ $\theta_{nn'} \overset{\text{def}}{=} \cos^{-1}(\hat{n}\cdot\hat{n}')$

FOURIER-SPACE REPRESENTATION $\phi(\vec{x}) = \int \frac{d^n k}{(2\pi)^n}\,\tilde{\phi}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$ | $\phi(\hat{n}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}\, Y_{\ell m}(\hat{n})$ $Y_{\ell m}(\hat{n}) = Y_{\ell m}(\theta,\phi)$

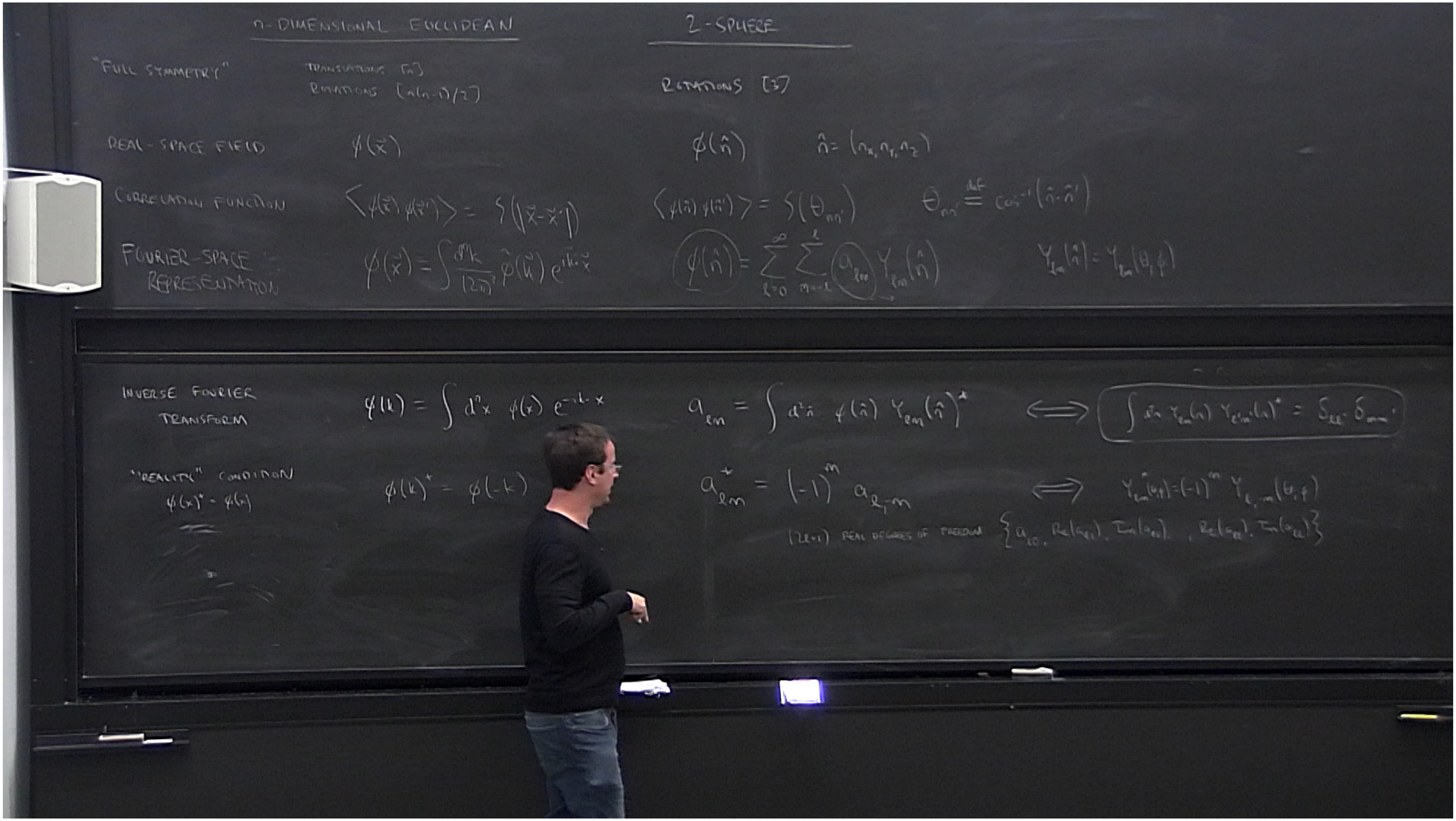
c_s = "SOUND SPEED"

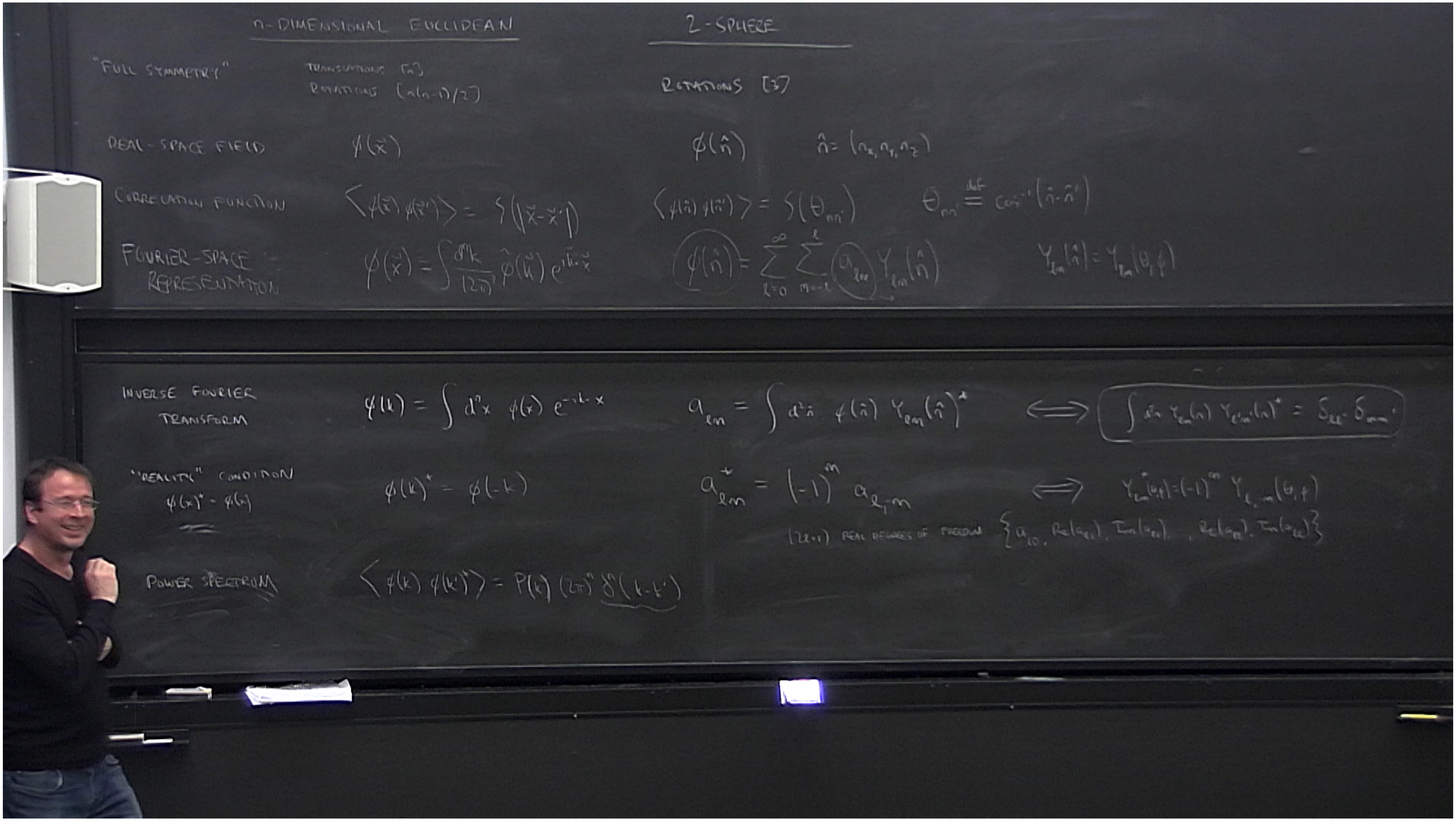
WITH THE FOLLOWING ... CONDITIONS

$\tilde{\phi}_t(k) = \int dx\ \phi(t,x)\, e^{-ikx}$

WAVE EQUATION BECOMES

$$\left(\frac{\partial^2}{\partial t^2} + c_s^2 k^2\right)\tilde{\phi}_t(k) = 0$$


n-DIMENSIONAL EUCLIDEAN
2-SPHERE
"FULL SYMMETRY"
TRANSLATIONS [n]
ROTATIONS [n(n-1)/2]
ROTATIONS [3]
REAL-SPACE FIELD
$\phi(\vec{x})$
$\phi(\hat{n})$
$\hat{n} = (n_x, n_y, n_z)$
CORRELATION FUNCTION
$\langle \phi(\vec{x}) \phi(\vec{x}') \rangle = \zeta(|\vec{x} - \vec{x}'|)$
$\langle \phi(\hat{n}) \phi(\hat{n}') \rangle = \zeta(\theta_{nn'})$
$\theta_{nn'} \stackrel{\text{def}}{=} \cos^{-1}(\hat{n} \cdot \hat{n}')$
FOURIER-SPACE REPRESENTATION
$\phi(\vec{x}) = \int \frac{d^n k}{(2\pi)^n} \tilde{\phi}(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$
$\phi(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{n})$
$Y_{\ell m}(\hat{n}) = Y_{\ell m}(\theta, \phi)$
INVERSE FOURIER TRANSFORM
$\phi(k) = \int d^n x \; \phi(x) \, e^{-ik\cdot x}$
$a_{\ell m} = \int d^2\hat{n} \; \phi(\hat{n}) \, Y_{\ell m}(\hat{n})^*$
$\Longleftrightarrow$
$\int d^2\hat{n} \; Y_{\ell m}(\hat{n}) \, Y_{\ell' m'}(\hat{n})^* = \delta_{\ell\ell'} \delta_{mm'}$
"REALITY" CONDITION
$\phi(x)^* = \phi(x)$
$\phi(k)^* = \phi(-k)$
$a_{\ell m}^* = (-1)^m a_{\ell, -m}$
$\Longleftrightarrow$
$Y_{\ell m}^*(\theta,\phi) = (-1)^m Y_{\ell,-m}(\theta,\phi)$
$(2\ell+1)$ REAL DEGREES OF FREEDOM $\{a_{\ell 0}, \mathrm{Re}(a_{\ell 1}), \mathrm{Im}(a_{\ell 1}), \ldots, \mathrm{Re}(a_{\ell\ell}), \mathrm{Im}(a_{\ell\ell})\}$


n-DIMENSIONAL EUCLIDEAN
2-SPHERE
"FULL SYMMETRY"
TRANSLATIONS [n]
ROTATIONS [n(n-1)/2]
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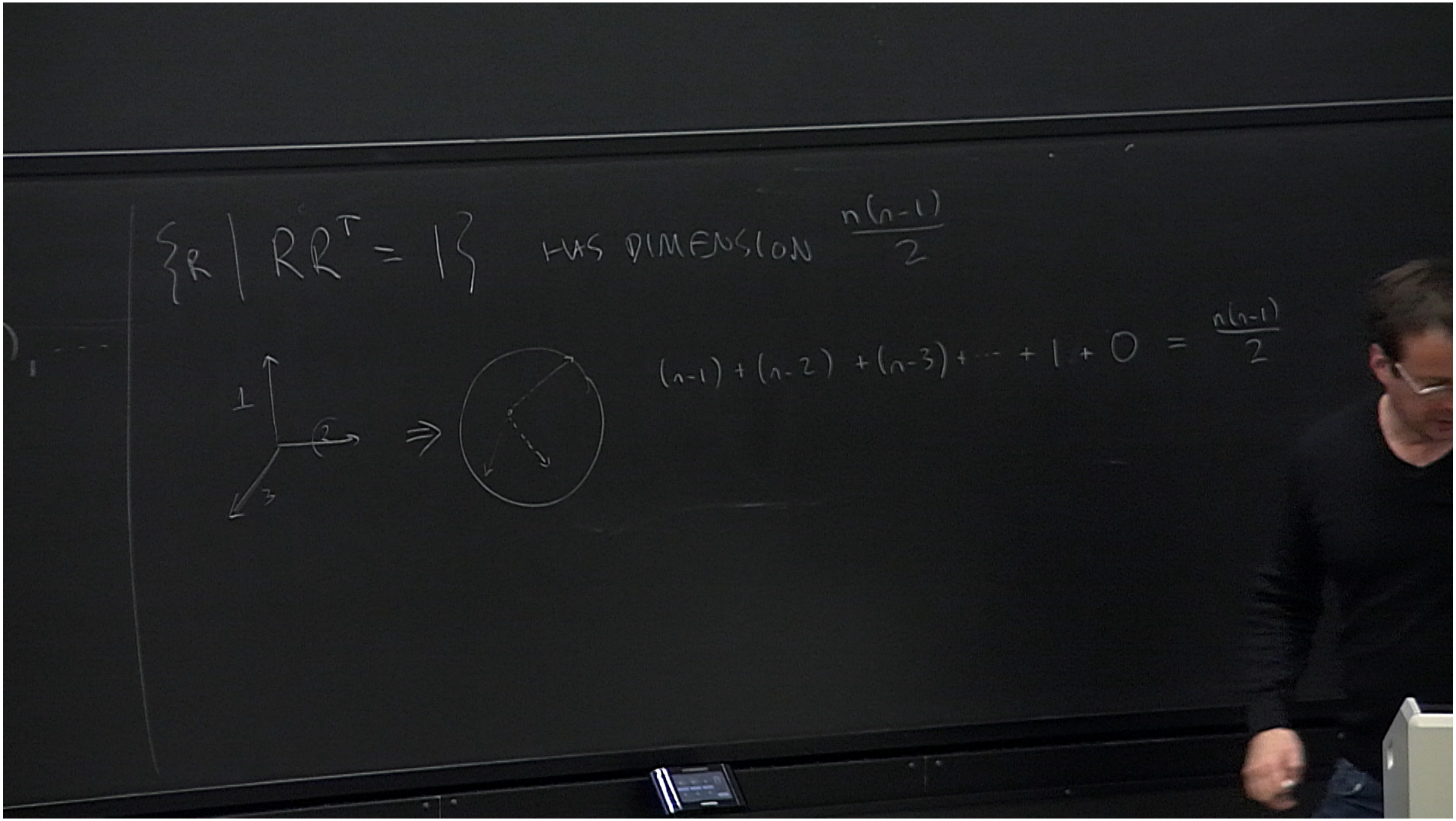
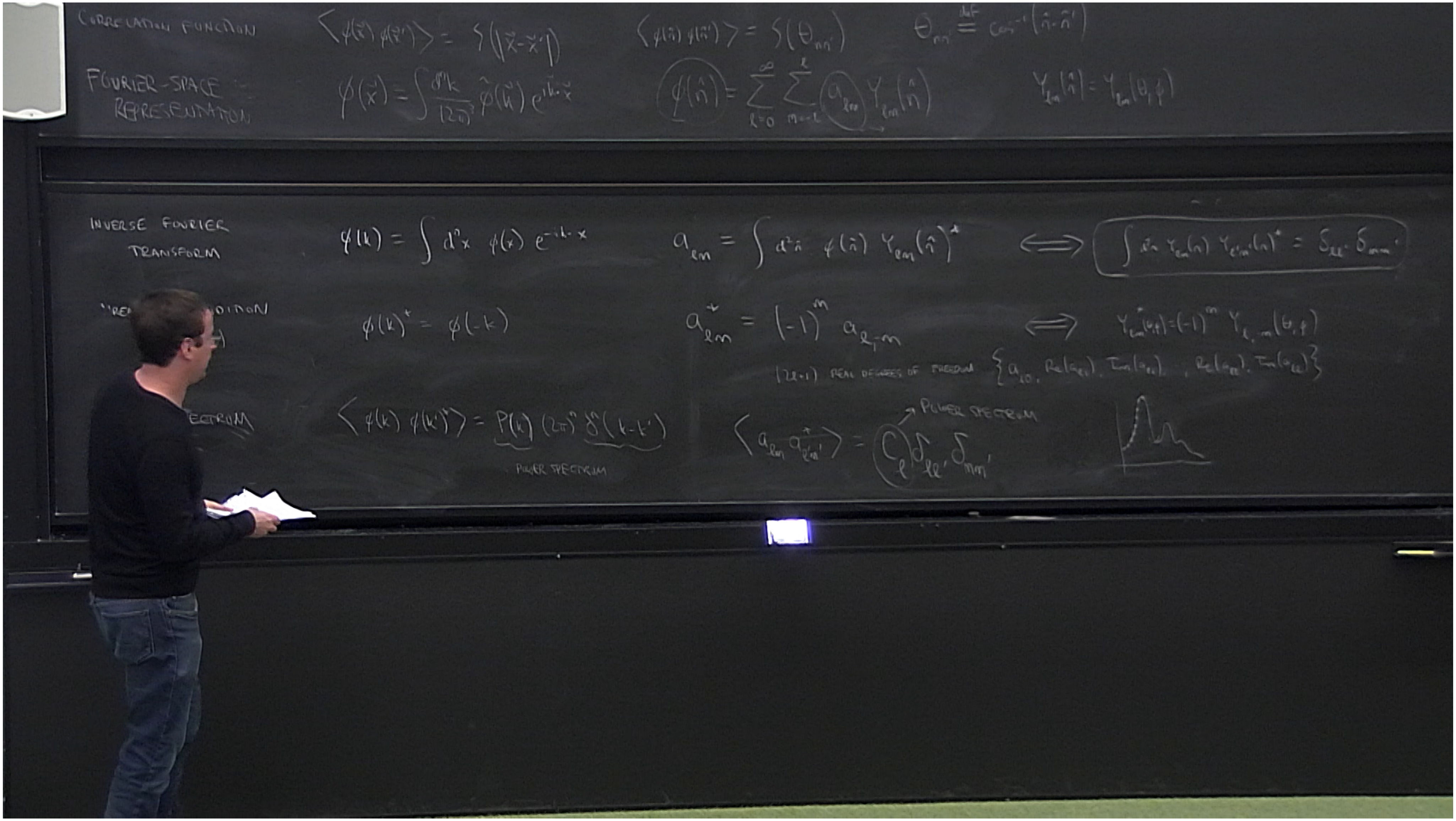
$\phi(\vec{x})$
$\phi(\hat{n})$
$\hat{n} = (n_x, n_y, n_z)$
CORRELATION FUNCTION
$\langle \phi(\vec{x}) \phi(\vec{x}') \rangle = \xi(|\vec{x} - \vec{x}'|)$
$\langle \phi(\hat{n}) \phi(\hat{n}') \rangle = \xi(\theta_{nn'})$
$\theta_{nn'} \overset{\text{def}}{=} \cos^{-1}(\hat{n} \cdot \hat{n}')$
FOURIER-SPACE REPRESENTATION
$\phi(\vec{x}) = \int \frac{d^n k}{(2\pi)^n} \tilde{\phi}(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$
$\phi(\hat{n}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\hat{n})$
$Y_{lm}(\hat{n}) = Y_{lm}(\theta, \phi)$
INVERSE FOURIER TRANSFORM
$\phi(k) = \int d^n x \; \phi(x) \, e^{-ik\cdot x}$
$a_{lm} = \int d^2\hat{n} \; \phi(\hat{n}) \, Y_{lm}(\hat{n})^*$
$\Longleftrightarrow \int d^2\hat{n} \; Y_{lm}(\hat{n}) Y_{l'm'}(\hat{n})^* = \delta_{ll'} \delta_{mm'}$
"REALITY" CONDITION
$\phi(x)^* = \phi(x)$
$\phi(k)^* = \phi(-k)$
$a_{lm}^* = (-1)^m a_{l,-m}$
$\Longleftrightarrow Y_{lm}^*(\theta,\phi) = (-1)^m Y_{l,-m}(\theta,\phi)$
(2l+1) REAL DEGREES OF FREEDOM $\{ a_{l0}, \mathrm{Re}(a_{l1}), \mathrm{Im}(a_{l1}), \ldots, \mathrm{Re}(a_{ll}), \mathrm{Im}(a_{ll}) \}$
POWER SPECTRUM
$\langle \phi(k) \phi(k')^* \rangle = P(k) (2\pi)^n \delta^n(k - k')$


{R | RR^T = 1} HAS DIMENSION n(n-1)/2
(n-1) + (n-2) + (n-3) + ... + 1 + 0 = n(n-1)/2
1
2
3


CORRELATION FUNCTION
$\langle \phi(\vec{x})\,\phi(\vec{x}')\rangle = \xi(|\vec{x}-\vec{x}'|)$
$\langle \phi(\hat{n})\,\phi(\hat{n}')\rangle = \xi(\theta_{nn'})$
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FOURIER-SPACE REPRESENTATION
$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\,\tilde{\phi}(\vec{k})\,e^{i\vec{k}\cdot\vec{x}}$
$\phi(\hat{n}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}\,Y_{\ell m}(\hat{n})$
$Y_{\ell m}(\hat{n}) = Y_{\ell m}(\theta,\phi)$
INVERSE FOURIER TRANSFORM
$\phi(k) = \int d^3x\;\phi(x)\,e^{-ik\cdot x}$
$a_{\ell m} = \int d^2\hat{n}\;\phi(\hat{n})\,Y_{\ell m}(\hat{n})^*$
$\Longleftrightarrow$
$\int d^2n\;Y_{\ell m}(n)\,Y_{\ell' m'}(n)^* = \delta_{\ell\ell'}\,\delta_{mm'}$
$\phi(k)^* = \phi(-k)$
$a^*_{\ell m} = (-1)^m\,a_{\ell,-m}$
$\Longleftrightarrow$
$Y^*_{\ell m}(\theta,\phi) = (-1)^m\,Y_{\ell,-m}(\theta,\phi)$
$(2\ell+1)$ REAL DEGREES OF FREEDOM $\{a_{\ell 0},\ \mathrm{Re}(a_{\ell 1}),\ \mathrm{Im}(a_{\ell 1}),\ \ldots,\ \mathrm{Re}(a_{\ell\ell}),\ \mathrm{Im}(a_{\ell\ell})\}$
ECTRUM
$\langle \phi(k)\,\phi(k')^*\rangle = P(k)\,(2\pi)^3\,\delta(k-k')$
POWER SPECTRUM
POWER SPECTRUM
$\langle a_{\ell m}\,a^*_{\ell' m'}\rangle = C_\ell\,\delta_{\ell\ell'}\,\delta_{mm'}$

INVERSE FOURIER TRANSFORM

$$\phi(k) = \int d^n x \; \phi(x)\, e^{-ik\cdot x}$$

$$a_{\ell m} =$$

"REALITY" CONDITION

$$\phi(x)^* = \phi(x)$$

$$\phi(k)^* = \phi(-k)$$

POWER SPECTRUM

$$\langle \phi(k)\, \phi(k')^* \rangle = \underbrace{P(k)}_{\text{POWER SPECTRUM}} (2\pi)^n \underbrace{\delta^n(k-k')}$$

$$\phi(\hat n) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}\, Y_{\ell m}(\hat n) \qquad Y_{\ell m}(\hat n) = Y_{\ell m}(\theta,\phi)$$

$$a_{\ell m} = \int d^2\hat n\; \phi(\hat n)\, Y_{\ell m}(\hat n)^* \quad \Longleftrightarrow \quad \boxed{\int d^2 n\; Y_{\ell m}(n)\, Y_{\ell' m'}(n)^* = \delta_{\ell\ell'}\,\delta_{mm'}}$$

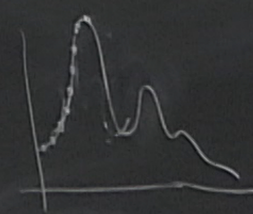
$$a^*_{\ell m} = (-1)^m\, a_{\ell,-m} \quad \Longleftrightarrow \quad Y^*_{\ell m}(\theta,\phi) = (-1)^m\, Y_{\ell,-m}(\theta,\phi)$$

$(2\ell+1)$ REAL DEGREES OF FREEDOM $\{a_{\ell 0},\ \mathrm{Re}(a_{\ell 1}),\ \mathrm{Im}(a_{\ell 1}),\ \dots,\ \mathrm{Re}(a_{\ell\ell}),\ \mathrm{Im}(a_{\ell\ell})\}$

$(k-k')$

POWER SPECTRUM

$$\langle a_{\ell m}\, a^*_{\ell' m'}\rangle = C_\ell\, \delta_{\ell\ell'}\,\delta_{mm'}$$



$X(\omega)\, X(\omega')^* \rangle = P(\omega)\,(2\pi)\,\delta(\omega-\omega')$