Title: String Theoretic Illuminations of Moonshine

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Abstract: Many of the rich interactions between mathematics and physics arise using general mathematical frameworks that describe a host of physical phenomena: from differential equations, to algebra, to topology and geometry. On the other hand, mathematics also possesses many examples of "exceptional objects": they constitute the finite set of leftovers that appear in numerous classification problems. For example, groups of symmetries in three dimensions appear in two infinite families (cyclic groups and dihedral groups of n-sided polygons) and the symmetry groups of the five Platonic solids--- the 'exceptional' structures.

The mathematical subject of moonshine refers to surprising relationships between other kinds of special/exceptional objects that arise from the theory of finite groups and from number theory. Increasingly, string theory has been a source of insights in and explanations for moonshine. It is even the source of new examples of moonshine that further implicate special objects in geometry. We will review moonshine, survey these developments, and highlight some of the (many!) exciting mysteries that remain.

String Theoretic Illuminations of Moonshine

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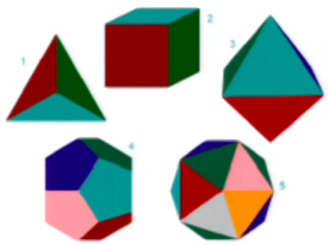
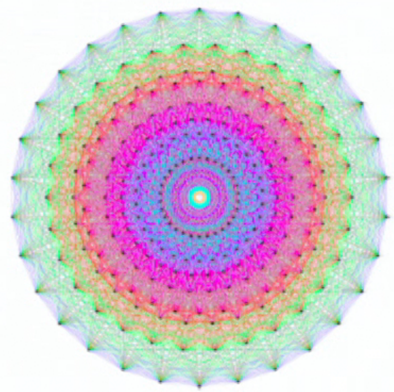
March 6, 2019

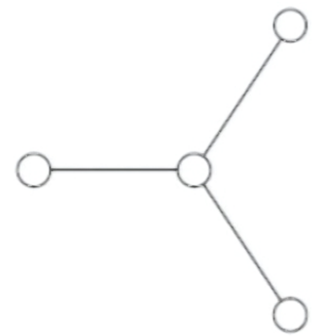
Perimeter Institute Colloquium

The Spirit of Moonshine

Today will be a celebration of some weird, rare mathematical structures and their appearance in physics.

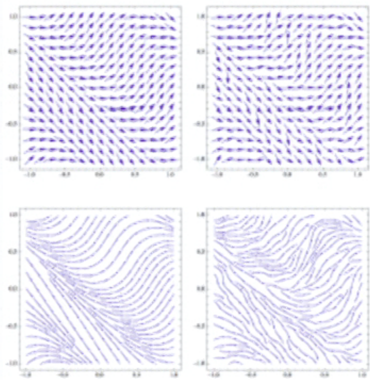




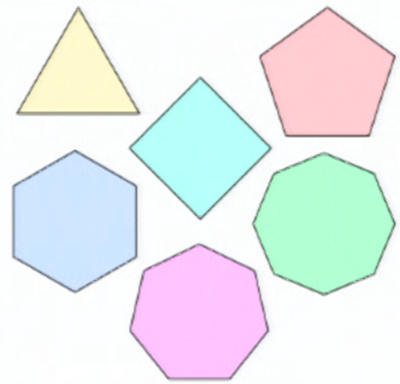


These oddballs are more often called exceptional.

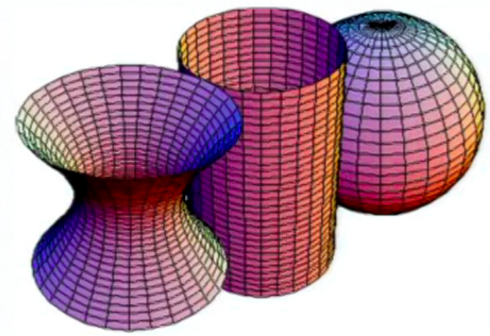
On the other hand, you already know that a lot of beautiful mathematics shows up in physics, e.g.:



analysis & calculus



group theory



differential geometry

But they are broad, uniform structures, and they describe a wide range of phenomena

Exceptional structures

There are all kinds of classification problems that mathematicians would like to solve.

The results of classifications can be divided into two types:

1. Several infinite families of objects
2. A finite number of exceptions

For instance, 3d rotation groups (finite subgroups of $SO(3)$):

- 2 infinite families: Cyclic group $\mathbb{Z}_n$, Dihedral group D_n
- 3 exceptions: Rotation groups of regular tetrahedron A_4, octahedron/cube S_4, icosahedron/dodecahedron A_5 (Platonic solids).

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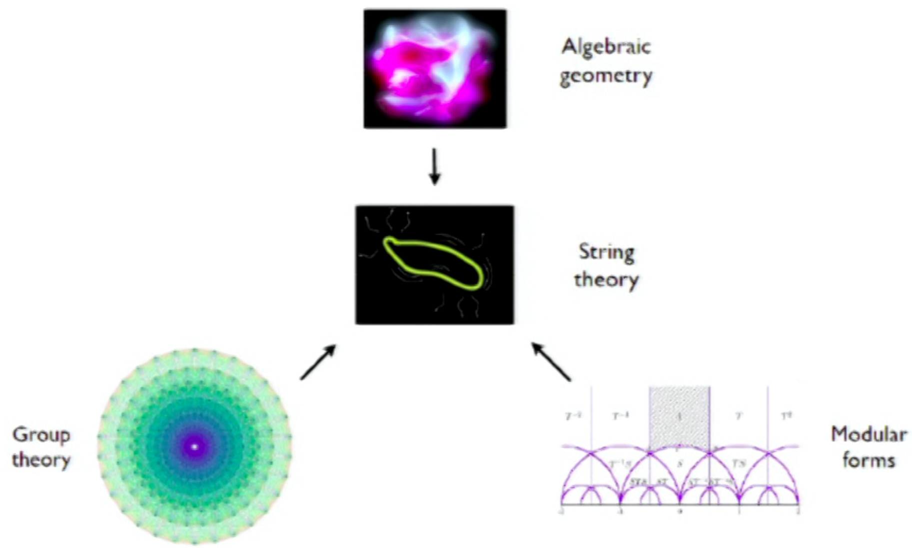
Moonshine: mysterious correspondences for exceptional/special objects

At least some instances of moonshine can be understood and unified via string theory.



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Benefits for physics

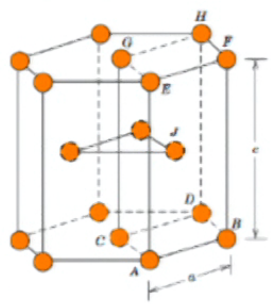
1. What is string theory?
2. Easier (but not easy!) sub-question:
   Can we characterize or organize (some of) the many solutions to string theory's equations of motion in some way?
   Perhaps using some hidden mathematical structure?
3. Moonshine observations in string theory point to large discrete symmetry groups, whose role in the theory is still being elucidated.
4. Ancillary benefit: learning a lot about special classes of states in string theory called BPS states, which are powerful tools for studying dynamics at strong coupling. More on this later...
5. More generally, physics/math interactions are powerful and productive for both sides!

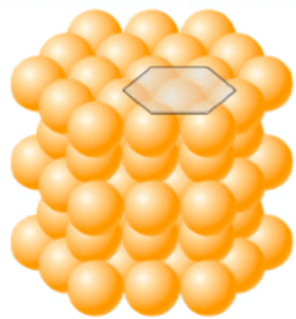
Ingredients for brewing moonshine, Pt 1

Symmetries and groups

Symmetries: powerful organizational principle for physics

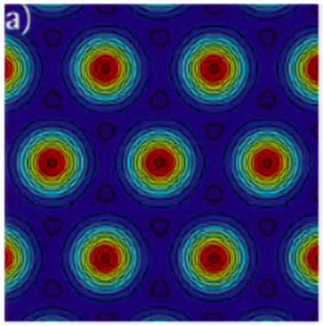
Many guises and origins: spacetime & internal, global & gauge, approximate & exact. . .





Callister & Rethwisch 5e

microscopic symmetries



https://www.st-andrews.ac.uk/ supermag/vortex.html

emergent symmetries

Today we will focus on 'microscopic' symmetries: property of **formulation** of the theory rather than a property of its **solutions**

- Symmetries of physical systems leave system invariant
- Organize excitations of a system according to how they transform, i.e. representation
- e.g. Symmetry: spatial rotations, Quantum number: spin J, $(2J+1)$-dimensional representation
- Codified in group theory:
  associative multiplication law, identity, inverse

So... what kind of groups can we classify?

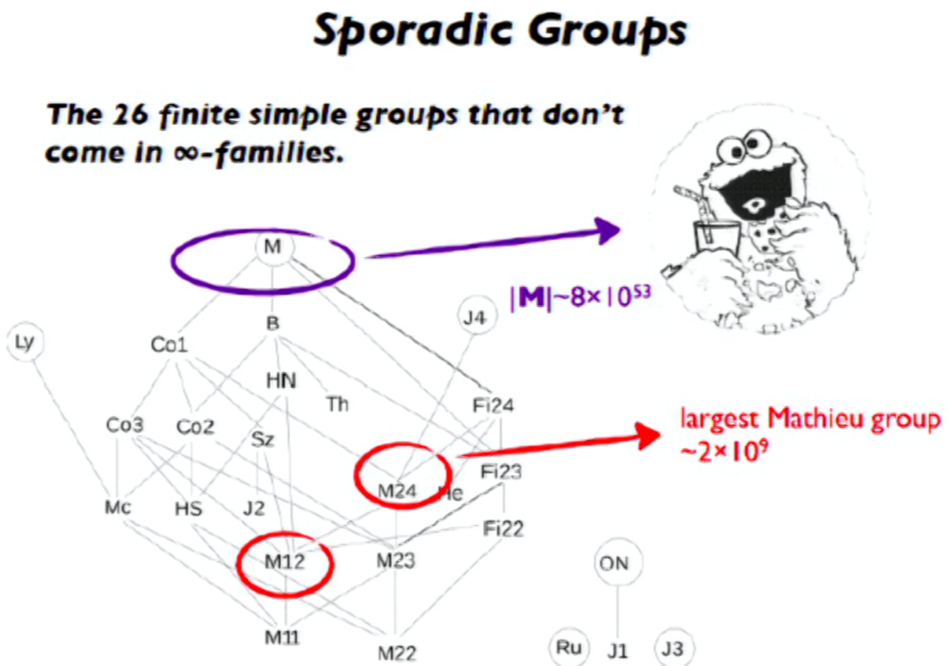
Finite simple groups

- Finite simple groups: finite groups with no nontrivial normal subgroups
- 'Prime numbers' of group theory

Classification results (2004, after many years and MANY contributions!):

1. 18 infinite families, including $\mathbb{Z}_p$ for prime p, $A_{n\geq 5}$, ...
2. 26 exceptional cases called the sporadic groups

Sporadic groups



Seemingly in a class by themselves. But deeply connected to huge amount of mathematical physics! In fact, some are basic symmetries of interesting string backgrounds.

Ingredients for brewing moonshine, Pt. 2

String theory primer

The basic constituents of a string theory (in perturbation theory around weak string coupling) are either closed loops or open threads of string.

They interact:

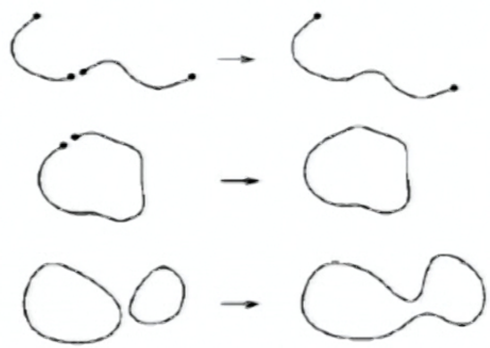
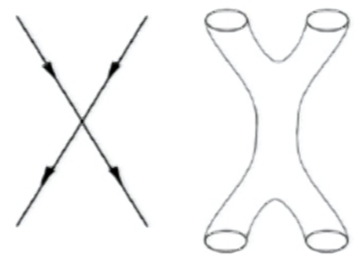


Fig.4: String interactions



Feynman diagrams →
Riemann surfaces
Worldlines → worldsheets

open → closed, not vice versa!
Today: **closed** strings

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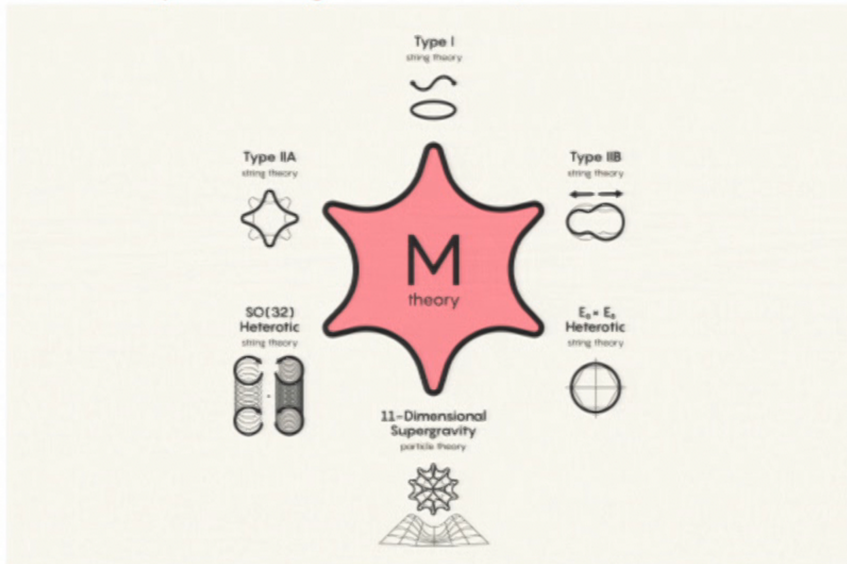
Critical string theories

Bosonic string: $d = 26$, but is unstable (tachyon)

Superstrings: $d = 10$, and stable!



Pic credit: *Olena Shmahalo/Quanta Magazine*

String dualities: different descriptions of equivalent physics

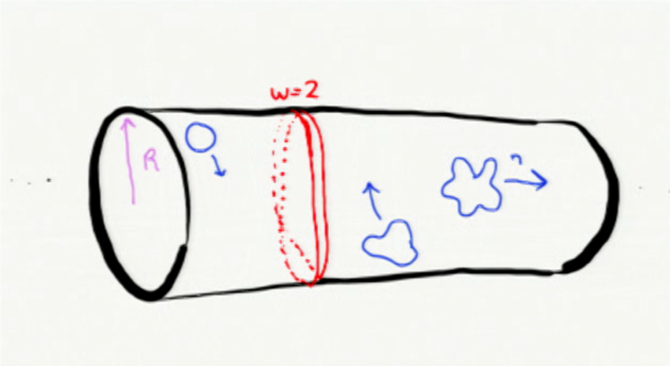
Compactification

Can make some of these dimensions very tiny and obtain a lower dimensional theory in a consistent way.

In the original idea of Kaluza-Klein, one goes from 5d to 4d by compactifying on a circle C and obtaining an effective theory on M.

In string theory, 10d $\rightarrow$ 4d on a **Calabi-Yau manifold**. Geometric properties of this space are just right to obtain a static, supersymmetric solution of the vacuum Einstein equations in 4d.

String states



Strings carry momentum, winding quantum numbers (n, w), and oscillator modes $N_L, N_R \to$ particle mass in noncompact directions:

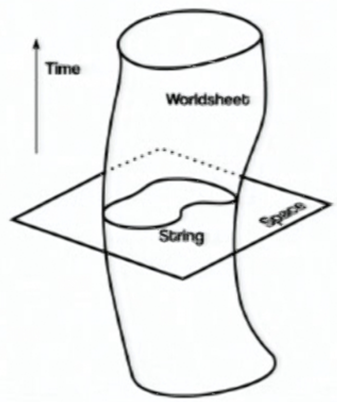
$$m^2 = (n/R)^2 + (wR/l_s^2)^2 + (2/l_s)(N_L + N_R - 2)$$

This was for a bosonic string compactified on a circle of radius R. Similar principles apply for Calabi-Yau compactifications.

Count the string states

Compute a partition function of string states, graded by mass:

$$Z = tr_{\mathcal{H}} e^{-\beta H} = \sum_n c_n q^n$$



Defined on a **torus** (pass to Euclidean time, take the trace).

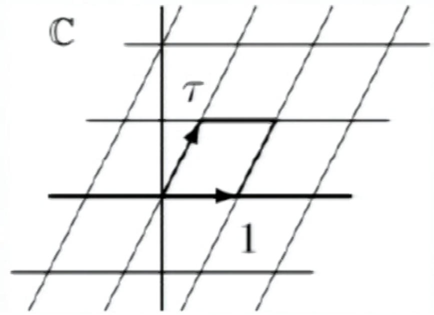
Recall: partition function of quantum particle $\leftrightarrow$ path integral for particle in periodic Euclidean time

$$\beta \sim 2\pi R_{time}$$

Ingredients for brewing moonshine, Pt. 3

Modular forms

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tau \in \mathbb{H}$



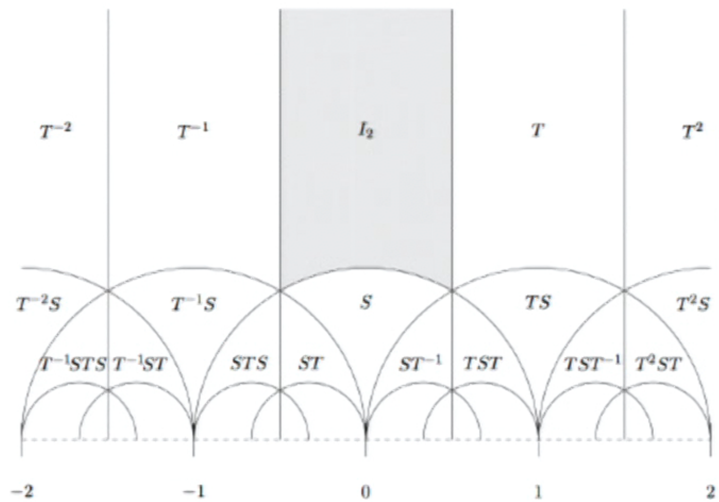
Modular function

$f(\frac{a\tau+d}{c\tau+d}) = f(\tau)$ (+ some analyticity conditions)

Generators:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Large diffeomorphisms or "mapping class group" of torus



Modular forms are prominent in number theory since their coefficients are often simple expressions in integers, yet the functions themselves are very constrained by modularity.

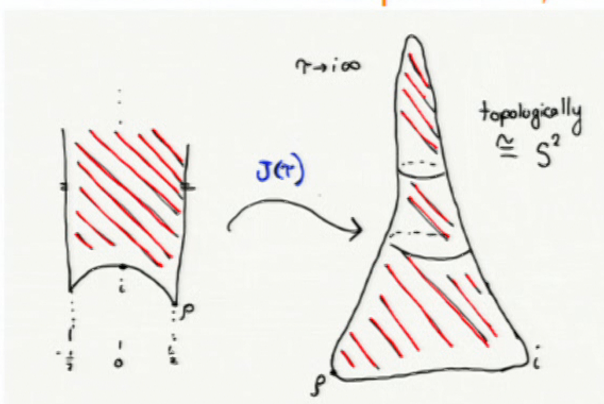
They are central ingredients in some very deep proofs, e.g.

- Andrew Wiles' proof of Fermat's last theorem
- Maryna Viazovska's proof of optimal sphere packing in 8 dimensions

Genus zero

$\Gamma \subseteq SL(2,\mathbb{R})$, consider $\bar{\mathbb{H}}/\Gamma$ and corresponding modular forms.
$\bar{\mathbb{H}}/\Gamma \simeq$ Riemann sphere $\leftrightarrow$ field of Γ-mod fxns has 1 generator $f(\tau)$

Then Γ is called **genus zero**. The special generator is sometimes called the Hauptmodul, or principal modulus.



$SL(2,\mathbb{Z})$ is genus zero.

Its generator is:

$$J(\tau) = J(\tau+1) = J(-1/\tau) = 1/q + 196884q + 21493760q^2 \ldots,$$
$$q := e^{2\pi i \tau}$$

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Modular physics

String theory partition functions 'live on' a torus.

'Large diffeomorphisms' are gauge symmetries to a string theory $\rightarrow$ observables **invariant**

i.e. a change of basis. Like parameterizations of a particle's worldline, partition function must be **independent**

$\therefore$ string theory produces modular functions!

Vintage Moonshine

Origin of moonshine

Modular invariant: $J(\tau) = \sum_{n=-1}^{\infty} c(n) q^n$ (McKay)
$c(n)$ decompose into dimns of irreps of Monster $\mathbb{M}$, largest sporadic simple group.

$$196884 = 1 + 196883$$

$$21493760 = 1 + 196883 + 21296876$$

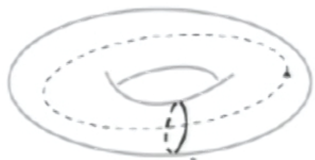
$$\vdots$$

Imagine an ∞-dim'l rep of Monster: $V^\natural = \oplus_{n=-1}^{\infty} V_n$,
$dim(V_n) = \chi_{V_n}(1) = c(n)$.
McKay-Thompson: $T_g(\tau) = Tr_{V^\natural}(g q^H) = \sum_n \chi_{V_n}(g) q^n$, $g \in \mathbb{M}$

Monstrous Moonshine conjectures: (Conway-Norton) MT series are Hauptmoduls for genus zero $\Gamma_g \subset SL(2, \mathbb{R})$. (Proof: Borcherds '92, Fields Medal '98).

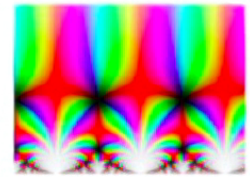
Monstrous Moonshine

Strings in the Leech lattice background:

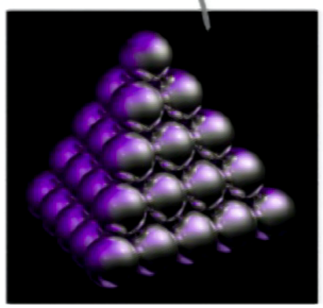


Modular Symmetry

The Partition Function = $J(\tau)$



Monstrous Moonshine



R^{24}/Λ_{Leech}

Sporadic Symmetry

Λ_{Leech} ↻ $Co_I \xrightarrow{Z/2}$ **Monster**

Uses 'orbifold conformal field theory/vertex operator algebra' (FLM) for $V^\natural$.

Note: This *doesn't* tell us why MT series are the special, rare mod fns of MM conjectures!

Origin of moonshine

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How to lift the shadows on genus zero?

Old story: based on a bosonic string construction (unstable!) Superstring theories are endowed with an extra symmetry, relating bosons and fermions, called **supersymmetry**.

Elementary aspects of the supersymmetry algebra privilege a certain subset of states called BPS states.

BPS states in a theory are counted by simpler analogues of partition functions called indices, which often can be computed exactly.

These indices are often independent of (some) parameters in the theory, such as the coupling constant. So: compute at weak coupling, extrapolate to strong coupling, gain insight re: dualities & nonpert. dynamics!

It's good to be super

SUSY algebra: $\{Q_A, Q_B\} = Q_A Q_B + Q_B Q_A = E\delta_{AB} - K_{AB}$
E energy-momentum operator, K_{AB} bosonic generator

BPS states: satisfy $E\delta_{AB} - K_{AB} = 0$, annihilated by some (linear combination) of the supercharges.

E.g. $\{Q^\dagger, Q\} = 2H$, other (anti)comms $= 0$

$$\langle\psi| \left\{Q^\dagger, Q\right\} |\psi\rangle =$$
$$|Q|\psi\rangle|^2 + |Q^\dagger|\psi\rangle|^2 \geq 0$$
$$\rightarrow H \geq 0$$

$\therefore$ SUSY is spontaneously broken if the vacuum has positive energy.
$Q|BPS\rangle = Q^\dagger|BPS\rangle = 0 \rightarrow |BPS\rangle$ a SUSY'c ground state.

Hunting Monsters

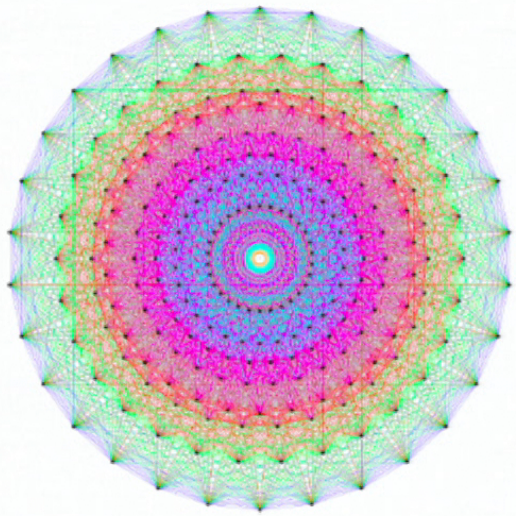
With Persson & Volpato ('16, '17):

1. Superstring compactification with $V^\natural$ building block[1]
2. BPS index, famous identity:
   $J(T) - J(U) = \prod(1 - p^m q^n)^{c(mn)}$, McKay-Thompson analogues. Related to Generalized Kac-Moody algebras
3. Γ_g of MM arise as T-duality groups. Study phase transitions.
4. Genus zero: study index in *decompactification* limits.

[1] Heterotic string on asymmetric $T^8 \times S^1$ orbifolds, T^8 at special point in moduli space

Other examples?

An instance of super-moonshine related to $Aut(\Lambda_{Leech}) \simeq Co_0$ (almost sporadic) (FLM, Duncan).
Moonshine module known $V^{s\natural}$, genus zero property proved but inherited from $\mathbb{M}$ (Duncan/Mack-Crane).



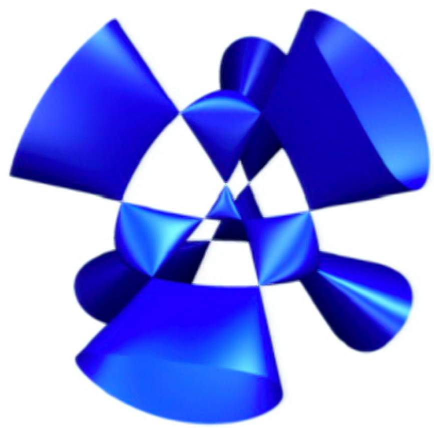
With Harrison & Volpato ('18):

1. Similar identities related to new superGKM (analogue of Borcherds' approach; Scheithauer)
2. Set up for analogous superstring construction, BPS indices & genus zero (WIP)

Young Moonshine

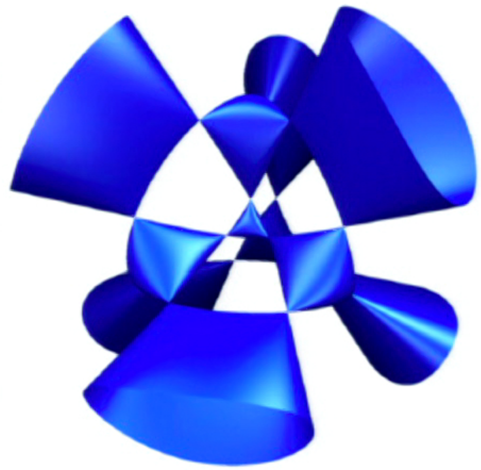
The moon shines on K3

Many superstring theory dualities arise after compactifying two theories on different spaces.
The K3 surface is ubiquitous in such dualities. It is the simplest nontrivial (compact) Calabi-Yau manifold. It is 4 real-dimensional.



Another special object, from algebraic geometry. One topological type, many shapes and sizes. *Connected* by continuous deformations: moduli space

We don't know any explicit metrics for compact CYs, including K3! This means we can't generically compute the partition function for a given K3 surface.



But we can compute a simpler BPS index called the elliptic genus for a superstring probing a K3 surface. Knows about some limited geometric/topological data. BPS states are lightest possible states of a given charge *therefore* cannot decay ∴ change size and shape of K3 and they persist.

Mathieu moonshine

Roughly:

1. $Z_{EG}(\tau) = \sum_{BPS} d_n q^n$ First computed '89 (Eguchi, Ooguri, Taormina, Yang)
2. Moonshine phenomenon observed in 2010! Sporadic group M_{24} (Eguchi, Ooguri Tachikawa)
3. $d_1 = 45 + 45,\ d_2 = 231 + 231,\ d_3 = 770 + 770, \ldots$
4. McKay-Thompson analogues computed: twining genera (Cheng, Gaberdiel/Hohenegger/Volpato, Eguchi/Hikami)

Puzzles & Proposals

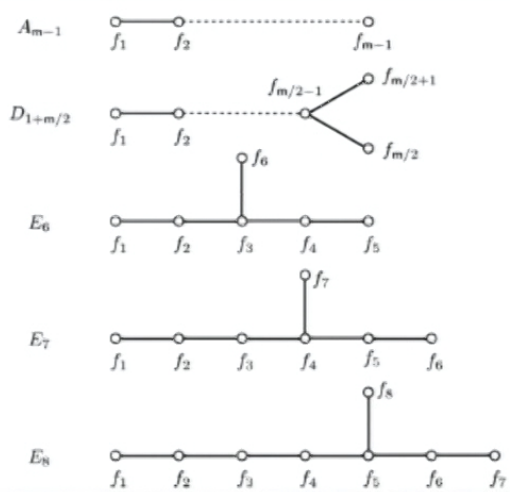
1. So, does string theory on K3 (maybe for some very special, symmetric K3 surface) have M_{24} symmetry? No! (Gaberdiel/Hohenegger/Volpato). Not all M_{24} group elements present, and some theories have symmetries outside M_{24} but inside Co_0.
2. Proposals to combine only *geometric* symmetries ($G \subset M_{23} \subset M_{24}$) of different K3s. (Taormina/Wendland, Gaberdiel/Keller/Paul, Wendland)
3. Connections between Conway moonshine and K3 geometry (Duncan/Mack-Crane, Cheng/Duncan/Harrison/Kachru, Harvey/Moore)
4. Perhaps it is only a symmetry of BPS sector? Perhaps we should look at other string theory contexts where K3 appears? Look at other 'duality frames'?

Umbral moonshine

23 new moonshines (Cheng/Duncan/Harvey). Mathieu a special case!

Symmetries: Niemeier lattices. Modular objects: mock modular forms

All related to symmetries of string theory on K3! (Cheng/Harrison, Cheng/Ferrari/Harrison/NMP, Cheng/Harrison/Volpato/Zimet)



With Volpato & Zimet ('17): Proved conjectures by CHVZ that certain subsets of twining genera from Umbral/Conway moonshines govern K3 twining genera fully. Using BPS states from $K3 \times T^2$ & orbifolds, connections to GKMs, enumerative geometry (Pixton/Oberdieck, Bryan/Oberdieck/Katz).

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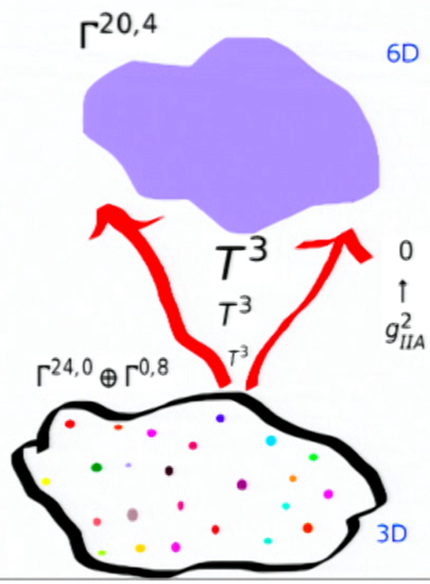
The moon shines on Flatland

With Kachru & Volpato ('16):

1. Superstrings on T^7 have Mathieu/umbral duality symmetries at special points in moduli space. Maximal symmetries.
2. Dual to another superstring on $K3 \times T^3$
3. But how to get the number theory side?



My perspective

Low-dimensional SUSY'c string vacua enjoy huge duality groups.

At *special points* in moduli space, these groups accommodate exotic, discrete groups.

Decompactify to higher dimensions & see remnants... Useful principle for organizing simple string vacua?

BPS quantities in string theory must be duality invariant and are automorphic objects.

They are also sensitive to (some) algebro-geometric data of the compactification manifold.

I hope we will understand all instances of moonshine with these ideas.

Conclusions

- Moonshine is a rich story connecting increasingly more areas of mathematics and mathematical physics, especially modular forms and finite group theory.
- Many moonshine phenomena have a natural, explanatory home in string theory.
- Numerous aspects of moonshine remain mysterious, and continue to hint at beautiful new structures that could help us understand string theory (dualities, algebras of BPS states, symmetries of string vacua, ...) and mathematics better!

Thank you!