Title: 3d Quantum Gravity: from tetrahedra to holography

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Abstract: 3d quantum gravity is a beautiful toy-model for 4d quantum gravity: it is much simpler, it does not have local degrees of freedom, yet retains enough complexity and subtlety to provide a non-trivial example of dynamical quantum geometry and open new directions of research in physics and mathematics. I will present the Ponzano-Regge model, introduced in 1968, built from tetrahedra â€œquantized" as 6j-symbols from the theory of recoupling of spins. I will show how it provides a discrete path integral for 3d quantum gravity, related to topological invariants and loop quantum gravity and other approaches to quantum gravity. It is also a perfect arena to investigate boundary theories and holographic dualities, with a beautiful duality with the 2d Ising model realized through a supersymmetry, and more.

The Ponzano-Regge model :
from tetrahedra to holography in 3d quantum gravity

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Perimeter Institute - March '19

Why 3d Quantum Gravity ?

The main reason :

Quantum Gravity

An excellent toy model for quantum gravity :

- most of the conceptual issues but a much simpler problem
- no gravitational wave, no graviton, no local degree of freedom
- a **topological field theory** of the BF type

↳ solvable theory, equivalent to Chern-Simons theory — Witten '89

↳ amplitudes depend entirely on topology & boundary state

↳ admits exact discretization at classical & quantum levels

Why 3d Quantum Gravity ?

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- a **topological field theory** of the BF type

Not only a toy model... with applications in :

- in symmetry-reduced situation (BHs, ...)

  & in condensed matter (topological phases, topological defects)

Why 3d Quantum Gravity ?

And on top of working in 3 dimensions,

further consider Euclidean signature $(+++)$

But still a quantum theory !

We study the path integral : $\int [\mathrm{d}g^{(3d)}]\, e^{iS_{\mathrm{grav}}[g]}$

Not « Euclidean gravity »

The Ponzano-Regge model for 3d Quantum Gravity

- Describe a « quantized 3d metric »
- Define the Ponzano-Regge model for 3d triangulation
- Discrete path integral for 3d gravity ?
- Topological Invariance

- What does the PR amplitude look like?
  - ↳ e.g. with topological defects defining particles

The Ponzano-Regge model for 3d Quantum Gravity

- Describe a « quantized 3d metric »
- Define the Ponzano-Regge model for 3d triangulation
- Discrete path integral for 3d gravity ?
- Topological Invariance

- What does the PR amplitude look like?
- Boundary theories & (quasi-local) Holography
  - ↪ a duality with the 2d Ising model on a 2-sphere boundary
  - ↪ towards AdS/CFT results on the 2-torus boundary

Building blocks of 3d quantized geometry

Ingredients for the Ponzano-Regge model :

- Consider a 3d triangulation, with edges, triangles and tetrahedra
- Dress them with data on their quantized geometry
- Build an amplitude from this data
- Define the Ponzano-Regge path integral for a 3d triangulation by summing over all possible data

Building blocks of 3d quantized geometry

Ingredients for the Ponzano-Regge model :

- Consider a 3d triangulation, with edges, triangles and tetrahedra

edge = classically a vector $\vec{X}_e \in \mathbb{R}^3$ transform under the group of 3d rotations

Building blocks of 3d quantized geometry

Ingredients for the Ponzano-Regge model :

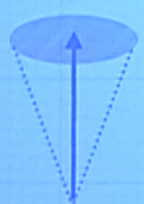
- Consider a 3d triangulation, with edges, triangles and tetrahedra

edge = classically a vector $\vec{X}_e \in \mathbb{R}^3$ transform under the group of 3d rotations

edge = quantized as state in an irreducible representation of $\mathrm{SU}(2)$

$|j, m\rangle \in \mathcal{V}^j$ **spin** $j \in \mathbb{N}/2$ $-j \leq m \leq +j$ $\dim \mathcal{V}^j = 2j+1$

$C = \vec{J}^2 = j(j+1)$



- a quantized vector with length $L_e = \sqrt{\langle \vec{J} \rangle^2} = j_e$

  and uncertainty $\delta = \sqrt{\langle \vec{J}^2 \rangle - \langle \vec{J} \rangle^2} = \sqrt{j}$

- can define coherent states as $g\,|j, j\rangle$ with $g \in \mathrm{SU}(2)$

Building blocks of 3d quantized geometry

Ingredients for the Ponzano-Regge model :

- Consider a 3d triangulation, with edges, triangles and tetrahedra

triangle = three vectors summing to 0

$$\vec{X}_1 + \vec{X}_2 + \vec{X}_3 = 0 \longrightarrow \vec{J}_1 + \vec{J}_2 + \vec{J}_3 = 0$$

$$\psi^{j_1,j_2,j_3} \in \mathrm{Inv}_{\mathrm{SU}(2)}\left[\mathcal{V}^{j_1} \otimes \mathcal{V}^{j_2} \otimes \mathcal{V}^{j_3}\right]$$

triangle = singlet in the tensor product of three spins

quantized triangle up to 3d rotations

unique state ψ^{j_1,j_2,j_3} if spins satisfy triangular inequalities $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$

coefficients given by 3j-symbols (Clebsh-Gordan coefficients)

$$\langle \{j_i, m_i\}_{i=1,2,3} | \psi^{j_1,j_2,j_3} \rangle = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$m_1 + m_2 + m_3 = 0$$

Building blocks of 3d quantized geometry

Ingredients for the Ponzano-Regge model :

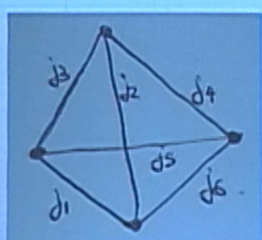
- Consider a 3d triangulation, with edges, triangles and tetrahedra

tetrahedron = 6 edges, 4 triangles

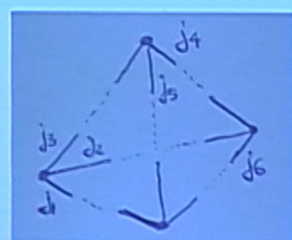
fully determined up to 3d rotations by the 6 edge lengths

$L_1, L_2, L_3, L_4, L_5, L_6$

quantum tetrahedron = 6 spins, 4 singlet states (3j-symbols)



dual



4 triangles :

(j_1, j_2, j_3)

(j_1, j_5, j_6)

(j_4, j_2, j_6)

(j_4, j_5, j_3)

Building blocks of 3d quantized geometry

Ingredients for the Ponzano-Regge model :

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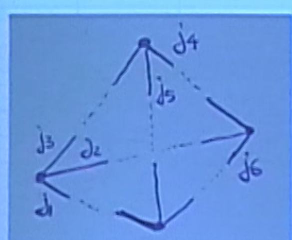

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \sum_{\{m_i\}} (-1)^{\sum_i j_i - m_i} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ -m_1 & m_5 & -m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ -m_4 & -m_2 & m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & -m_5 & -m_3 \end{pmatrix}$$

6j-symbol

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Building the Ponzano-Regge amplitudes

Ingredients for the Ponzano-Regge model

- Consider a 3d triangulation, with edges, triangles and tetrahedra
- Dress them with data on their quantized geometry

edge ⟶ **spin** $j_e \in \mathbb{N}/2$

triangle ⟶ **singlet state** given by 3j-symbol $\begin{pmatrix} j_{e_{t(1)}} & j_{e_{t(2)}} & j_{e_{t(3)}} \\ m_{e_{t(1)}} & m_{e_{t(2)}} & m_{e_{t(3)}} \end{pmatrix}$

tetrahedron ⟶ **trace over tensor of 4 singlet states**

given by 6j-symbol $\begin{Bmatrix} j_1^T & j_2^T & j_3^T \\ j_4^T & j_5^T & j_6^T \end{Bmatrix}$

Building the Ponzano-Regge amplitudes

Ingredients for the Ponzano-Regge model

- Consider a 3d triangulation, with edges, triangles and tetrahedra
- Dress them with data on their quantized geometry
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- Define the Ponzano-Regge path integral for a 3d triangulation by summing over all possible data

$$\mathcal{A}_\Delta[\{j_e\}] = \prod_{T\in\Delta} (-1)^{j_1^T+j_2^T+j_3^T+j_4^T+j_5^T+j_6^T}\{6j\}_T$$

Ponzano & Regge '68

$$\mathcal{Z}_\Delta = \sum_{\{j_e\}} \prod_e (-1)^{2j_e}(2j_e+1)\,\mathcal{A}_\Delta[\{j_e\}] \longrightarrow \mathcal{Z}_\Delta[j_{\partial\Delta}]$$

for fixed spins on the boundary

Dressing geometry with algebra

Standard method in mathematical physics

to **probe geometry** of (discrete) manifolds

by **dressing** geometrical elements **with algebraic data**

↳ knot invariants, (topological) invariants, ...

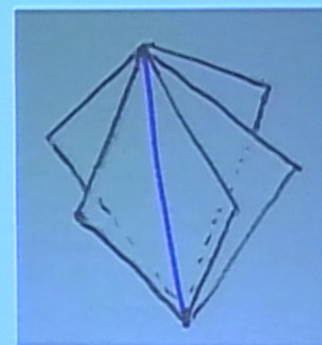
Ponzano-Regge as quantum Regge calculus

And is it related at all with quantum gravity ?

Discretizing 3d Gravity : Regge calculus

Regge calculus = discretization of gravity in terms of edge lengths

$$S_{\text{Regge}}[\{l\}] = \sum_{e \in \Delta}^{\text{T}} l_e \phi_e[\{l\}]$$



$$\phi_e = \sum_{t \ni e} \theta_e^t$$ deficit angle around edge

eqn of motion $\phi_e^{(classical)} = 0$ **is discrete equivalent of Ricci-flat**

in arbitrary dim, deficit angle around cells of co-dim 2

Ponzano-Regge as quantum Regge calculus

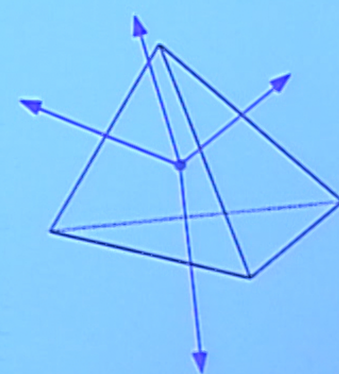
And is it related at all with quantum gravity ?

Asymptotics of 6j-symbols for large spins i.e. in semi-classical limit :

$$\{6j\} \underset{j_e \gg 1}{\sim} \frac{1}{\sqrt{V[j_e]}} \cos\left(S_{\text{Regge}}[j_e] + \frac{\pi}{4}\right)$$

in terms of Regge action for the tetrahedron :

$$S_{\text{Regge}}[j_e] = \sum_{e=1}^{6} j_e \theta_e[\{j\}]$$



with $\theta_e[\{j\}]$ (external) dihedral angles

Discretizing 3d Gravity : Regge calculus

Regge calculus = discretization of gravity in terms of edge lengths

$$\mathcal{A}_\Delta[\{j_e\}] \underset{j\gg 1}{\sim} \sum_{\epsilon_T=\pm} e^{\sum_T \epsilon_T \sum_e j_e \theta_e[\{j\}]}$$

↳ **Ponzano-Regge model = path integral for quantized Regge calculus**

Illustrates general feature of quantum gravity :

Two layers of quantization: → **quantized geometry** + **path integral**

implies two types of quantum corrections to classical theory

Ponzano-Regge: Tunelling through signature change

A very interesting subtlety ...

6 lengths satisfying the correct triangular inequalities do not ensure the existence of a classical tetrahedron

Cayley-Menger determinant $$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & j_4^2 & j_5^2 & j_6^2 \\ 1 & j_4^2 & 0 & j_3^2 & j_2^2 \\ 1 & j_5^2 & j_3^2 & 0 & j_1^1 \\ 1 & j_6^2 & j_2^2 & j_1^2 & 0 \end{vmatrix}$$

- If $V^2 > 0$ then usual oscillating asymptotics in $e^{iS_{\mathrm{Regge}}[j]}$
- If $V^2 < 0$ then new evanescent asymptotics in $e^{-S_{\mathrm{Regge}}^{\mathrm{Lorentzian}}[j]}$

→ **Ponzano-Regge contains exp-suppressed Lorenztian configurations**

Topological Invariance of Ponzano-Regge

But is there more quantum gravity in Ponzano-Regge than a semi-classical limit ?

↓

- Topological invariant
- Realizes the exact quantization of 3d gravity as a BF theory

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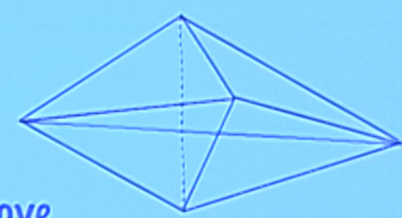
Topological Invariance of Ponzano-Regge

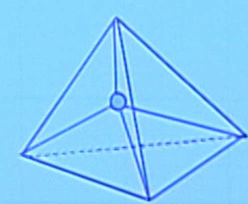
Topological invariance as invariance under Pachner moves :

- 2-3 Pachner move



add or remove
an edge

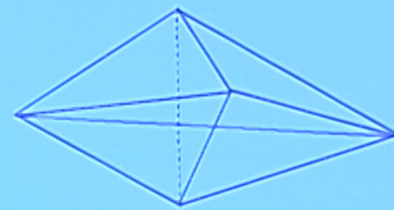
- 1-4 Pachner move



add or remove
a vertex

Topological Invariance of Ponzano-Regge

- **2-3 Pachner move**



This is the **Biedenharn-Elliot identity**

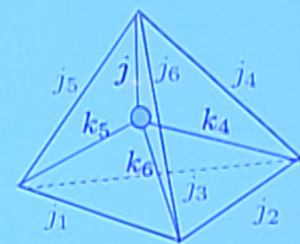
or pentagonal identity
for the associator of
a monoidal category

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ k_4 & k_5 & k_6 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_4 & l_5 & l_6 \end{Bmatrix} = \sum_j (-1)^{j+\sum_i j_i+k_i+l_i} \begin{Bmatrix} k_4 & l_4 & j \\ l_5 & k_5 & j_3 \end{Bmatrix}\begin{Bmatrix} k_5 & l_5 & j \\ l_6 & k_6 & j_1 \end{Bmatrix}\begin{Bmatrix} k_6 & l_6 & j \\ l_4 & k_4 & j_2 \end{Bmatrix}$$

↪ leads to 2nd order recursion relation for 6j-symbol
(useful for numerics & asymptotics)

Topological Invariance of Ponzano-Regge

- **1-4 Pachner move**



« Tent move »

Derived by action of holonomy operator (Wilson loop) on 6j-symbol or by using orthogonality of 6j-symbol on Biedenharn-Elliot

$$(2j+1)\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \sum_{k_4,k_5,k_6} (-1)^{j+\sum_i j_i+k_i}(2k_4+1)(2k_5+1)(2k_6+1)$$

$$\begin{Bmatrix} k_5 & j_5 & j \\ j_4 & k_4 & j_3 \end{Bmatrix}\begin{Bmatrix} k_6 & j_6 & j \\ j_5 & k_5 & j_1 \end{Bmatrix}\begin{Bmatrix} k_4 & j_4 & j \\ j_6 & k_6 & j_2 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ k_4 & k_5 & k_6 \end{Bmatrix}$$

↳ new point refining tetrahedra is inside or « outside »

↳ Here, one of the new spins is fixed ...

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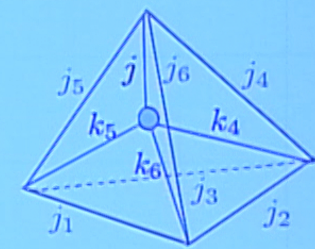
Topological Invariance of Ponzano-Regge

- **1-4 Pachner move**



$$(2j+1)\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \sum_{k_4,k_5,k_6} (-1)^{j+\sum_i j_i+k_i}(2k_4+1)(2k_5+1)(2k_6+1)$$
$$\begin{Bmatrix} k_5 & j_5 & j \\ j_4 & k_4 & j_3 \end{Bmatrix}\begin{Bmatrix} k_6 & j_6 & j \\ j_5 & k_5 & j_1 \end{Bmatrix}\begin{Bmatrix} k_4 & j_4 & j \\ j_6 & k_6 & j_2 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ k_4 & k_5 & k_6 \end{Bmatrix}$$

↪ Here, one of the new spins is fixed … **Leads to divergences !!**

If we compute PR partition function for triangulation with a vertex inside :

$$\sum_{j,k_4,k_5,k_6} (-1)^{j+\sum_i j_i+k_i}(2j+1)(2k_4+1)(2k_5+1)(2k_6+1)$$
$$\begin{Bmatrix} k_5 & j_5 & j \\ j_4 & k_4 & j_3 \end{Bmatrix}\begin{Bmatrix} k_6 & j_6 & j \\ j_5 & k_5 & j_1 \end{Bmatrix}\begin{Bmatrix} k_4 & j_4 & j \\ j_6 & k_6 & j_2 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ k_4 & k_5 & k_6 \end{Bmatrix} = \underbrace{\left(\sum_j (2j+1)^2\right)}_{\text{Infinite factor}}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

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Topological Invariance of Ponzano-Regge

The PR amplitude diverges as soon as there is a vertex in the bulk !

but we can « gauge-fix » by collapsing triangulation (or fixing spins) along maximal tree ...?

So :
- What symmetry are we gauge-fixing ?
- How to get a well-defined model ?
- What topological invariant does it define ?

↳ Ponzano-Regge as discretization of topological BF theory

3d quantum gravity as a theory of flat connections

Let's take a step back ...

3d gravity as a topological BF theory:

$$S[A,e] = \int_{\mathcal{M}} \mathrm{Tr}\, e \wedge F[A] = \int_{\mathcal{M}} \delta_{ij}\epsilon^{abc}\, e^i_a\, F^j_{bc}[A]$$

- Triad e 1-form with value in $\mathfrak{su}(2)$ Lie algebra $\qquad g_{ab} = e^i_a e^i_b$
- $\mathrm{SU}(2)$ Connection A with curvature $F[A] = dA + A \wedge A$

↳ TQFT with no local degree of freedom

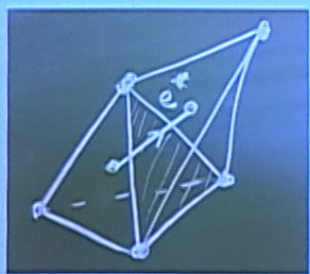
↳ Path integral over flat connections :

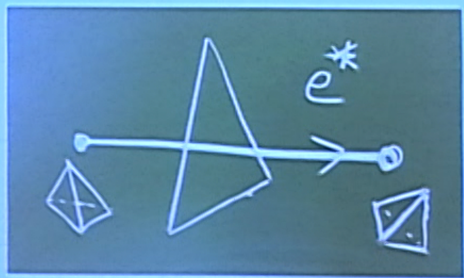
$$Z_{BF} = \int \mathrm{d}e\mathrm{d}A\, e^{iS[A,e]} = \int \mathrm{d}A\, \delta(F[A])$$

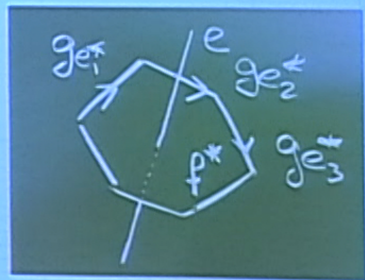
Ponzano-Regge as discretized BF theory

Topological field theory ⟹ Can be discretized exactly

1. Choose a 3d triangulation or 3d cellular decomposition
2. Discretize triad along edges $X_e \in \mathfrak{su}(2) \sim \mathbb{R}^3$
3. Look at dual 2-complex, the « spinfoam »
4. Discretize connection along dual edges $g_{e^*} \in \mathrm{SU}(2)$
5. Consider curvature as holonomies around dual faces $G_e = G_{f^*} = \overrightarrow{\prod_{e^* \in \partial f^*}} g_{e^*}$







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Ponzano-Regge as discretized BF theory

We discretize the BF path integral :

$$Z_{BF} = \int \mathrm{d}e\mathrm{d}A\, e^{iS[A,e]} = \int \mathrm{d}A\, \delta(F[A])$$

$$Z_{PR} = \int \mathrm{d}X_e \mathrm{d}g_{e^*}\, e^{\langle X_e, G_e \rangle} = \int \mathrm{d}g_{e^*} \prod_{f^*} \delta(G_{f^*})$$

- **Decompose SU(2)-distributions as sum over spins** — **Plancherel formula**

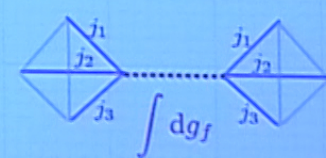
$$\delta(G_{f^*}) = \delta(G_e) = \sum_{j_e \in \mathbb{N}/2} (2j_e+1)\chi_{j_e}(G_e) = \sum_{j_e \in \mathbb{N}/2} (2j_e+1)\mathrm{Tr}\left[\prod_{f \ni e} D^{j_e}(g_f)\right]$$

- **Integrate over group elements to get 3nj-symbols of spin recoupling**

each triangle has 3 edges

$$\int \mathrm{d}g_f\, D^{j_{e_1}}(g_f) D^{j_{e_2}}(g_f) D^{j_{e_3}}(g_f) = |\psi^{j_{e_1} j_{e_2} j_{e_3}}\rangle\langle\psi^{j_{e_1} j_{e_2} j_{e_3}}|$$

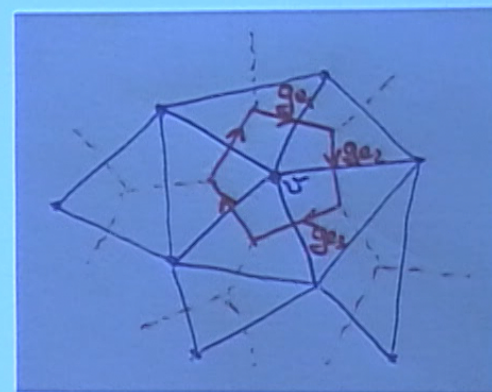
group averaging = projector over singlet states



the 2d version of the Ponzano-Regge model

To see things clearer... let's go down to 2d !

- group elements between triangles
- holonomies around vertices



$$Z_{2d} = \int \mathrm{d}g_e \prod_v \delta(G_v) \qquad G_v = \overrightarrow{\prod_{e \ni v}} g_e$$

vertex ⟶ **spin** $j_v \in \mathbb{N}/2$

edge ⟶ **singlet state** between 2 spins $\psi^{j,k} = \frac{\delta_{j,k}}{\sqrt{2j+1}} \sum_m |j,m\rangle\langle j,m|$

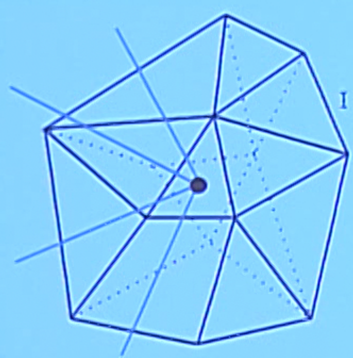
triangle ⟶ **trace over tensor of 3 singlet states** $\delta_{j_{t(1)},j_{t(2)},j_{t(3)}} \frac{(2j+1)}{\sqrt{2j+1}^3}$

Overall, all the spins are equal : topological invariant = Euler characteristic

$$Z_{2d} = \sum_j (2j+1)^{V+F-E} \qquad \chi_{\text{Euler}} = V - E + F = 2 - 2g$$

Ponzano-Regge as a topological invariant

Let's come back to 3d ! $Z_{PR} = \int \mathrm{d}g_f \prod_e \delta(G_e)$ $G_e = \overrightarrow{\prod_{f\ni e}} g_f$



We have bubbles around each vertex

with flatness around every (dual) face on the bubbles

$\delta(G_e)$ distributions responsible for both :

- **topological invariance**
- **divergences due to redundancies** (Bianchi identity)

Possible to gauge-fix... Freidel, Louapre '04

integrate over face variables edge-weights with gauge-fixing at vertices

↪ Ponzano-Regge amplitude = Reidemeister torsion for twisted cohomology

$$Z_{PR}[\Delta] = \int \mathrm{d}A_{\Delta^*}\, \tau[A_{\Delta^*}]$$

over moduli of flat discrete connections

Barrett, Guzman '09

Ponzano-Regge as a topological invariant

Let's come back to 3d ! $Z_{PR} = \int \mathrm{d}g_f \prod_e \delta(G_e)$ $\qquad G_e = \overrightarrow{\prod_{f \ni e}} g_f$

↪ Ponzano-Regge amplitude = Reidemeister torsion for twisted cohomology

$$Z_{PR}[\Delta] = \int \mathrm{d}\mathcal{A}_{\Delta^*}\, \tau[\mathcal{A}_{\Delta^*}]$$

over moduli of flat discrete connections

Barrett, Guzman '09

↪ Fits with Ray-Singer analytic torsion for BF theory in the continuum

$$Z_{BF} = \int \mathrm{d}E \mathrm{d}A\, e^{i \int E \wedge F[A]}$$

Action invariant under shift symmetry:

$$E \to E + \mathrm{d}_A \phi$$

From BRST symmetry for gauge-fixed BF theory :

$$Z_{BF} = \int \mathrm{d}\mathcal{A}\, \mathrm{Tor}[\mathcal{A}]$$

over moduli of flat connections

$$\mathrm{Tor}[\mathcal{A}] = \sqrt{\frac{\det\big((\mathrm{d}^0_{\mathcal{A}})^\dagger \mathrm{d}^0{}_{\mathcal{A}}\big)\, \det\big((\mathrm{d}^2_{\mathcal{A}})^\dagger \mathrm{d}^2{}_{\mathcal{A}}\big)}{\det\big((\mathrm{d}^1_{\mathcal{A}})^\dagger \mathrm{d}^1{}_{\mathcal{A}}\big)}}$$

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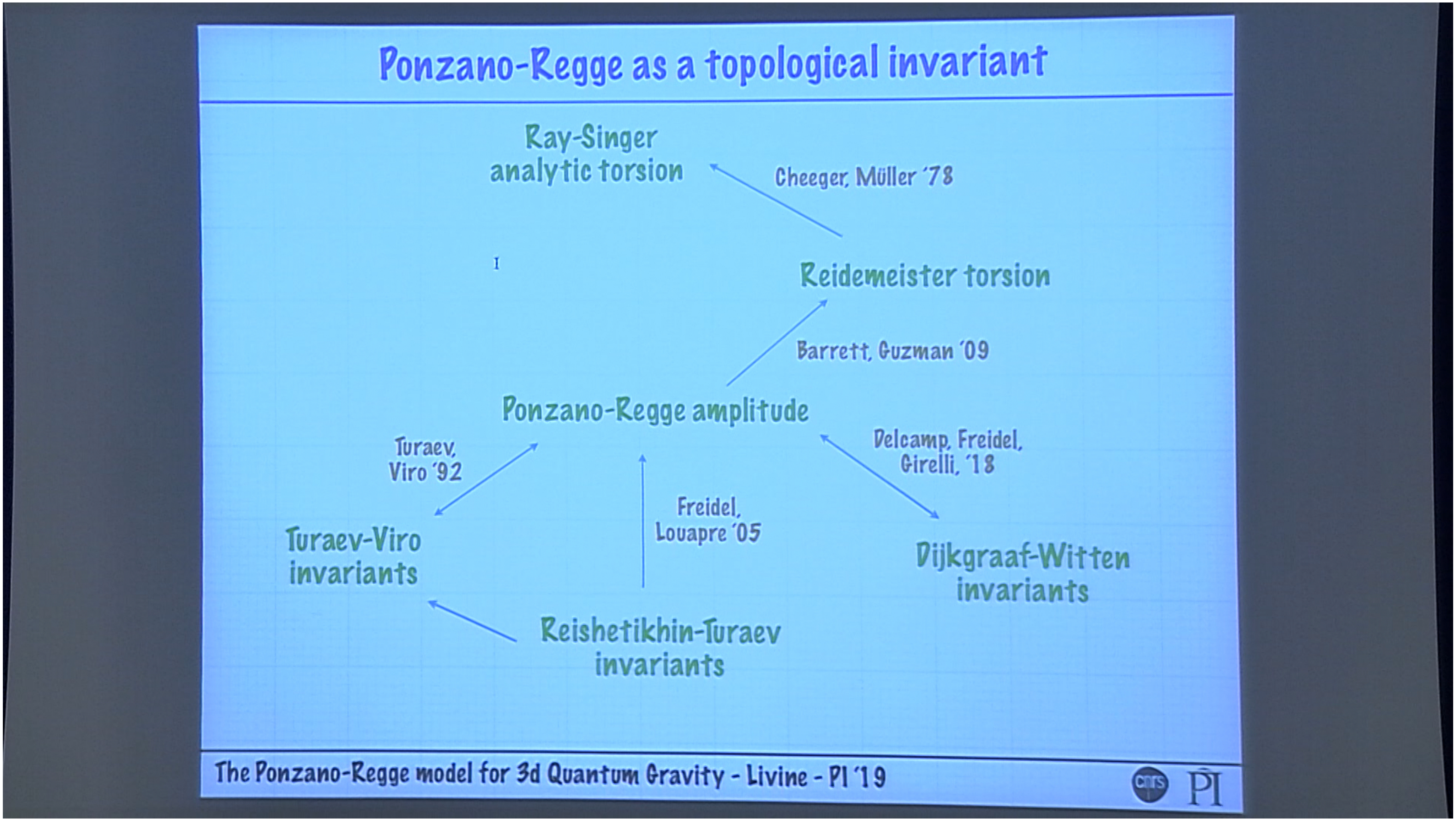
Ponzano-Regge as a topological invariant
Ray-Singer analytic torsion
Cheeger, Müller '78
Reidemeister torsion
Barrett, Guzman '09
Ponzano-Regge amplitude
Turaev, Viro '92
Delcamp, Freidel, Girelli, '18
Freidel, Louapre '05
Turaev-Viro invariants
Dijkgraaf-Witten invariants
Reishetikhin-Turaev invariants
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PI

Quantum Gravity from BF theory

Generalization to gravity in higher dimension

outlines quantization program from BF theory

$$S_{grav}[e,A] = \int \star(e \wedge e) \wedge F[A]$$

$$\rightarrow \quad S_{BF}[B,A,\phi] = \int B \wedge F[A] + \phi\, \mathcal{C}_{simplicity}[B]$$

↪ « spinfoam » framework

quantize BF theory as a topological state-sum,
then impose simplicity constraints on algebraic data at quantum level

Part II

So... in practice ?

Evaluation of observables on the 3-sphere

Start with **trivial topology but add curvature defects :**

- triangulation of 3-sphere
- defects around chosen edges

$$Z_{PR}[\Delta,\Gamma,\{\theta_e\}_{e\in\Gamma}] = \int \mathrm{d}g_f \prod_{e\in\Gamma}\delta_{\theta_e}(G_e)\prod_{e\notin\Gamma}\delta(G_e)$$

$$= \sum_{\{j_e\}}\prod_{e\notin\Gamma}(2j_e+1)\prod_{e\in\Gamma}\frac{\sin(2j_e+1)\frac{\theta_e}{2}}{\sin\frac{\theta_e}{2}}\prod_T\{6j\}_T$$

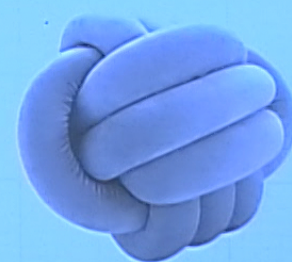
two perspectives

Knot invariants

Put curvature defect $\theta_e=\theta$ along sequence of edges, or tube, or knot

$$Z_{PR}[\mathcal{S}^3,\mathcal{K},\theta] = \frac{4\sin^2\frac{\theta}{2}}{|A_{\mathcal{K}}(e^{i\theta})|^2}$$

Barrett, Guzman '09



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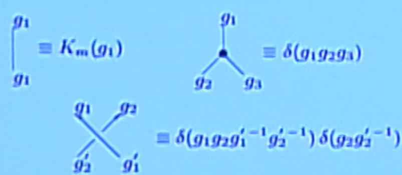

two perspectives → Feynman diagrams of a braided NCQFT

- Effective QFT for matter after integrating out QG effects
- Particle momenta live in SU(2)

↳ QG Phenomenology, DSR, relative locality

Curvature defects are massive particles $m_e = \frac{\theta_e}{2\pi}m_{\text{Planck}}$

$$Z_{PR}[\mathcal{S}^3,\Gamma,\{\theta_e\}] = I_\Gamma[\{\theta_e\}]$$

Freidel, L '05

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two perspectives

Knot invariants ⟷ **Feynman diagrams of a braided NCQFT**

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Freidel, L '05

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Part III

and with boundaries ... ?

Holographic theory

Theory is topological, with no local degree of freedom, so everything gets projected onto the boundary

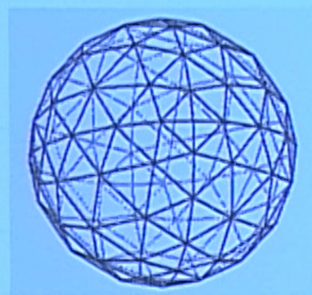
↳ invariant under coarse-graining and refinement of bulk triangulation, but behavior under change of boundary triangulation ?

↳ dynamics of the theory encoded in boundary theory

So let's look at 3d manifolds with boundary !

Spin network evaluation for planar graphs

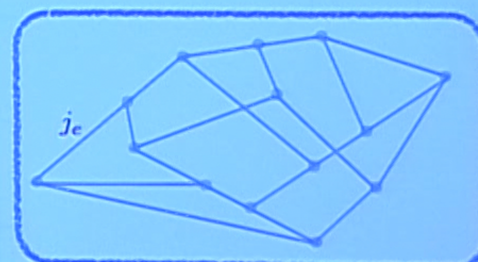
Let's start with the trivial topology of a 3-ball



- Boundary = 2-sphere
- Consider 3-valent planar connected oriented boundary graph $\Gamma = \partial\Delta$
- PR amplitude becomes projector on flat connection

$$Z^{\Delta}_{PR}[\Gamma, \{j_e\}_{e\in\Gamma}] = \langle \mathbb{I} | \psi_\Gamma^{\{j_e\}} \rangle$$

This defines the **evaluation of a spin network** obtained by simple gluing of 3j-symbols :



$$s^\Gamma(\{j_e\}) = \psi^\Gamma_{\{j_e\}}(\mathbb{1}) = \sum_{\{m_e\}} \prod_e (-1)^{j_e - m_e} \prod_v \begin{pmatrix} j_{e_1^v} & j_{e_2^v} & j_{e_3^v} \\ \epsilon^v_{e_1} m_{e_1^v} & \epsilon^v_{e_2} m_{e_2^v} & \epsilon^v_{e_3} m_{e_3^v} \end{pmatrix}$$

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Coherent boundary states & Generating functions

Two equivalent ways to repackage the boundary amplitude

Generating function

Coherent boundary state with superpositions of spins

We introduce a variable for each edge and define:

$$Z_\Gamma^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v-2j_e)!}}\, s^\Gamma(\{j_e\}) \prod_e Y_e^{2j_e}$$

We use the generating function for 3j-symbols to write it as a Gaussian integral :

$$Z_\Gamma^{Spin}(\{Y_e\}) = \int \prod_{ev} \frac{d^2 z_{ev} d^2 w_{ev}}{\pi^2}\, e^{-\sum_{ev}(|z_{ev}|^2+|w_{ev}|^2)}$$
$$e^{-\sum_e (\bar{z}_{s(e)}\bar{w}_{t(e)} - \bar{w}_{s(e)}\bar{z}_{t(e)}) + \sum_\alpha X_\alpha (z_{s(\alpha)} w_{t(\alpha)} - w_{s(\alpha)} z_{t(\alpha)})}$$

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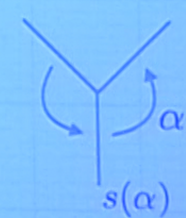
We use the generating function for 3j-symbols ...

$$\sum_{j_e, m_e} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \sqrt{(J+1)!} \prod_e \frac{Y_e^{j_e} z_e^{j_e+m_e} w_e^{j_e-m_e}}{\sqrt{(J-2j_e)!(j_e-m_e)!(j_e+m_e)!}}$$

$$= \exp \sum_\alpha X_\alpha \left(z_{s(\alpha)} w_{t(\alpha)} - w_{s(\alpha)} z_{t(\alpha)}\right)$$



$$X_\alpha = \sqrt{Y_{s(\alpha)} Y_{t(\alpha)}}$$

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Duality with 2d Ising

The Gaussian integral can be computed explicitly :

$$(Z_f)^2 \, Z_\Gamma^{Spin} = 1 \qquad Z_f = \sum_{\gamma \in \mathcal{G}} \prod_{\alpha \in \gamma} X_\alpha = \sum_{\gamma \in \mathcal{G}} \prod_{e \in \gamma} Y_e$$

$$(Z^{Ising})^2 \, Z^{Spin} \; = \; 2^{2V} \prod_e \cosh(y_e)^2$$

$$Y_e = \tanh y_e$$

Westbury '98

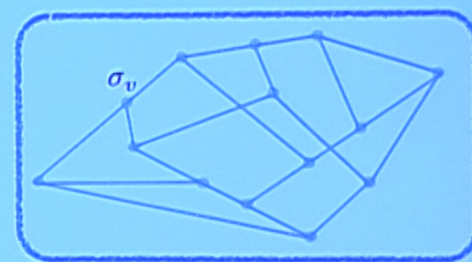
↳ **Duality between Ising model & PR amplitudes**

↳ can be realized by a supersymmetry !

Back to the Ising Model

On same graph, put « spins » on vertices: $\sigma_v = \pm 1 \in \mathbb{Z}_2$

$$Z_\Gamma^{Ising}(\{y_e\}) = \sum_\sigma \exp_{\text{I}} \left(\sum_e y_e \sigma_{s(e)} \sigma_{t(e)} \right)$$



Can define high temperature expansion...

$$Z_\Gamma^{Ising}(\{y_e\}) = \left(\prod_e \cosh(y_e)\right) \sum_\sigma \prod_e (1 + \tanh(y_e)\sigma_{s(e)}\sigma_{t(e)})$$

... as sum over loops:

$$Z_\Gamma^{Ising}(\{y_e\}) = 2^V \left(\prod_e \cosh(y_e)\right) \sum_{\gamma \in \mathcal{G}} \prod_{e \in \gamma} Y_e \quad \text{with} \quad Y_e = \tanh y_e$$

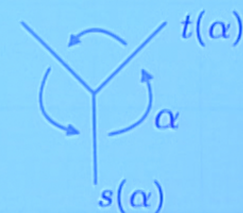
↪ can be written as a fermionic integral

The Ising Model as a Fermion Path Integral

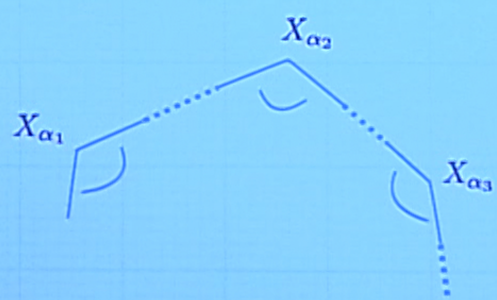
Here explicitly: $$Z_\Gamma^{Ising}(\{y_e\}) = 2^V \prod_e \cosh(y_e)\, Z_f(\{X_\alpha\})$$

$$Z_f(\{X_\alpha\}) = \int \prod_{ev} d\psi_{ev}\, \exp\left(\sum_e \psi_{s(e)}\psi_{t(e)} + \sum_\alpha X_\alpha\, \psi_{s(\alpha)}\psi_{t(\alpha)} \right)$$

with angle couplings: $$X_\alpha = \sqrt{Y_{s(\alpha)}Y_{t(\alpha)}}$$



↪ We glue angles along edges to form loops :



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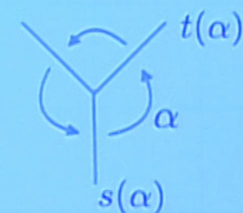

↪ We glue angles along edges to form loops

↪ We compute the square of the Ising partition function :

$$(Z_f)^2 = \int \prod_{ev} [d\psi d\eta d\bar{\psi} d\bar{\eta}]_e^v \, e^{\sum_{e,v} \psi_e^v \bar{\eta}_e^v + \bar{\psi}_e^v \eta_e^v}$$

$$e^{-\sum_e \bar{\psi}_{s(e)}\bar{\psi}_{t(e)} + \bar{\eta}_{s(e)}\bar{\eta}_{t(e)}}\, e^{\sum_\alpha X_\alpha (\psi_{s(\alpha)}\psi_{t(\alpha)} + \eta_{s(\alpha)}\eta_{t(\alpha)})}$$

Supersymmetry on the boundary

We can introduce a meta-theory combining

- Ising model $\longleftrightarrow$ Fermions
- Spin networks $\longleftrightarrow$ Bosons

$$\mathcal{Z}_\Gamma = (Z_f)^2 Z^{Spin} = \int \mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\psi\,\mathrm{d}\eta\, e^{S[z,w,\psi,\eta_{ev}]}$$

$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_\alpha X_\alpha S_\alpha$$

$$\begin{aligned}
K_{e,v} &= |z_e^v|^2 + |w_e^v|^2 - \psi_e^v \bar{\eta}_e^v - \bar{\psi}_e^v \eta_e^v \\
S_e &= \bar{z}_{s(e)} \bar{w}_{t(e)} - \bar{w}_{s(e)} \bar{z}_{t(e)} + \bar{\psi}_{s(e)} \bar{\psi}_{t(e)} + \bar{\eta}_{s(e)} \bar{\eta}_{t(e)} \\
S_\alpha &= z_{s(\alpha)} w_{t(\alpha)} - w_{s(\alpha)} z_{t(\alpha)} + \psi_{s(\alpha)} \psi_{t(\alpha)} + \eta_{s(\alpha)} \eta_{t(\alpha)}
\end{aligned}$$

We define a supersymmetry generator, acting on each half-edge i=(e,v) :

$$\begin{aligned}
Q z_i &= \psi_i \\
Q w_i &= \eta_i \\
Q \psi_i &= w_i \\
Q \eta_i &= -z_i
\end{aligned}$$

Action S is both Q-closed and Q-exact, so supersymmetry localizes the integral:

$$\frac{\partial \mathcal{Z}_\Gamma}{\partial \lambda_{e,v}} = \frac{\partial \mathcal{Z}_\Gamma}{\partial \mu_e} = \frac{\partial \mathcal{Z}_\Gamma}{\partial X_\alpha} = 0$$

↳ Duality by supersymmetry

Bonzom, Costantino, L '13

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CNRS PI

Duality with 2d Ising

The Gaussian integral can be computed explicitly :

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$$(Z^{Ising})^2\, Z^{Spin} \;=\; 2^{2V} \prod_e \cosh(y_e)^2$$

↳ Duality between Ising model & PR amplitudes

↳ Ising -> Phase transitions for 3d QG ?

↳ QG-> Ising : Geometric formula of Ising's zeroes

Stat Phys → Q Grav → Stat Phys

Critical Ising from Ponzano-Regge

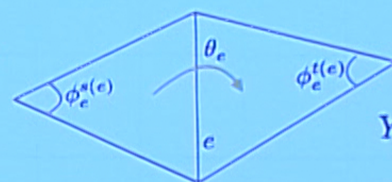
$$\left(Z^{Ising}\right)^{-2} = Z^{Spin} = \sum_{\{j_e\}} \dots$$

- Can study large spin asymptotics of series
- Poles of PR amplitude give Zeroes of Ising

$$Z_\Gamma^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v-2j_e)!}}\, s^\Gamma(\{j_e\}) \prod_e Y_e^{2j_e}$$

Bonzom, Costantino, L

Stationary point in spins j's when couplings Y's are given by the angles of a dual triangulation



$$Y_e^c = e^{\epsilon \frac{1}{2}\theta_e} \sqrt{\tan\frac{\phi_e^{s(e)}}{2} \tan\frac{\phi_e^{t(e)}}{2}}$$

New geometric formula for Fisher zeroes

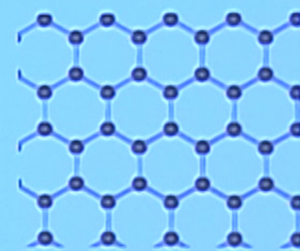
Admissible geometric couplings Y_e^{geom} → Scale invariant saddle points in j_e → Pole in the generating function $Z^{Spin} \to \infty$ → Zero of Ising i.e. critical coupling

Critical Ising from Ponzano-Regge

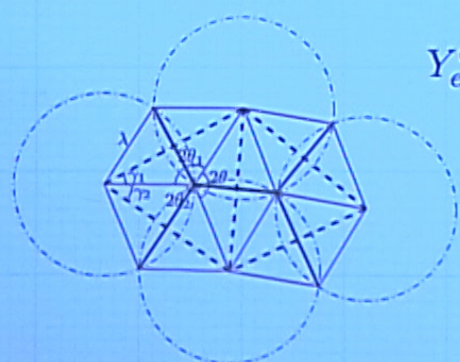
Numerical applications:

- **Regular honeycomb network**

$$Y^c = \frac{1}{\sqrt{3}}$$



- **Isoradial graphs**



$$Y_e^c = \tan\frac{\gamma_e}{2}$$

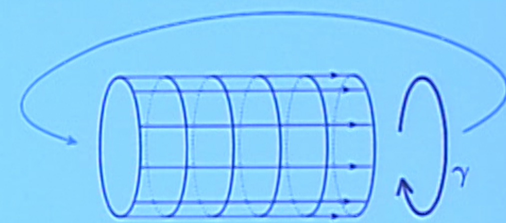
What's new here :

- **beyond isoradial**
- **planar but not flat**

Exploring the next topology

Ponzano-Regge on twisted cylinder with 2-torus boundary

→ towards AdS_3/CFT_2



Two types of boundary states on square lattice on toric boundary :

- **Fixed spin state**

  map to 6-vertex model
  & Heisenberg XXX spin chain

  ↓

  integrability of
  PR boundary theory ?

- **Coherent state with spin superposition**

  allows exact resummation !

$$\mathcal{Z}^{PR}[\lambda,\gamma] \propto \prod_{p=1}^{N_x-1} \frac{1}{\cosh N\lambda - \cos(p\gamma + iN\lambda)}$$

  regularized version of BMS character,
  related to Dedekind eta function

Dittrich, Goeller, L, Riello '18

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Et voilà: what's next for Ponzano-Regge ?

- Effective NCQFT for matter coupled to 3d QG: renormalisation, ...
- Duality with Ising : coupled supersymmetric theories, continuum limit for critical and non-critical in terms of WZW theories, ...
- Boundary theories on the torus : integrability, boundary symmetries (discrete current algebra and Virasoro ?)
- Holography: CFT and non-critical QFT on quasi-local boundary ?

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