Title: Fusion Hall algebra and shuffle conjectures

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Collection: Cohomological Hall Algebras in Mathematics and Physics

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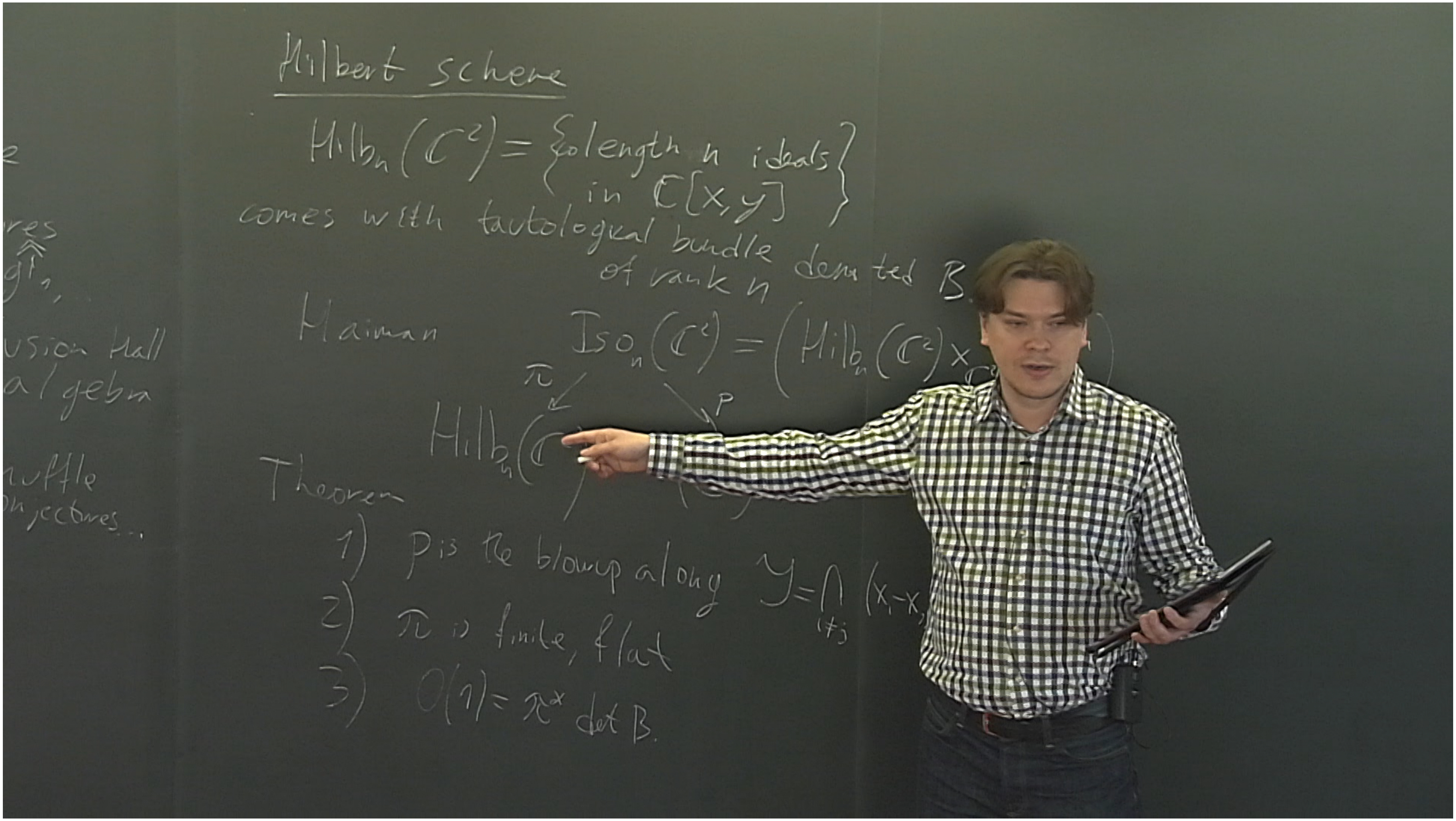
Abstract: The classical Hall algebra of the category of representations of one-loop quiver is isomorphic to the ring of symmetric functions, and Hall-Littlewood polynomials arise naturally as the images of objects. I will talk about a second "fusion" product on this algebra, whose structure constants are given by counting of bundles with nilpotent endomorphisms on P^1 with restrictions at 0, 1 and infinity. The two products together make up a structure closely related to the elliptic Hall algebra. In the situations when bundles can be explicitly enumerated, I will explain how this leads to q,t-identities conjectured by combinatorists, such as the shuffle conjecture and its generalizations. This is a joint project with Erik Carlsson.

Plan

1) Hilbert scheme
2) Algebraic structures
   ~ EHA, $\hat{\hat{gl}}_1$, ...
3) Emergence of Fusion Hall algebra
4) Applications, shuffle conjectures...


Hilbert scheme
Hilb_n(C^2) = {length n ideals in C[x,y]}
comes with tautological bundle of rank n denoted B.
Haiman
Iso_n(C^2) = (Hilb_n(C^2) x
π
p
Hilb_n(C
Theorem
1) p is the blowup along Y = ∩_{i≠j} (x_i - x_j
2) π is finite, flat
3) O(1) = π^* det B.

...eme

...ctures
..., $g^{\uparrow}{}_{n}$, ...

Fusion Hall algebra

shuffle conjectures...

Hilbert scheme

$$\mathrm{Hilb}_n(\mathbb{C}^2) = \left\{ \begin{array}{c} \text{length } n \text{ ideals} \\ \text{in } \mathbb{C}[x,y] \end{array} \right\}$$

comes with tautological bundle denoted $\mathcal{B}$.
of rank n

Haiman $\quad \mathrm{Iso}_n(\mathbb{C}^2) = \left(\mathrm{Hilb}_n(\mathbb{C}^2) \times_{\mathbb{C}^{2n}/S_n} \mathbb{C}^{2n} \right)_{red}$

$$\begin{array}{ccc} & \mathrm{Iso}_n(\mathbb{C}^2) & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle p} \\ \mathrm{Hilb}_n(\mathbb{C}^2) & & (\mathbb{C}^2)^n \end{array}$$

Theorem

1) p is the blowup along $\mathcal{Y} = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j)$
2) π is finite, flat
3) $\mathcal{O}(1) = \pi^* \det \mathcal{B}$.

by 2)
is a ve...
is a bu...
e...

by 2) $\pi_* \mathcal{O} =: P$

is a vector bundle of rank $n!$

is a bundle of rings

endowed with a

regular S_n-rep.

$\mathbb{C}^{2n}/S_n \quad \mathbb{C}^{2n}\big)_{red}$

$=: P$

dle of rank $n!$

rings

with a

S_n-rep.

Algebraic structures

Fock space $= \bigoplus_{n \geq 0} K_{\mathbb{C}^\times_q \times \mathbb{C}^\times_t}(\mathrm{Hilb}_n \mathbb{C}^2) = F$

BKR map $F \to \mathrm{Sym} \otimes \mathbb{Q}(q,t)$

symmetric functions

$\mathcal{E}$ equivariant vector bundle $\to \chi\left(R\mathcal{H}om(P, \mathcal{E}) \otimes \mathrm{sign}\right)$

bigraded vector space w/ S_n-action

χ sends irreducible S_n-rep for partition λ in degree (i,j) $\to$ $q^i t^j s_\lambda$

Schur function

BKR

Algebraic structures:

1) Nakajima operators (given by certain correspondences)

$$K_{\mathbb{C}_q^\times \times \mathbb{C}_t^\times}(\mathrm{Hilb}_n) \to K(\mathrm{Hilb}_{n+k})$$

form an action of Heisenberg

On Sym these are easy

- multiplication operators
- the adjoints with respect to the Hall scalar products.

these are easy

tion operators

nts with respect
to the Hall scalar
product.

2) tensor product

$\mathcal{E}, \mathcal{E}' \longrightarrow \mathcal{E} \otimes \mathcal{E}'$

is not easy on Sym.

$K(\mathrm{Hilb}_n) \times K(\mathrm{Hilb}_n) \to K(\mathrm{Hilb}_n)$

f. d. vector space.

Let Z_λ be the torus fixed point for partition λ → ideal generated by $x^i y^j : (i,j) \notin \lambda$

$\mathcal{O}_{Z_\lambda} \in K(\mathrm{Hil}$

$\mathcal{O}_{Z_\lambda}$ are ortho

$\mathcal{O}_{Z_\lambda} \otimes \mathcal{O}_{Z_\lambda} = \Lambda$

$|\lambda| = n$

It turns out

$$\mathrm{BKR}(\mathcal{O}_{Z_\lambda}) \sim \tilde{H}_\lambda[X; q, t]$$

multiple

modified Macdonald

$$X = (x_1, x_2, \ldots)$$

$$\chi(\text{“}P|_{Z_\lambda}\text{”})$$

Z_λ

$|\lambda| = n$

onal

$L \otimes \mathcal{O}_{Z_\lambda}$

tangent
bundle.

it turns out

multiple

$$BKR(\mathcal{O}_{Z_\lambda}) \sim \tilde{H}_\lambda[X; q, t]$$

modified Macdonald

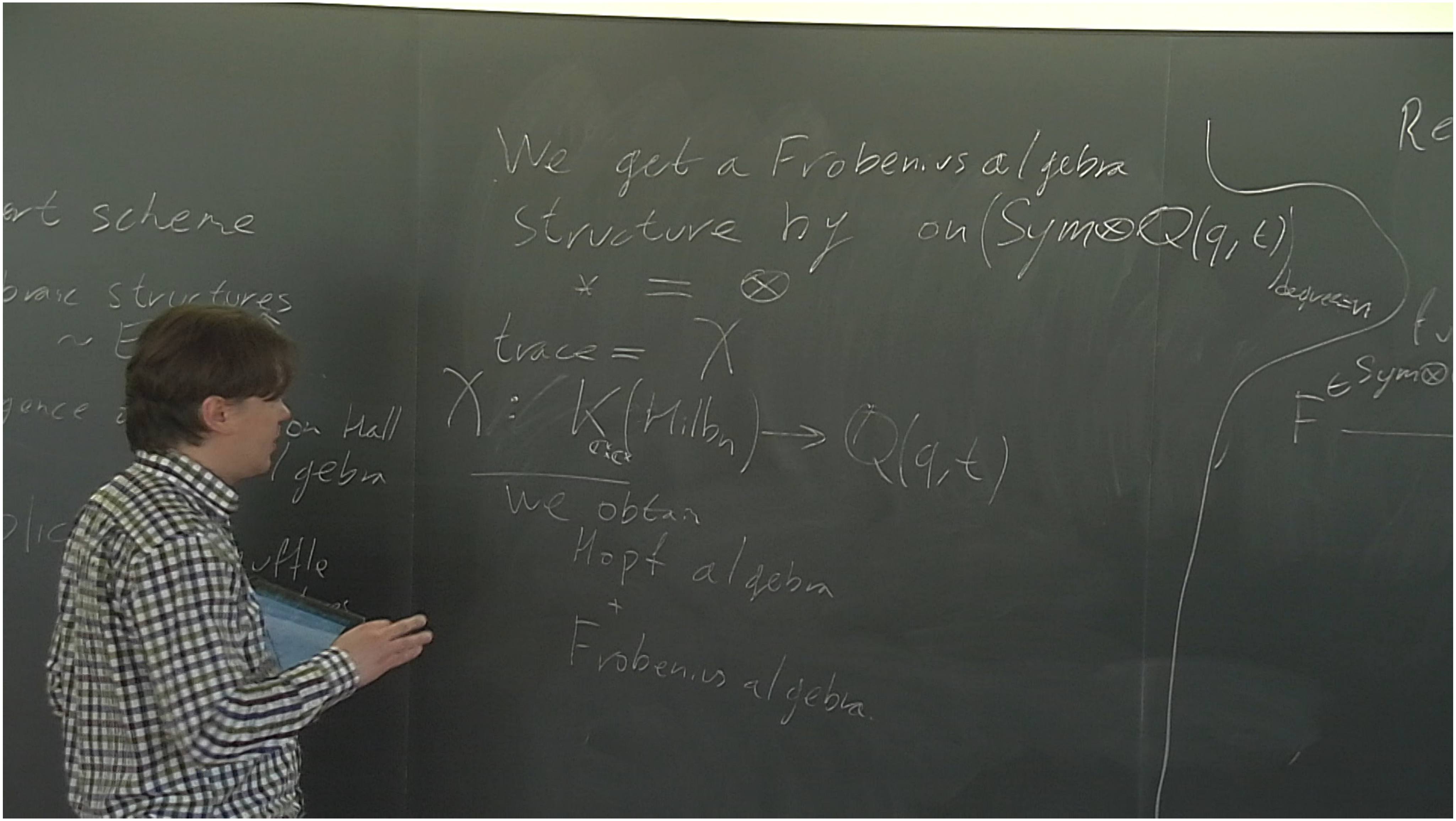
$$X = (x_1, x_2, \dots)$$

$$\chi(\text{``}P|_{Z_\lambda}\text{''})$$

to compute

$$BKR\left(BKR^{-1}(f) \otimes BKR^{-1}(g)\right)$$

we need to express f, g
in the Macdonald basis,
add the products of coefficients


We get a Frobenius algebra
structure by on (Sym⊗Q(q,t))
degree=n
* = ⊗
trace = χ
χ : K(Hilb_n) → Q(q,t)
We obtain
Hopf algebra
+
Frobenius algebra.

ize

metric

us.

ult

Remark 2

consider the algebra of operators generated by the Heisenberg operators

and $\otimes \Lambda^k B$

$\cong$ half of the elliptic Hall algebra

a.k.a shuffle algebra

a.k.a toroidal gl_1

$-t$

(q,t)

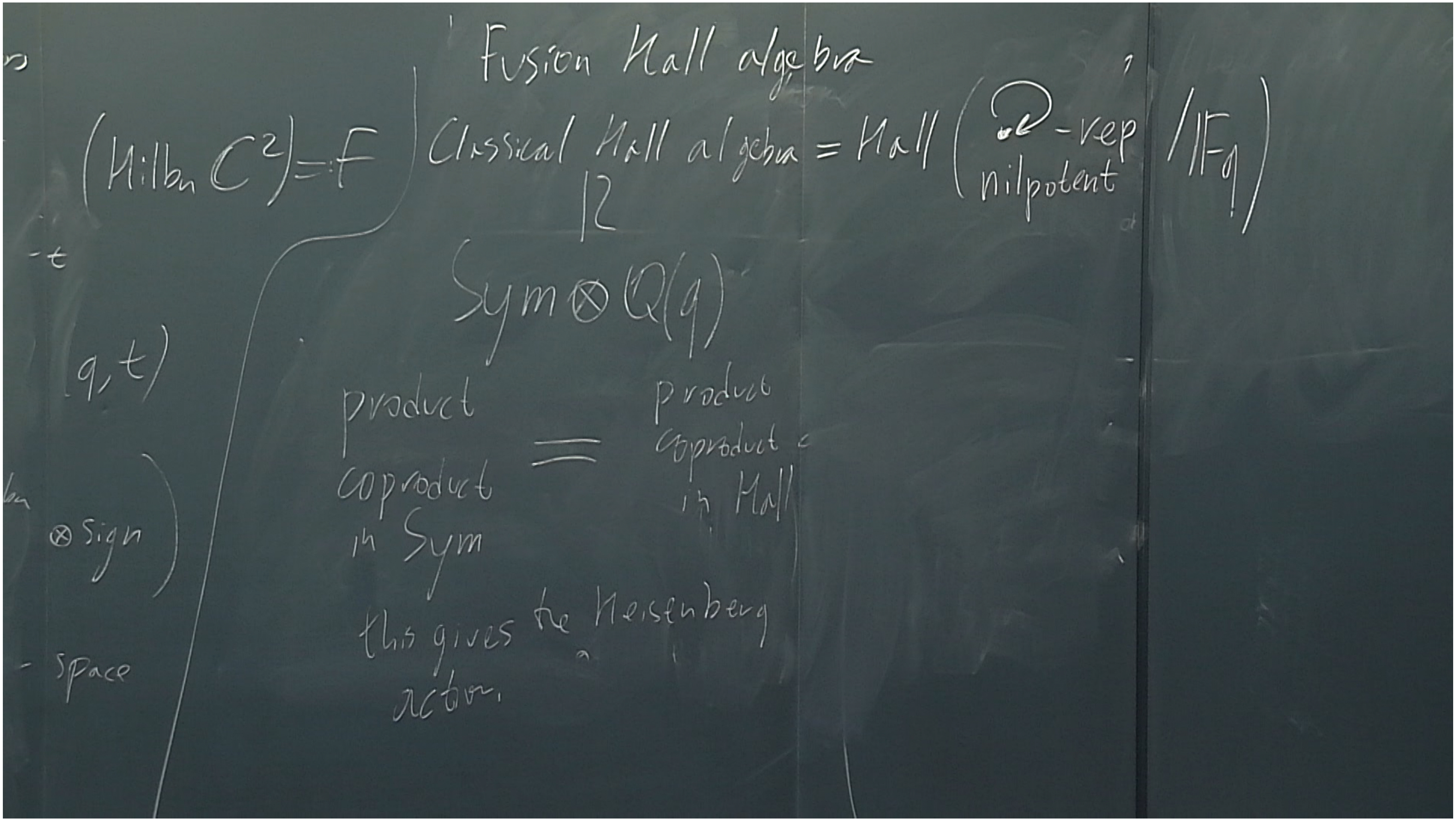
$\otimes$ sign

- space

$(\mathrm{Hilb}_n \mathbb{C}^2) =: F$

Fusi

Classical


Fusion Hall algebra
$(\mathrm{Hilb}_n \mathbb{C}^2) =: F$
Classical Hall algebra = Hall (nilpotent -rep / $\mathbb{F}_q$)
$\cong$
Sym $\otimes$ $\mathbb{Q}(q)$
product coproduct in Sym
=
product coproduct in Hall
this gives the Heisenberg action.
(q, t)
$\otimes$ sign
space

eme

uctures

$\dots, \hat{g}, \dots$

Fusion Hall algebra

shuffle conjectures...

Idea

∂-bundles on $C/\mathbb{F}_q$ projective smooth curve

Let $s_1, \dots, s_m \in C(\mathbb{F}_q)$

$$\Rightarrow \left\{ (\mathcal{E}, \theta) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ vb on } C \\ \theta : \mathcal{E} \to \mathcal{E} \\ \text{nilpotent} \end{array} \right\}$$

$$\underbrace{\mathrm{Hall}(\partial) \otimes \mathrm{Hall}(\partial) \otimes \mathrm{Hall}(\partial)}_{m}$$

We

Bun

$(\mathbb{F}_q)$

if we take all bundles upstairs, pushforward is infinite.

We need a "measure" on $\mathrm{Bun}(C,$ [illegible]$)$

we take all
undles upstairs,
h forward is infinite,
ed a "measure" on
$\mathbb{C}$)

Def

a measure on $\mathrm{Bun}(C)_\tau$ is a full subcategory of $\mathrm{Bun}(C)$ such that

1) if $X \in \tau \Rightarrow$ any $X_0 \subset X$, X_0 is in τ

2) $0 \to X \to X' \to X'' \to 0$
$X, X'' \in \tau \Rightarrow X' \in \tau$

$(\mathrm{Hilb}_n \mathbb{C}^2) =$

$(C))_\tau$

ory of $\mathrm{Bun}(C)$

$X_0 \subset X$

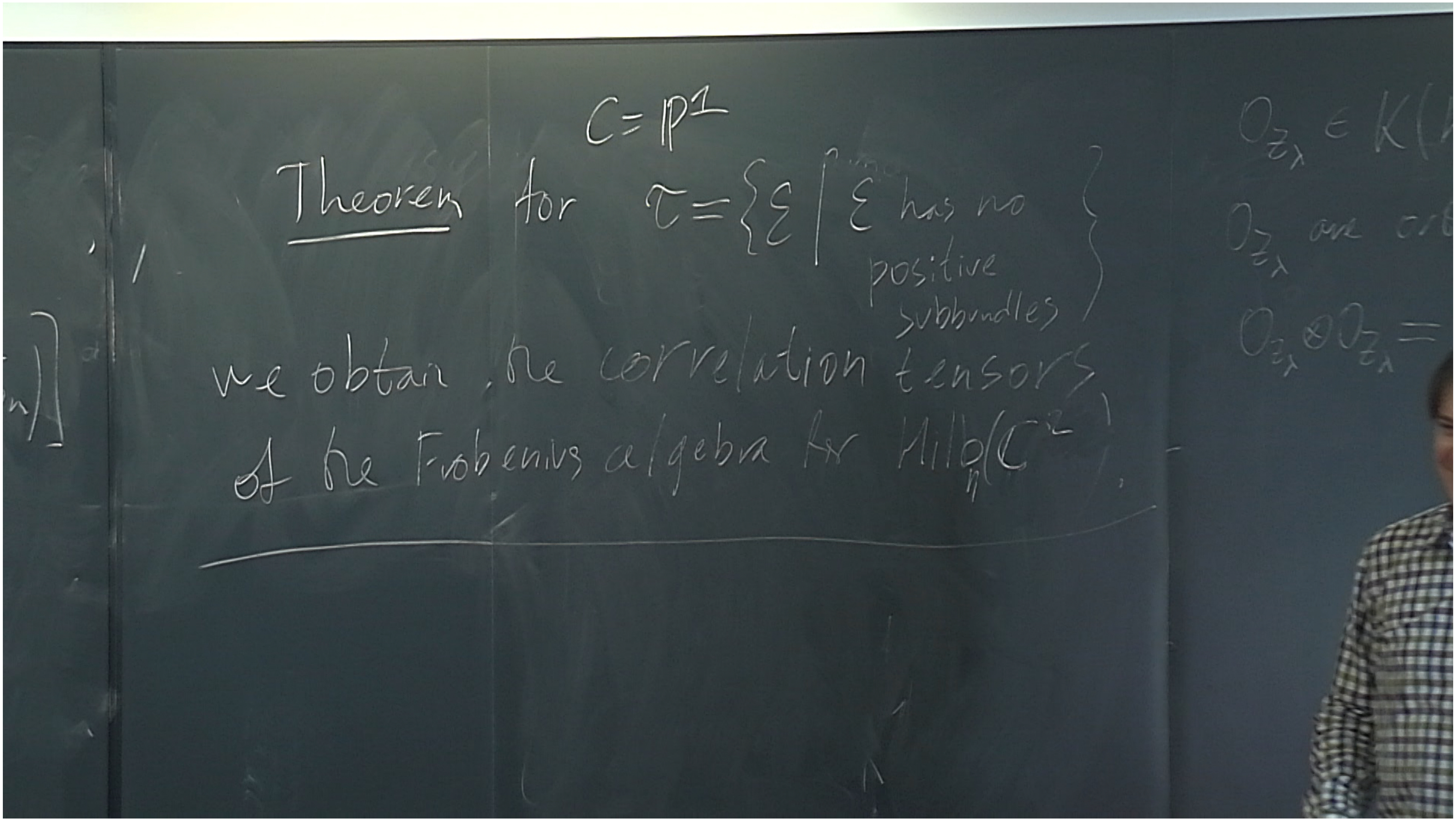
is in τ

Given such a measure

we write

$$\sum_{\substack{(\mathcal{E},\theta)\\ \mathcal{E}\in\tau}} \frac{t^{-\deg \mathcal{E}}}{|\mathrm{Aut}(\mathcal{E})|} \left[\mathcal{E}|_{S_1}, \theta(S_1)\right] \otimes \dots \left[\mathcal{E}|_{S_m}, \theta(S_m)\right]$$

we obtain element of $\mathbb{R}[[t]] \otimes \mathrm{Hall}^{\otimes m}$


$C = \mathbb{P}^1$
Theorem for $\tau = \{ \mathcal{E} \mid \mathcal{E}$ has no positive subbundles $\}$
we obtain the correlation tensors of the Frobenius algebra for $Hilb_n(\mathbb{C}^2)$.
$\mathcal{O}_{Z_\lambda} \in K($
$\mathcal{O}_{Z_\lambda}$ are
$\mathcal{O}_{Z_\lambda} \otimes \mathcal{O}_{Z_\lambda} =$

$C = \mathbb{P}^1$

Theorem for $\tau = \{ \mathcal{E} \mid \mathcal{E}$ has no positive subbundles $\}$

we obtain the correlation tensors of the Frobenius algebra for $\mathrm{Hilb}_n(\mathbb{C}^2)$.

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for C arbitrary, given τ as in $\mathbb{P}^1$.

$\exists\ F_C \in \mathrm{Hall} \otimes \mathbb{Q}(q,t)$ such that

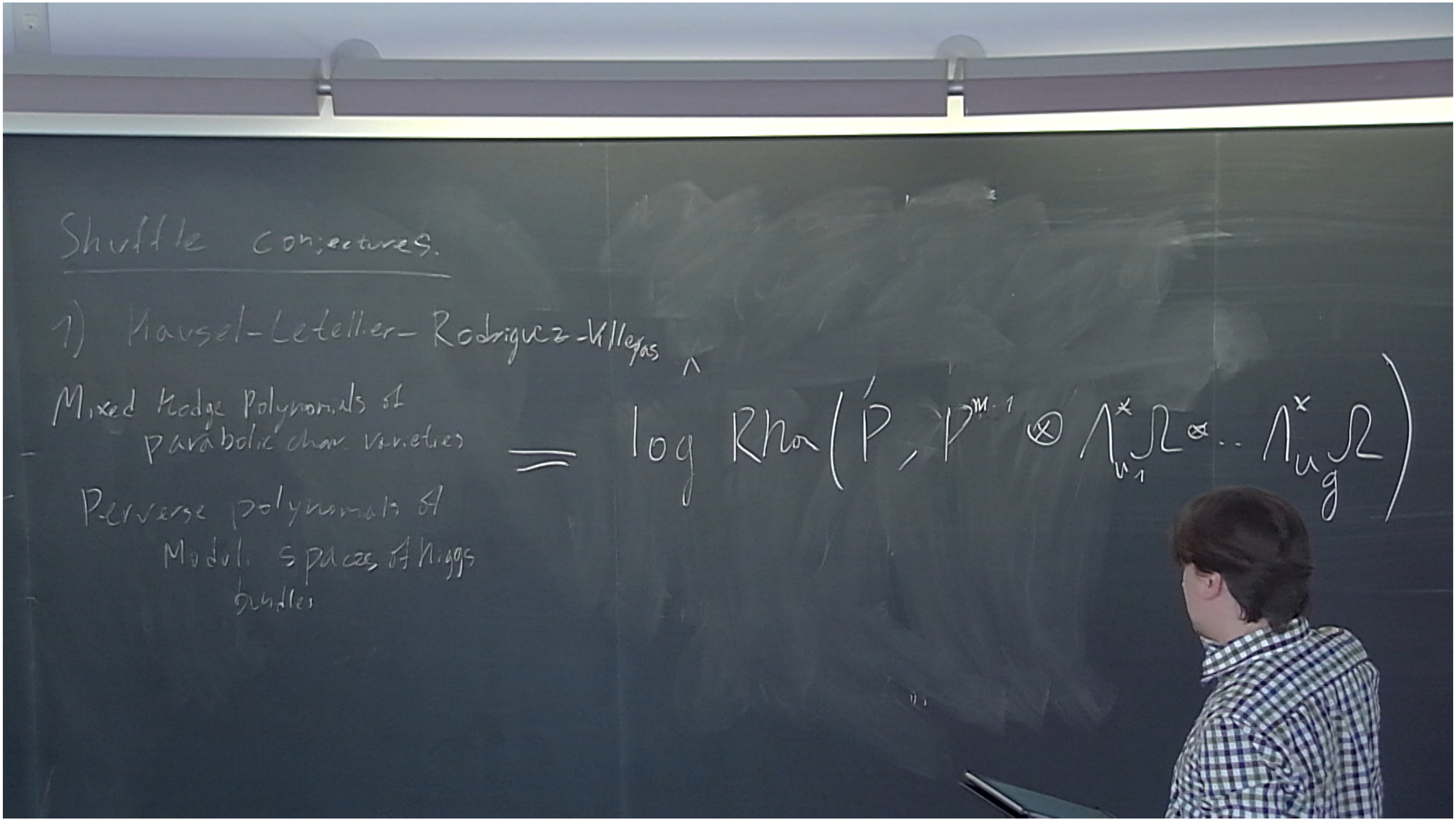
$$\Omega^*_C = \Omega^*_{\mathbb{P}^1} \cdot F_C .$$

Shuffle conjectures.

1) Hausel-Letellier-Rodriguez-Villegas

Mixed Hodge Polynomials of parabolic char. varieties $=$ log Rh

Perverse polynomials of Moduli spaces of Higgs bundles


Shuffle conjectures.
1) Hausel-Letellier-Rodriguez-Villegas
Mixed Hodge Polynomials of parabolic char. varieties
Perverse polynomials of Moduli spaces of Higgs bundles
= log Rhom(P, P ⊗ Λ*Ω ⊗ .. Λ*Ω)