Title: On one example of a chiral Lie group

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Collection: Cohomological Hall Algebras in Mathematics and Physics

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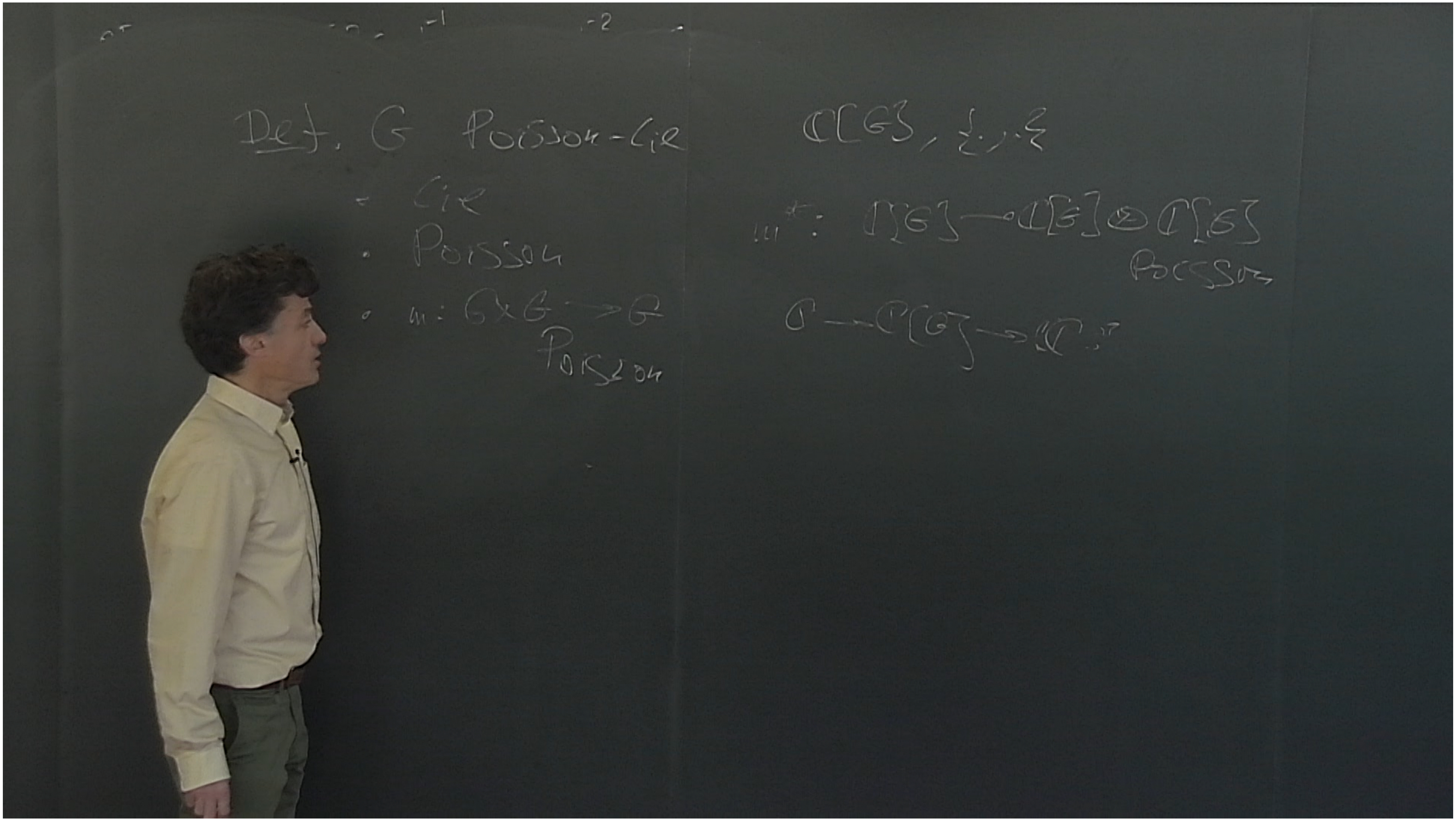
Abstract: We quantize the Khesin-Zakharevich Poisson-Lie group of pseudo-differential symbols. This is a joint work with A.Linshaw


Def. G Poisson-Lie
• Lie
• Poisson
• m: G×G → G
Poisson
ℂ[G], {·,·}
m*: ℂ[G] → ℂ[G] ⊗ ℂ[G]
Poisson

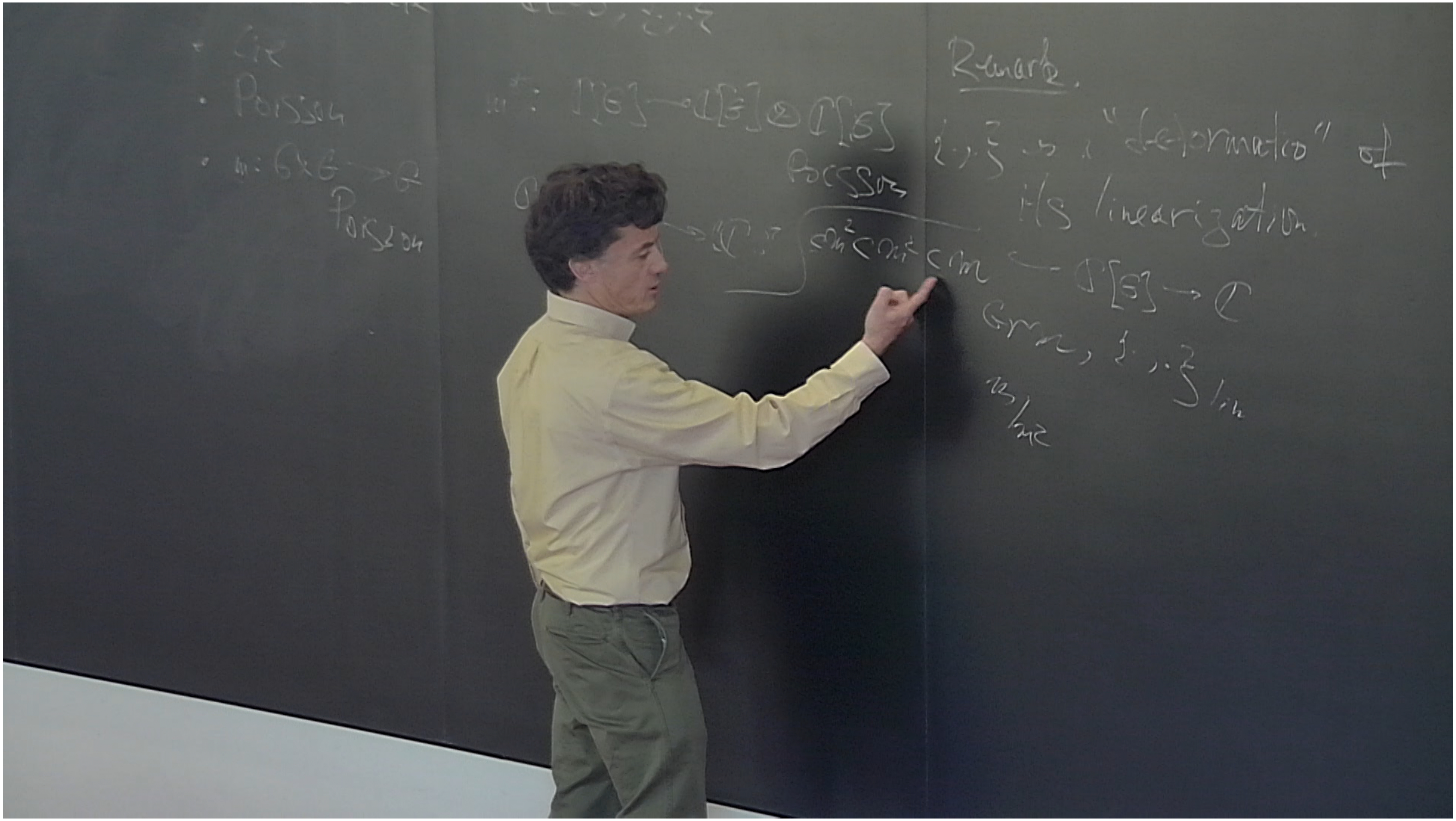
Def. G Poisson-Lie

- Lie
- Poisson
- $m : G \times G \to G$ Poisson

$\mathbb{C}[G]$, $\{\cdot,\cdot\}$

$m^*: \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ Poisson

$G \to \mathbb{C}[G] \to \mathbb{C}[\cdot\,;\cdot]$


Remark.
Poisson
its linearization.

Def. G Poisson-Lie

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Poisson

$\mathbb{C} \to \mathbb{C}[G] \to \mathbb{C}[\ldots]$ $\mathfrak{m}^2 \subset \mathfrak{m}^1 \subset \mathfrak{m}$

Remark.

$\{\cdot,\cdot\}$

i/s

Grm

$\mathbb{C}[G], \{\cdot,\cdot\}$

$m^*: \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$

Poisson

$\mathbb{C} \to \mathbb{C}[G] \to \mathbb{C}[[\cdot]] \quad \mathfrak{m}^2 \subset \mathfrak{m}^2 \subset \mathfrak{m}$

Remark.

$\{\cdot,\cdot\}$ is a "deformation" of its linearization

$\mathfrak{m} \hookrightarrow \mathbb{C}[G] \to \mathbb{C}$

$Gr\,\mathfrak{m}, \{\cdot,\cdot\}_{lin}$

$\mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$

$\mathfrak{g}^*, \mathfrak{g}^*$

Example: ΨDO

$$\left\{ \partial^{\lambda} + u_1(z)\partial^{\lambda-1} + u_2(z)\partial^{\lambda-2} + \dots, \ u_j \in \mathbb{C}((z)), \lambda \in \mathbb{C} \right\}$$

$$(\partial^{\lambda} + u_1\partial^{\lambda-1} + \dots)(\partial^{\mu} + v_1\partial^{\mu-1} + \dots) = \dots\dots$$

$$(\partial^{\lambda} + \dots)^{-1} = \partial^{-\lambda}\Big(1 + \underbrace{\sum_{j=1}^{\infty} u_j \partial^{-j}}\Big)^{-1} \quad \underline{\underline{\text{geom.}}} \ \text{series}$$

$G_{\Psi DO}$: group.

$\mathbb{C}\}$

$\ldots$

lom.

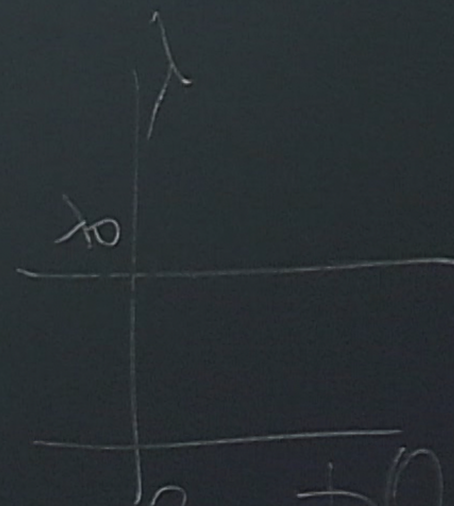
rivls

$$\lambda, \quad \int u_j(z)\, z^m\, dz \quad : \text{"coordinates"}$$

$$\mathbb{C}[G_{\psi DO}]$$

$$\{\lambda = \lambda_0\}$$

$$\{\lambda = n,\ u_{n+1} = u_{n+2} = \dots = 0\} = DO_n$$



$$\{u_i z^m, u_j z^n\}(L) = \mathrm{Res}_{\substack{z=0\\ \partial=0}} \left(\left(L\, \partial^{-\lambda+i-1} z^m \right)_+ L - L \left(\partial^{-\lambda+i-1} z^m L \right)_+ \partial^{-\lambda+j-1} z^n \right)$$

1. $\{u_1 z^m, u_1 z^n\} = \delta_{n,-m}\, n\lambda$

2. $\{$ fnctn vanish on $\{\lambda - h, u_{n+1} = \cdots = 0\}\}$ is a Poisson Ideal

Thm (Khesin – Zakharevich)

- $G_{\Psi DO}$ is a Poisson-Lie group w.r.t. $\{\cdot,\cdot\}_{GD}$

son Ideal

- $\{\lambda=\lambda_0\}$, $\{\lambda=h, u_{n+1}=u_{n+2}=\dots=0\}$ are Poisson Submanifolds

- $T_e^* G_{\Psi DO} = D_{C^*}$

§ Poisson–Lie Group

$\mathbb{C}[G_{\Psi DO}]$

$\mathbb{C}[u_1, u_2, \dots] \quad [\lambda] = \mathbb{C}[G_{\Psi DO}]^c$

$\{u_1 z^n, u_1 z^m\} = \dots$

$\{u_1(z), u_1(w)\} = \delta(z-w)^{(1)}$

$\partial^{-1} + u_1 \partial^{-2} \dots$

$u\lambda$

$\mathbb{C}[G_{\Psi DO}]^c$ Poisson

$\Delta : \mathbb{C}[\ ]^c \longrightarrow \mathbb{C}[\dots]^c \otimes \mathbb{C}[\dots]^c$ Poisson

$\mathbb{C} \to \mathbb{C}[\ ]^c \longrightarrow \mathbb{C}$

$u_i \to 0$

$\lambda \to 0$

$8(z-w)$

u_2

poisson

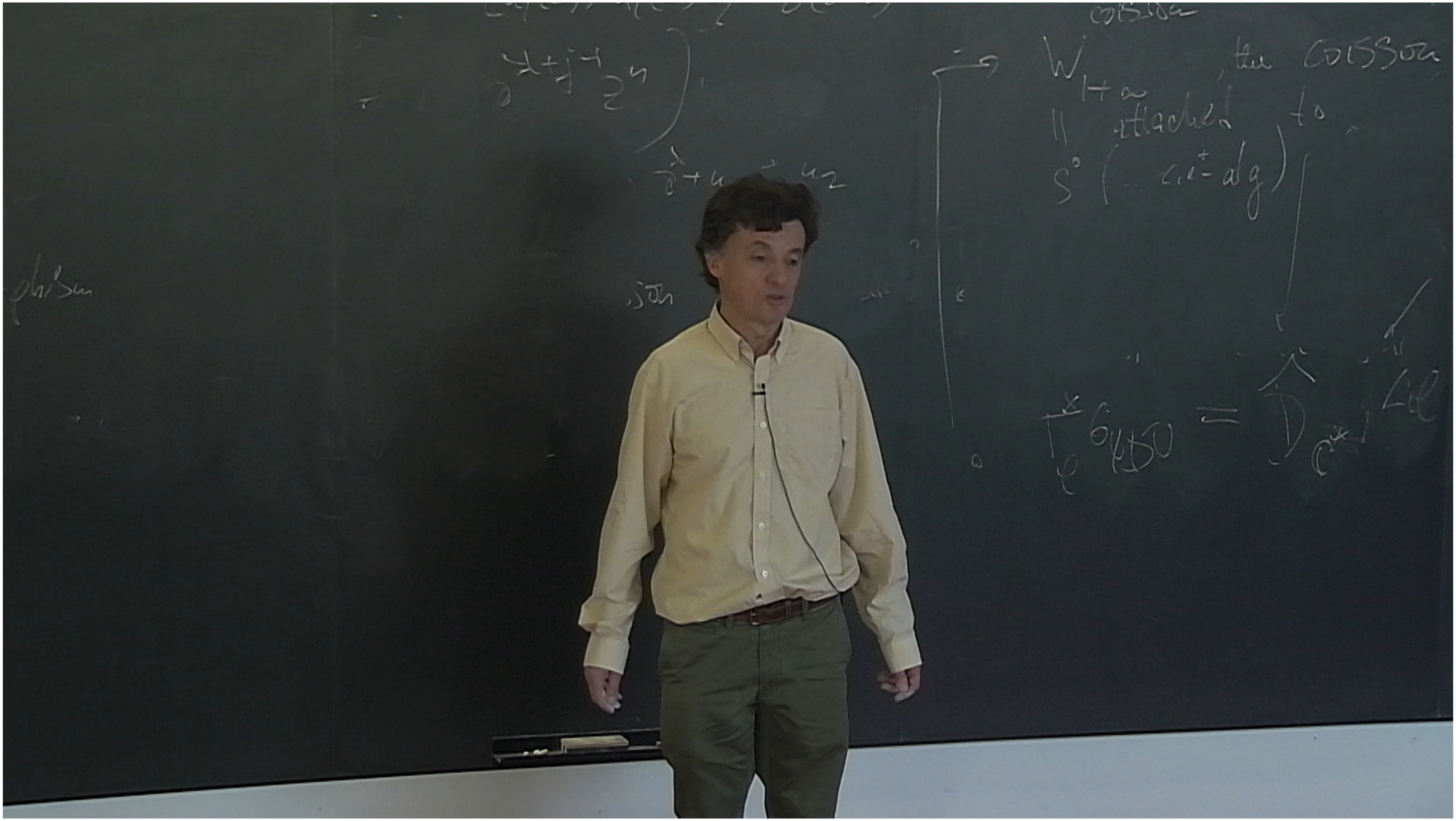
$W_{1+\infty}$, the Poisson algebra

$\|$ attached to

$S^0(\ldots \text{Lie}^+\text{-alg})$

Lie (Lie^+-algebra)

$\Gamma_e^{\times} G_{LHSO} = \hat{D}_{\mathbb{C}^\times}^{\text{Lie}}$



§ Quantization

Def. V is a vertex bi-algebra if

V is a VA [BD, 3.4.16]

$\Delta : V \longrightarrow \underline{V \otimes V}$ a coass. VA morphism

$\mathbb{1} \hookrightarrow V \longrightarrow \mathbb{C}$

$W(\mathfrak{gl}_N)$ à la Lukyanov.

$[BD, 3.4.16]$

$\mathcal{H}(\mathfrak{t}_n)$ $\quad I_1(z) \quad \ldots \quad I_n(z)$

$$\begin{bmatrix} t_1 & \cdots & 0 \\ 0 & \cdots & t_n \end{bmatrix}$$

A morphism

$$I_i(z) I_j(w) = \delta_{ij} \frac{t_n}{(z-w)^2} + \cdots$$

$$(\partial - t_1(z)) \cdots (\partial - t_n(z)) =$$

$W(\mathfrak{gl}_N)$ à la Lukyanov '88

$\mathcal{H}(\mathfrak{t}_n) \qquad I_1(z) \quad \cdots \quad I_n(z)$

$$\begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix}$$

$$I_i(z)\, I_j(w) = \delta_{ij} \frac{t_n}{(z-w)^2} + \cdots$$

$$(\partial - I_1(z)) \cdots (\partial - I_n(z)) = \partial^n + {}^nU_1(z)\,\partial^{n-1} + \cdots + {}^nU_j(z)\,\partial^{n-j} + \cdots {}^nU_n(z)$$

$\Longleftrightarrow$ smooth $t_n \subseteq \mathbb{C}[DO_n]$

$$DO_1^{\times n} \longrightarrow DO_n \curvearrowleft \text{Poisson}$$

$$\dots j(z)\,\partial^{n-j} + \dots + {}^{n}O_n(z)$$

Def. $W_n(t_n) = \langle {}^{n}O_j(z),\ 1 \le j \le n \rangle \subset \mathcal{H}$

Thm. $W_n(t_n)$ is a quantization of $\mathbb{C}[DO_n]^c$

$W(\mathfrak{gl}_r)$ à la Lubyanov '88

$\{W_n(\hbar), n \geq 2\}$: $\forall n \leq m$: $W_n(\hbar) \to W_m(\hbar)$

${}^nU_{ij}(z)$ ${}^nU_{ij}(n) =: P_{ij}(n)$ is a polynomial in n if $n >> 0$

$\Rightarrow$ $\lim_n W_n(\hbar)$ carries a $\mathbb{Z}$-parameter VA structure, is a vertex $\mathbb{C}[\nu, \hbar]$-algebra.

Denote $\mathcal{L}(\nu, \hbar)$

$W(\mathfrak{gl}_N)$ à la Lubyanov '88

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$\{W_n(\hbar), n \geq 2\}$: $\forall n \leq m$: $W_n(\hbar) \to W_m(\hbar)$

$^nU_j(z)$ $^nU_j(w) =: P_{ij}(n)$ is a polynomial in n if $n \gg 0$

$\Rightarrow$ $\lim_n W_n(\hbar)$ carries a $\mathbb{Z}$-parameter VA structure.

Denote $\mathcal{L}(\frac{\nu}{n}, \hbar)$ is a vertex $\mathbb{C}[\nu/\hbar]$-algebra.

Structure:

$$W_{n+m}(t) \longrightarrow W_n(t) \otimes W_m(t)$$

$$\underbrace{(\ )\cdots(\ )}_{n}\underbrace{(\ )\cdots(\ )}_{m}$$

$$\longmapsto \mathcal{L}(V,t) \longrightarrow \mathcal{L}(V,t) \otimes \mathcal{L}(V,t)$$

Counit:

$\mathcal{L}(V, \frac{t}{h}) \simeq \mathbb{C}[U_1, U_2, \ldots][[t/h]] \xrightarrow{\varepsilon} \mathbb{C}$

'88

$W_n(\frac{t}{h}) \longrightarrow W_m(\frac{t}{h})$

= a poly ... in n if n >> 0

Thm.

(1) ($\mathcal{L}(V, \frac{t}{h}), \Delta, \varepsilon$) is a vertex bi-algebra structure.

$\mathbb{C}[V, t/h]$-algebra

(2) mod($\frac{t}{h}$) $\rightsquigarrow \mathbb{C}[G_4 D_0]$

(3)

'88

Recall:

$\mathcal{L}(\nu, \frac{t}{h}) \xrightarrow{\sim} \mathbb{C}[u_1, u_2, \ldots][\nu, t_h] \xrightarrow{\varepsilon} \mathbb{C}$ $\quad W_n(\frac{t}{h}) \longrightarrow W_m(\frac{t}{h})$

= a polyn ... in n if $n >> 0$

Thm.

(1) $(\mathcal{L}(\nu, t_h), \Delta, \varepsilon)$ is a vertex bi-algebra

structure.

$\mathbb{C}[\nu, t_h]$-algebra

(2) mod $(\frac{t}{h}) \longmapsto \mathbb{C}[G_{4D0}]^{c}$ $\mathbb{Z}$

(3) $\exists$ $\mathcal{L}(\nu, t_h) \longrightarrow W_n(\frac{t}{n})$ $\quad n \geq 3$

$\nu \longmapsto n$

Recall:

$\mathcal{L}(\nu, \frac{1}{h}) \cong \mathbb{C}[u_1, u_2, \dots][[\frac{1}{h}]] \xrightarrow{e} \mathbb{C}$

Thm.

(1) $(\mathcal{L}(\nu,\hbar), \Delta, \varepsilon)$ is a vertex bi-algebra

(2) mod $(\frac{1}{h})$ $\mapsto \mathbb{C}[G_4 DO]^{c}$

$\mathcal{L}(\nu,\hbar) \twoheadrightarrow W_n(\frac{1}{n})$ $n \geq 3$

(3) $\exists$

$\nu \mapsto n$

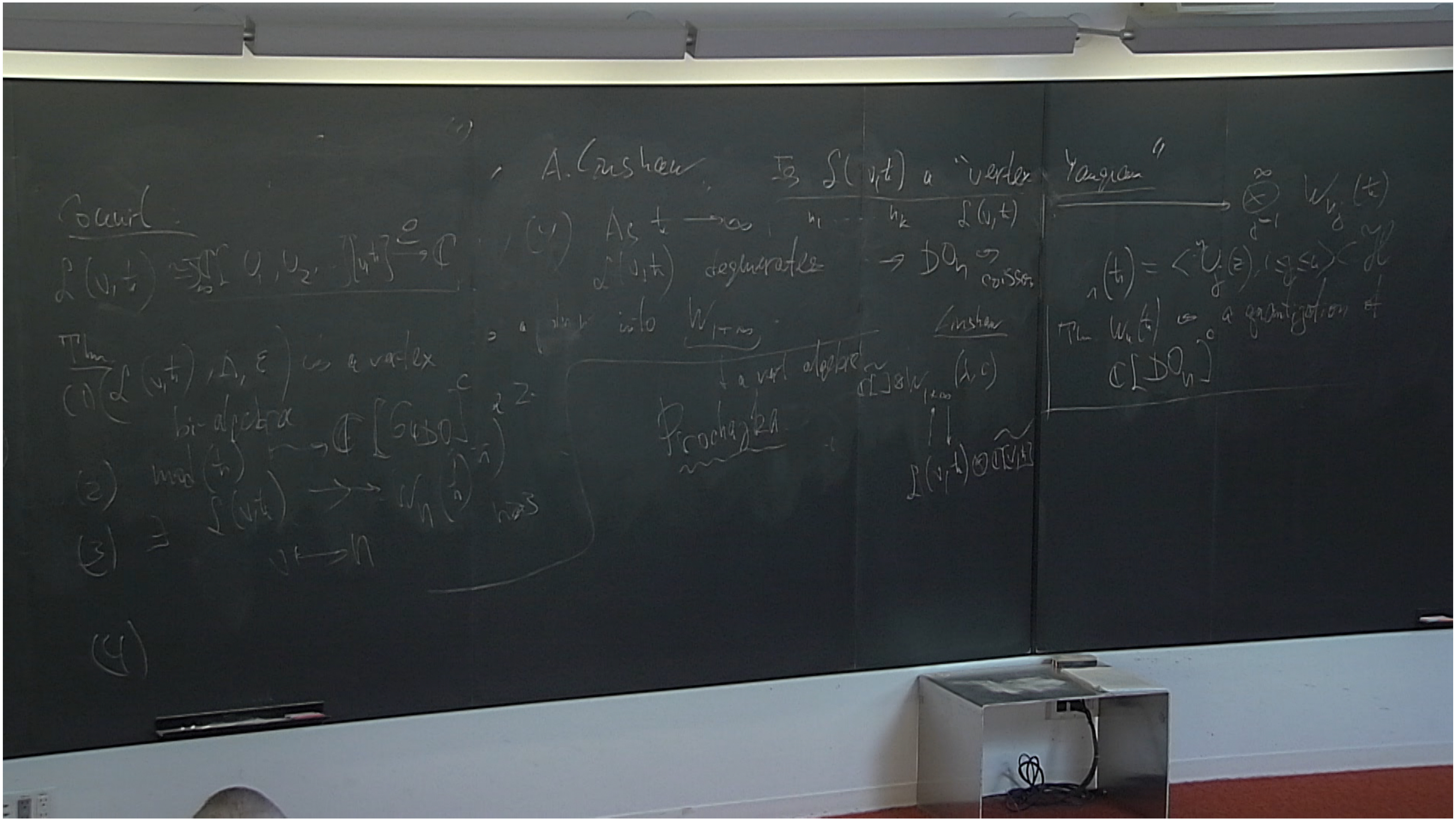
(4)

(4) As $\hbar \to \infty$ $\Rightarrow$ mod $\hbar$

$\mathcal{L}(\nu,\hbar)$ degenerates $\to$ DO

a ... into $W_{1+\infty}$

a vert. algebra


A. Linshaw
Thm
is a vertex
bi-algebra
As t → ∞
degenerates
Prochazka
Yangian