Title: PSI 2018/2019 - Explorations in Cosmology - Lecture 3

Speakers: Kendrick Smith

Collection: PSI 2018/2019 - Explorations in Cosmology (Smith)

Date: April 17, 2019 - 11:30 AM

URL: http://pirsa.org/19040065

$\det(C) \geq 0$

WEIGHT

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}$$

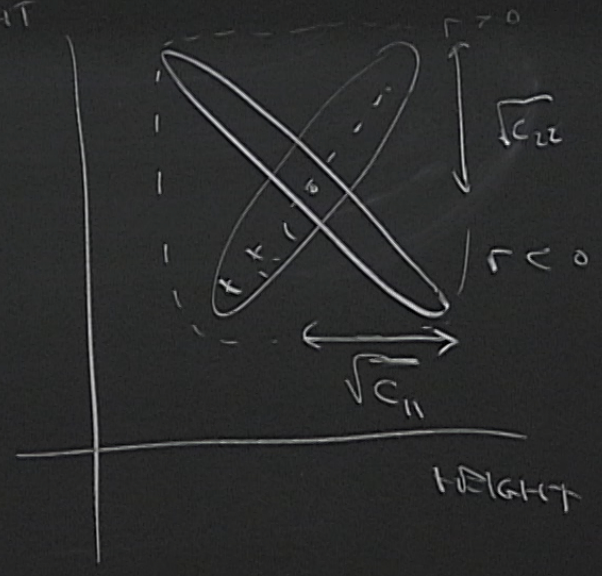
$C_{ii} = \mathrm{VAR}(X_i)$

$= \langle (X_i - \bar{X}_i)^2 \rangle$

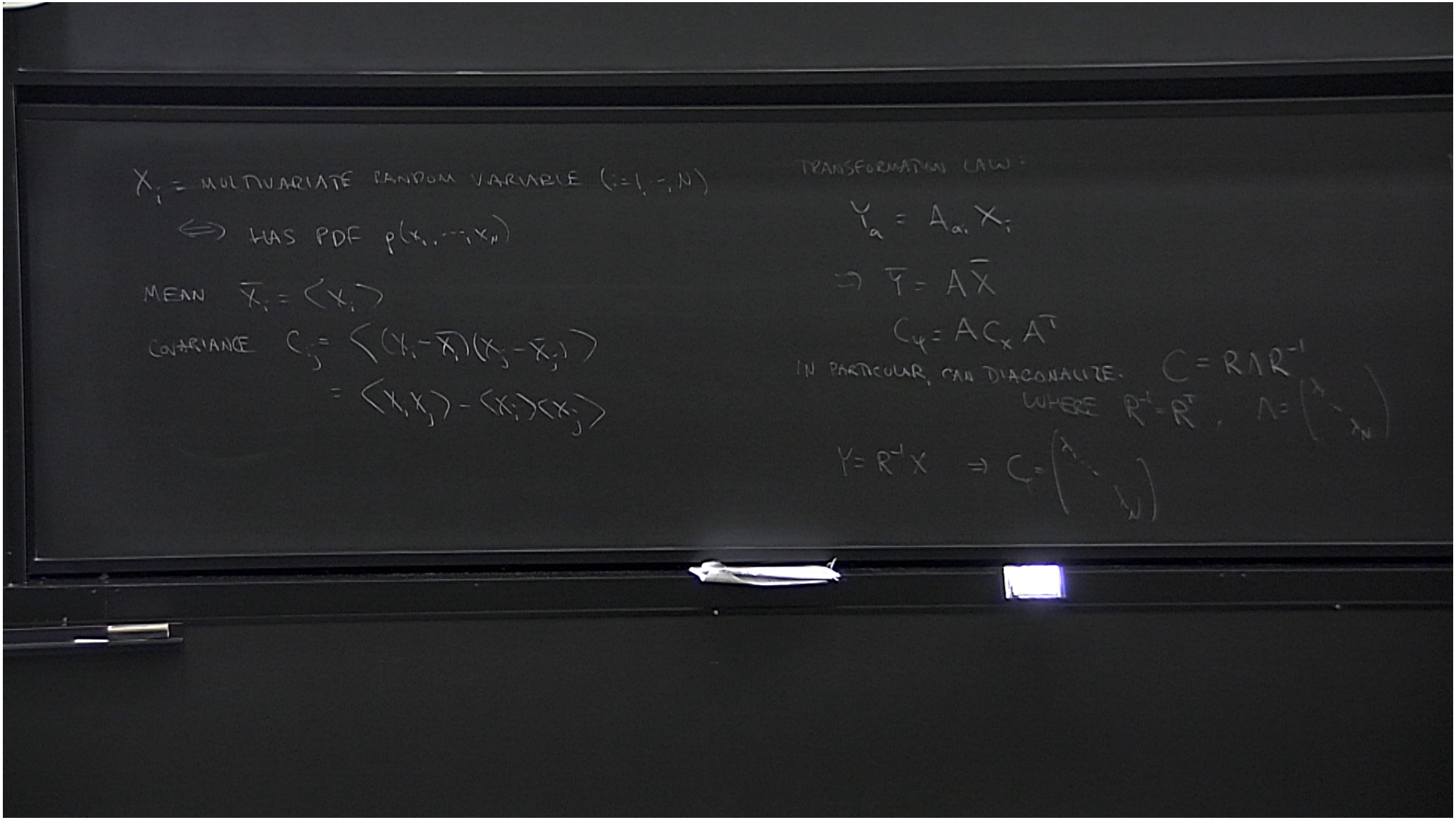
$$r_{ij} = \frac{C_{ij}}{\sqrt{C_{ii} C_{jj}}}$$

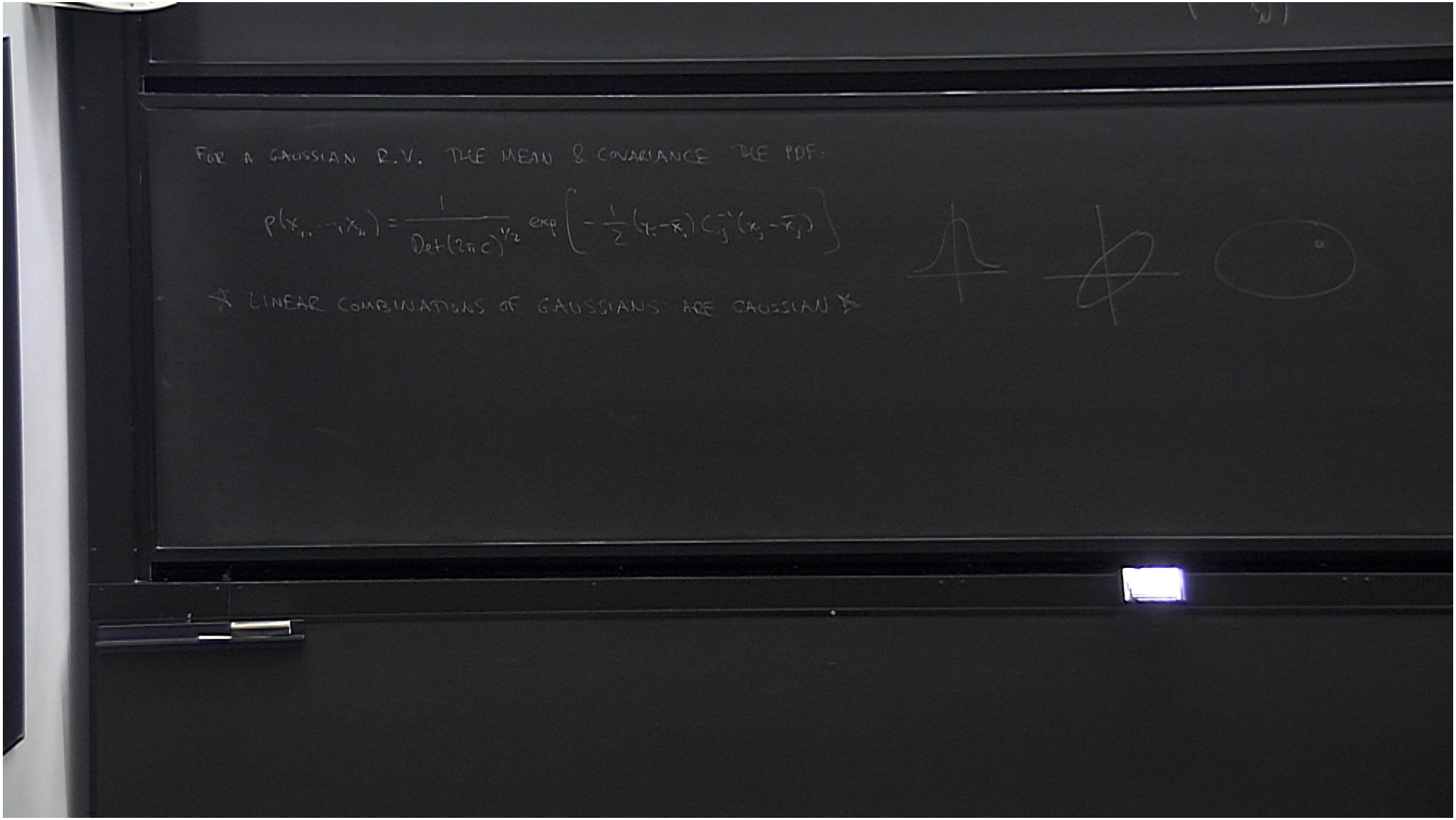
"CORRELATION COEFFICIENT"

$-1 \leq r \leq 1$

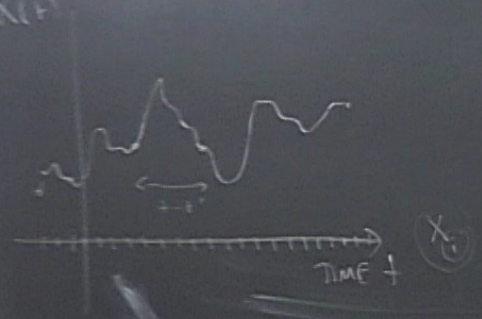


$r \to 1$


X_i = MULTIVARIATE RANDOM VARIABLE $(i = 1, \cdots, N)$
$\Leftrightarrow$ HAS PDF $p(x_1, \cdots, x_N)$
MEAN $\bar{X}_i = \langle X_i \rangle$
COVARIANCE $C_{ij} = \langle (X_i - \bar{X}_i)(X_j - \bar{X}_j) \rangle$
$= \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle$
TRANSFORMATION LAW:
$Y_a = A_{ai} X_i$
$\Rightarrow \bar{Y} = A\bar{X}$
$C_Y = A C_X A^T$
IN PARTICULAR, CAN DIAGONALIZE. $C = R \Lambda R^{-1}$
WHERE $R^{-1} = R^T$, $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$
$Y = R^{-1} X \Rightarrow C_Y = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$


FOR A GAUSSIAN R.V. THE MEAN & COVARIANCE THE PDF:
$p(x_1,\dots,x_n) = \frac{1}{Det(2\pi C)^{1/2}} \exp\left[-\frac{1}{2}(x_i-\bar{x}_i)C_{ij}^{-1}(x_j-\bar{x}_j)\right]$
★ LINEAR COMBINATIONS OF GAUSSIANS ARE GAUSSIAN ★

GAUSSIAN RANDOM FIELDS IN 1-D



TIME SYMMETRY ASSUMED

MULTIVARIATE RV X_i $\longleftrightarrow$ RANDOM FIELD $X(t)$

MEAN $\bar{X}_i$ $\longleftrightarrow$ $\bar{X}(t)$

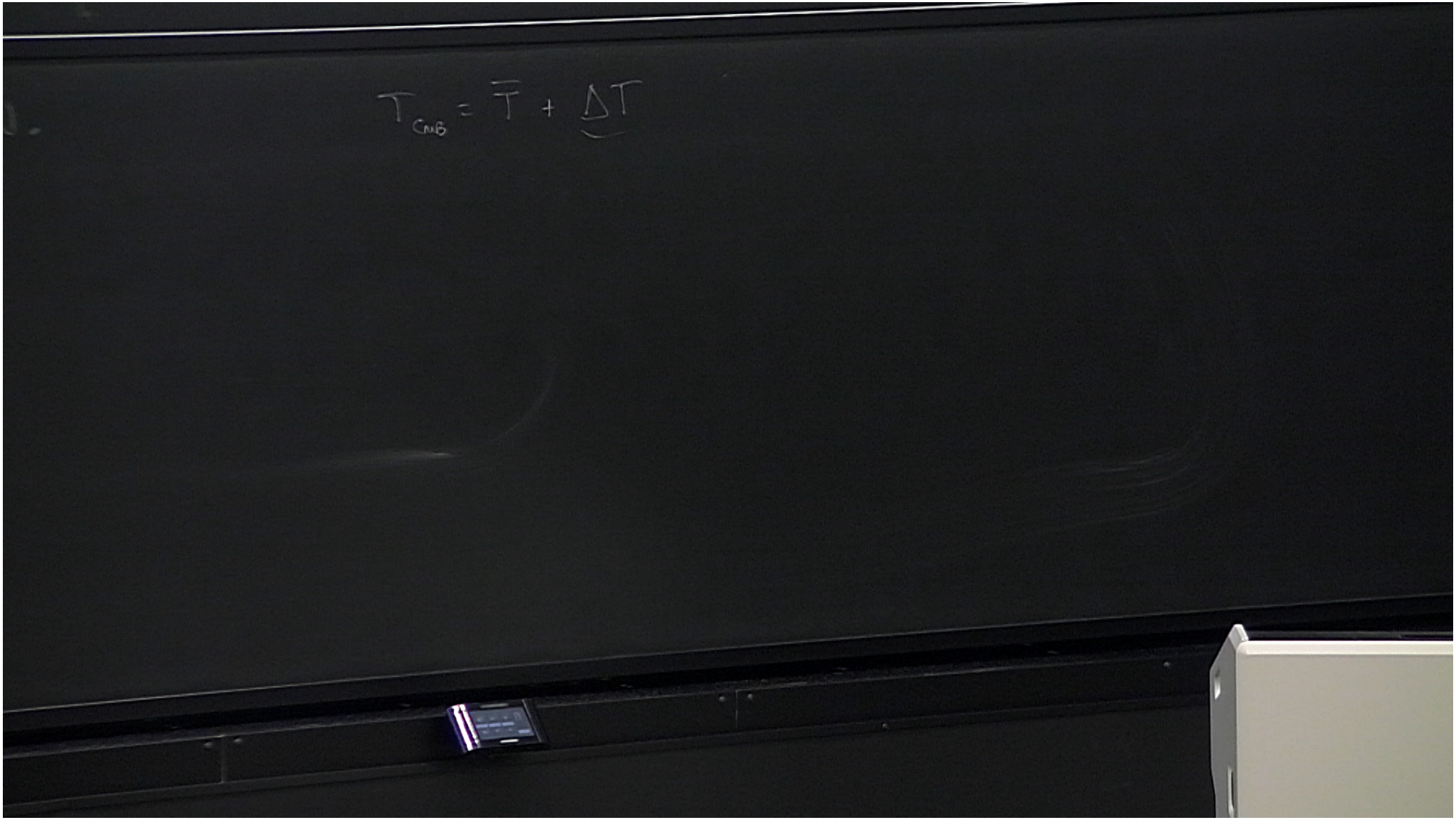
COVARIANCE C_{ij} $\longleftrightarrow$ TWO-POINT FUNCTION

$= \langle (X_i - \bar{X}_i)(X_j - \bar{X}_j) \rangle$ $\qquad C(t,t') = \langle (X(t) - \bar{X}(t))(X(t') - \bar{X}(t')) \rangle$

TIME TRANSLATION SYMMETRY ASSUMED

$\bar{X}(t) = \bar{X}$ [CONSTANT.]

$C(t,t') = S(t-t')$ → "CORRELATION FUNCTION"


$T_{CMB} = \bar{T} + \Delta T$

IN ADDITION, WE'LL ASSUME $\bar{X} = 0$ FROM NOW ON.

SO FROM NOW ON:

$$\langle X(t) \rangle = 0$$

$$\langle X(t)\, X(t') \rangle = \delta(t - t')$$

MY FOURIER CONVENTIONS:

$$X(t) = \int \frac{d\omega}{2\pi} \tilde{X}(\omega)\, e^{i\omega t}$$

$$\tilde{X}(\omega) = \int dt\, X(t)\, e^{-i\omega t}$$

FOURIE

NOW ON.

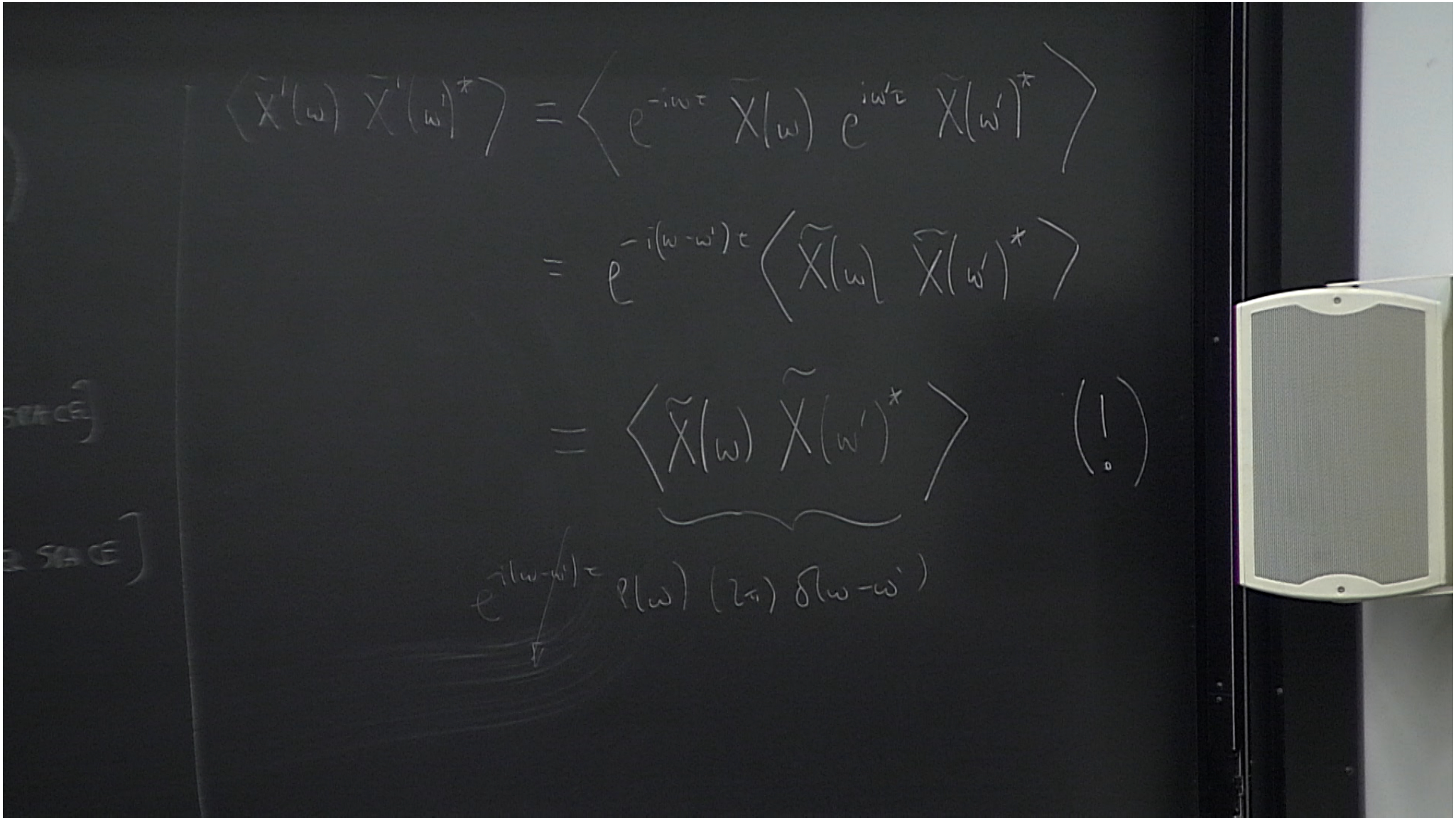
FOURIER-SPACE TWO-POINT FUNCTION:

$$\langle \tilde{X}(\omega)\, \tilde{X}(\omega')^* \rangle = \underbrace{P(\omega)}_{\text{"POWER SPECTRUM"}} (2\pi)\, \delta(\omega - \omega')$$

TIME TRANSLATION: $X'(t) = X(t+\tau)$ [REAL SPACE]

$\tilde{X}'(\omega) = e^{-i\omega\tau}\, \tilde{X}(\omega)$ [FOURIER SPACE]

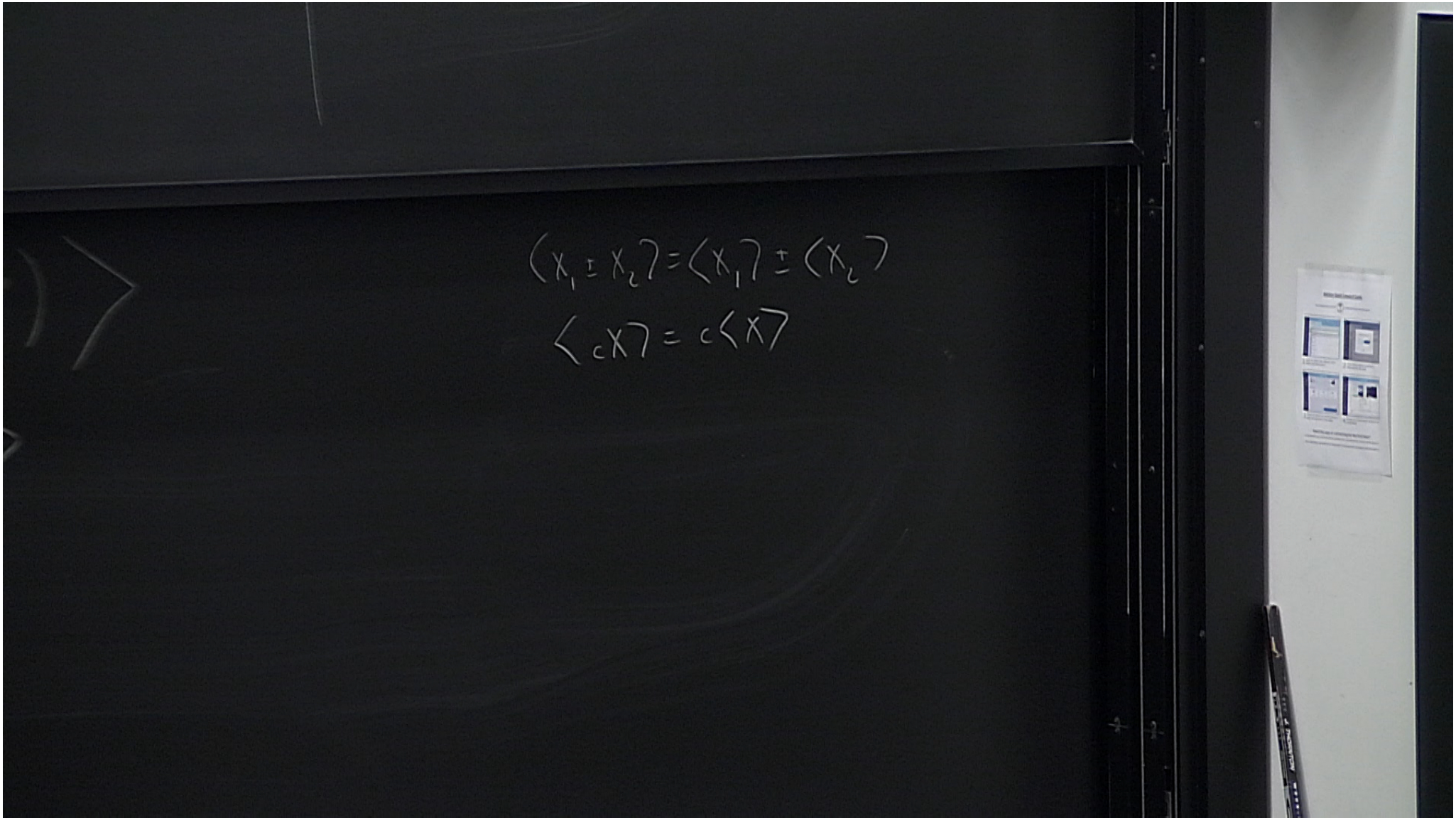
$\langle \tilde{X}'($


$\langle \tilde{X}'(\omega)\, \tilde{X}'(\omega')^* \rangle = \langle e^{-i\omega\tau}\, \tilde{X}(\omega)\, e^{i\omega'\tau}\, \tilde{X}(\omega')^* \rangle$
$= e^{-i(\omega-\omega')\tau} \langle \tilde{X}(\omega)\, \tilde{X}(\omega')^* \rangle$
$= \langle \tilde{X}(\omega)\, \tilde{X}(\omega')^* \rangle$ (!)
$e^{-i(\omega-\omega')\tau}\, P(\omega)\, (2\pi)\, \delta(\omega-\omega')$

TIME TRANSLATION SYMMETRY

$$\langle X(t)\, X(t')\rangle = S(t-t')$$

$$\langle X(\omega)\, X(\omega')^* \rangle = P(\omega)\,(2\pi)\,\delta(\omega-\omega')$$


$\langle X_1 \pm X_2 \rangle = \langle X_1 \rangle \pm \langle X_2 \rangle$
$\langle cX \rangle = c\langle X \rangle$

$$\langle X(\omega)\, X(\omega')^* \rangle = \left\langle \left(\int dt\, X(t)\, e^{-i\omega t} \right) \left(\int dt'\, X(t')\, e^{i\omega' t'} \right) \right\rangle$$

$$= \int dt\, dt'\, e^{-i\omega t + i\omega' t'} \left\langle X(t)\, X(t') \right\rangle$$

$$= \int dt\, dt'\, e^{-i\omega t + i\omega' t'} \xi(t - t') \qquad t = t' + \tau$$

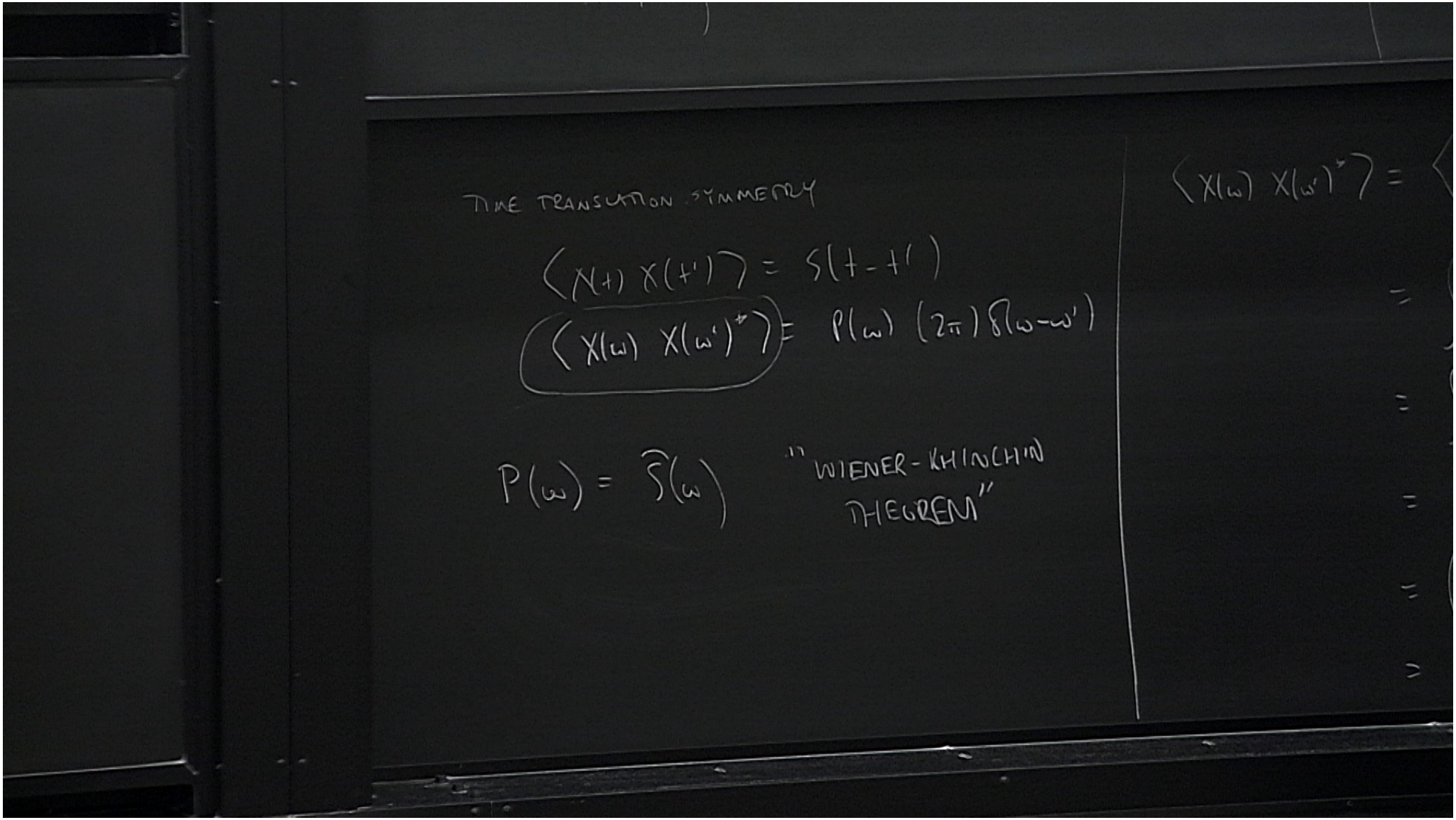
$$= \int d\tau\, dt'\, e^{-i\omega\tau} e^{-i(\omega - \omega')t'} \xi(\tau)$$

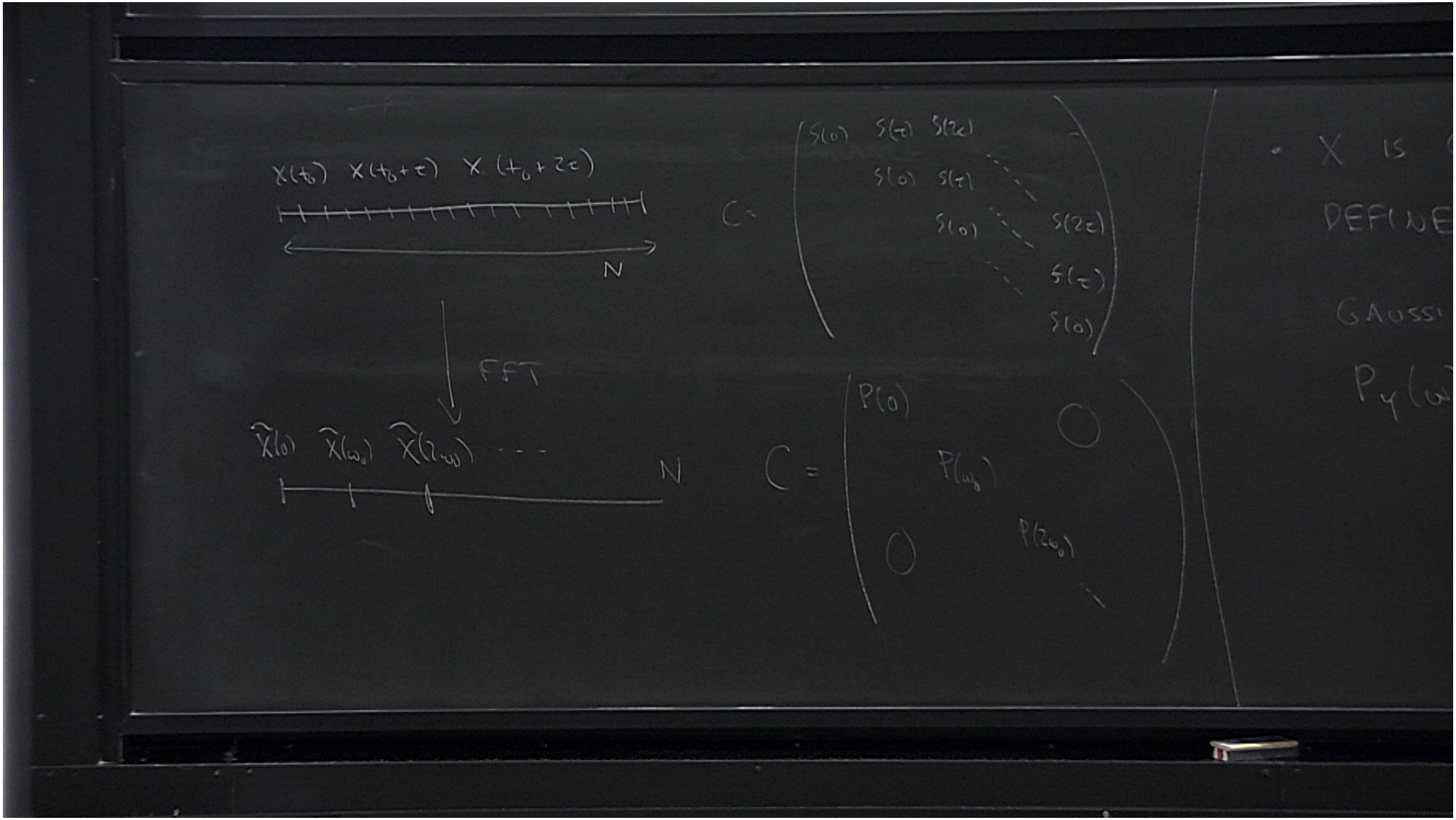
$$= \left(\int d\tau\, e^{-i\omega\tau} \xi(\tau) \right) \left(\int dt'\, e^{-i(\omega - \omega')t'} \right)$$

$$= \tilde{\xi}(\omega)\, (2\pi)\, \delta(\omega - \omega')$$

$\langle X_1 \pm X_2 \rangle = \langle$

$\langle cX \rangle =$


TIME TRANSLATION SYMMETRY
⟨X(t) X(t')⟩ = S(t-t')
⟨X(ω) X(ω')*⟩ = P(ω) (2π) δ(ω-ω')
P(ω) = S(ω)
"WIENER-KHINCHIN THEOREM"
⟨X(ω) X(ω')*⟩ =


$X(t_0)$ $X(t_0+\tau)$ $X(t_0+2\tau)$
N
FFT
$\hat{X}(0)$ $\hat{X}(\omega_0)$ $\hat{X}(2\omega_0)$ ...
N
$C =$
$S(0)$ $S(\tau)$ $S(2\tau)$
$S(0)$ $S(\tau)$
$S(0)$
$S(2\tau)$
$S(\tau)$
$S(0)$
$C =$
$P(0)$
$P(\omega_0)$
$P(2\omega_0)$
X IS
DEFINE
GAUSS
$P_Y(\omega$

- X is GRF w/ P.S. $P_X(\omega)$

DEFINE $Y(t) = \frac{dX}{dt} \iff \tilde{Y}(\omega) = i\omega \tilde{X}(\omega)$

GAUSSIAN? ✓

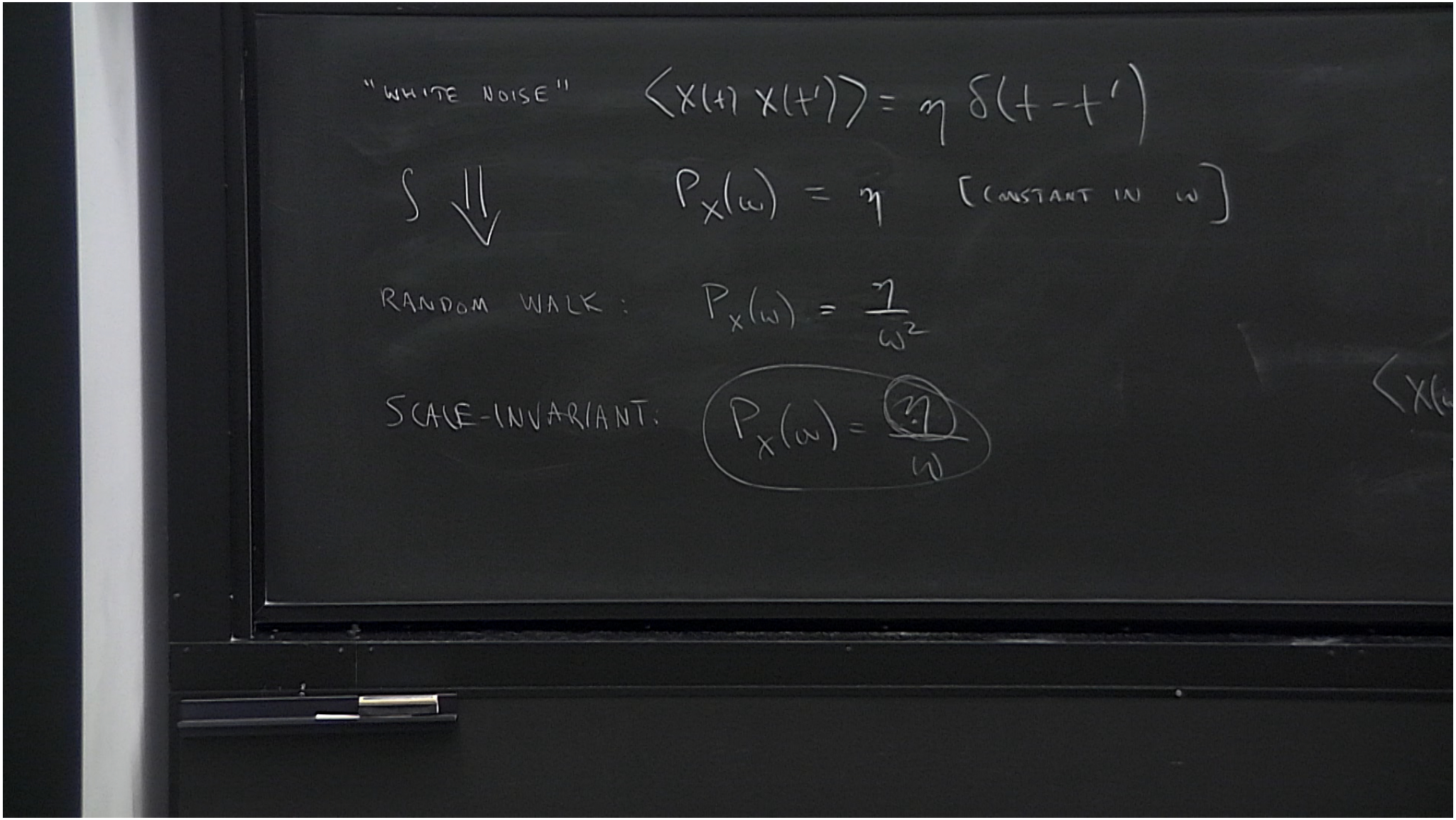
$P_Y(\omega) = \omega^2 P_X(\omega)$

$\langle Y(\omega) Y(\omega')^* \rangle = \langle (i\omega) \tilde{X}(\omega) (-i\omega') \tilde{X}(\omega')^* \rangle$

$= \omega\omega' \langle \tilde{X}(\omega) \tilde{X}(\omega')^* \rangle$

$= \omega\omega' P_X(\omega) (2\pi) \delta(\omega - \omega')$

$P_Y(\omega) = \omega^2 P_X(\omega)$


"WHITE NOISE" $\langle X(t)\,X(t')\rangle = \eta\,\delta(t-t')$
$P_x(\omega) = \eta$ [CONSTANT IN ω]
RANDOM WALK: $P_x(\omega) = \frac{\eta}{\omega^2}$
SCALE-INVARIANT: $P_x(\omega) = \frac{\eta}{\omega}$

$\delta(t-t')$

[CONSTANT IN ω]

$X(t)$

$\hat{X}(\omega) = \int dt\ X(t)\, e^{-i\omega t}$ $\qquad$ $[X]$ $\qquad$ $[X] \cdot [\text{TIME}]$

$P_X(\omega)$ .. HAS UNITS $[X^2] \cdot [\text{TIME}]$

$$\langle \hat{X}(\omega)\, \hat{X}(\omega')^* \rangle = \underbrace{P(\omega)}\ \underbrace{(2\pi)\,\delta(\omega-\omega')}$$

$[X]^2\ [\text{TIME}]^2$ $\qquad$ $[X^2][\text{TIME}]$ $\qquad$ $[\text{TIME}]$