Title: Topological Holography Course - Lecture 5

Speakers: Kevin Costello

Collection: Topological Holography Course (Costello)

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Loday Quillen Tsygan

A associative graded algebra

$A \otimes gl_N$ graded Lie algebra

$C^*(A \otimes gl_N)$ dg comm. algebra

$\longrightarrow$ Local ops of gl_N gauge theory are of this form

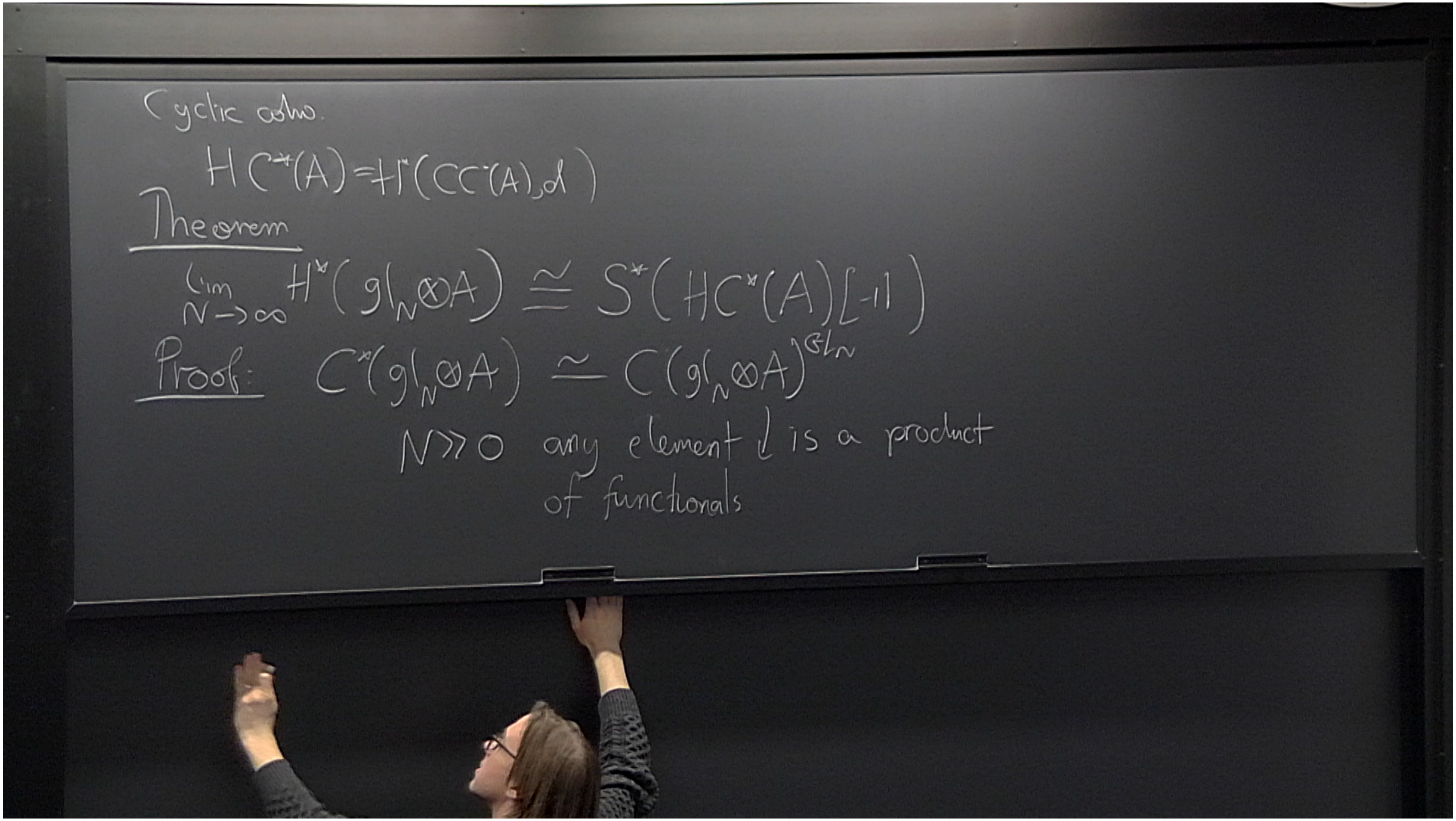
$\hookrightarrow$ Local ops of $\mathfrak{gl}_N$ gauge theory are of this form

Define $CC^n(A) = \{$ graded cyc. symmetric/anti maps $A^{\otimes n} \longrightarrow \mathbb{C} \}$

$d : CC^n(A) \longrightarrow CC^{n+1}(A)$

If $\varphi : A^{\otimes n} \longrightarrow \mathbb{C}$

$$(d\varphi)(a_1, \dots, a_{n+1}) = \varphi(a_1 a_2, a_3, \dots, a_{n+1}) \mp \varphi(a_1, a_2 a_3, \dots) \pm \varphi(a_n a_1, a_2, \dots, a_{n-1})$$


Cyclic coho.
$HC^*(A) = H^*(CC^*(A), d)$
Theorem
$\lim_{N \to \infty} H^*(gl_N \otimes A) \cong S^*(HC^*(A)[-1])$
Proof: $C^*(gl_N \otimes A) \simeq C(gl_N \otimes A)^{GL_N}$
$N \gg 0$ any element ℓ is a product
of functionals

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$N \gg 0$ any element l is a product of functionals

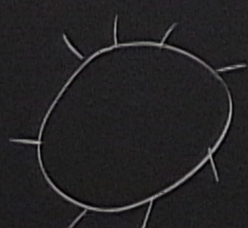
which send

$(a_1 \otimes m_1, \dots, a_n \otimes m_n) \longrightarrow$ Antisym. of

$a_i \in A$ $\qquad \varphi(a_1, \dots, a_n) \operatorname{Tr} m_1 \dots m_n$

$m_i \in gl_N$ $\qquad \varphi \in CC^n(A)$

Computation Lie algebra cochains differential = cyclic differential



$\pm \varphi(a_n a_1, a_2, \ldots, a_{n-1})$

Suppose A = free commutative alg on even variables x_i ? space-time
odd variables ε_α

Then, $HC^*(A)$ can be computed as the coho. of the cochain complex

$\mathbb{C}[\partial_{x_i}, \varepsilon_\alpha, \eta^i, \partial_{\varepsilon_\alpha}, t]$

$$\pm \varphi(a_n a_1, a_2, \ldots, a_{n-1})$$

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Then, $HC^*(A)$ can be computed as the coho. of the cochain complex

$$\mathbb{C}[\partial_{x_i}, \varepsilon_\alpha, \underset{\text{odd}}{\eta^i}, \underset{\text{even}}{\partial_{\varepsilon_\alpha}}, t]$$

differential is

$$\pm \varphi(a_n a_1, a_2, \ldots, a_{n-1})$$

Suppose A = free commutative alg on even variables x_i (? space-time)

odd variables ε_α

Then, $HC^*(A)$ can be computed as the coho. of the cochain complex

$$\mathbb{C}[\partial_{x_i}, \varepsilon_\alpha, \eta^i, \gamma^\alpha, t]$$

(η^i odd, γ^α even)

differential is $d = \left(\sum \partial_{x_i} \frac{\partial}{\partial \eta^i} + \frac{\partial}{\partial \varepsilon_\alpha} \frac{\partial}{\partial \gamma^\alpha} \right) t$

$$\pm \varphi(a_n a_1, a_2, \ldots, a_{n-1})$$

? space-time

Suppose A = free commutative alg on even variables x_i

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differential is $d = \left(\sum \partial_{x_i} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \varepsilon_\alpha} \frac{\partial}{\partial \gamma^\alpha} \right) t$

$$\pm\, \varphi(a_n a_1, a_2, \dots, a_{n-1})$$

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Then, $HC^*(A)$ can be computed as the coho. of the cochain complex

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η^i odd, γ^α even, t even

differential is $d = \left(\sum \partial_{x_i} \frac{\partial}{\partial \eta^i} + \frac{\partial}{\partial \varepsilon_\alpha} \frac{\partial}{\partial \gamma^\alpha} \right) t$

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$A = \mathbb{C}[x_1, \dots, x_n]$

Then $HC^*(A)$

$= \mathbb{C}[\partial^i, \eta^i, t]$

$d\eta^i = t\,\partial^i$

differential is $d = \left(\sum \partial_{x_i} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \varepsilon_\alpha} \frac{\partial}{\partial \gamma \alpha} \right) t$

$A = \mathbb{C}[x_1, \dots, x_n]$

Then $HC^*(A) \Longleftrightarrow C^*(A \otimes gl_N)$

$= \mathbb{C}[\partial^i, \eta^i, t]$

$d\eta^i = t\,\partial^i$

$\partial^{i_1} \dots \partial^{i_n} \to$ linear functional

$f(x_i)\, m \longmapsto (\partial^{i_1} \dots \partial^{i_n} f)(0)\, \mathrm{Tr}\, m$

differential is $d = \left(\sum \partial_{x_i} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \varepsilon_\alpha} \frac{\partial}{\partial \gamma_\alpha} \right) t$

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Then $HC^*(A) \Longleftrightarrow C^*(A \otimes gl_N)$

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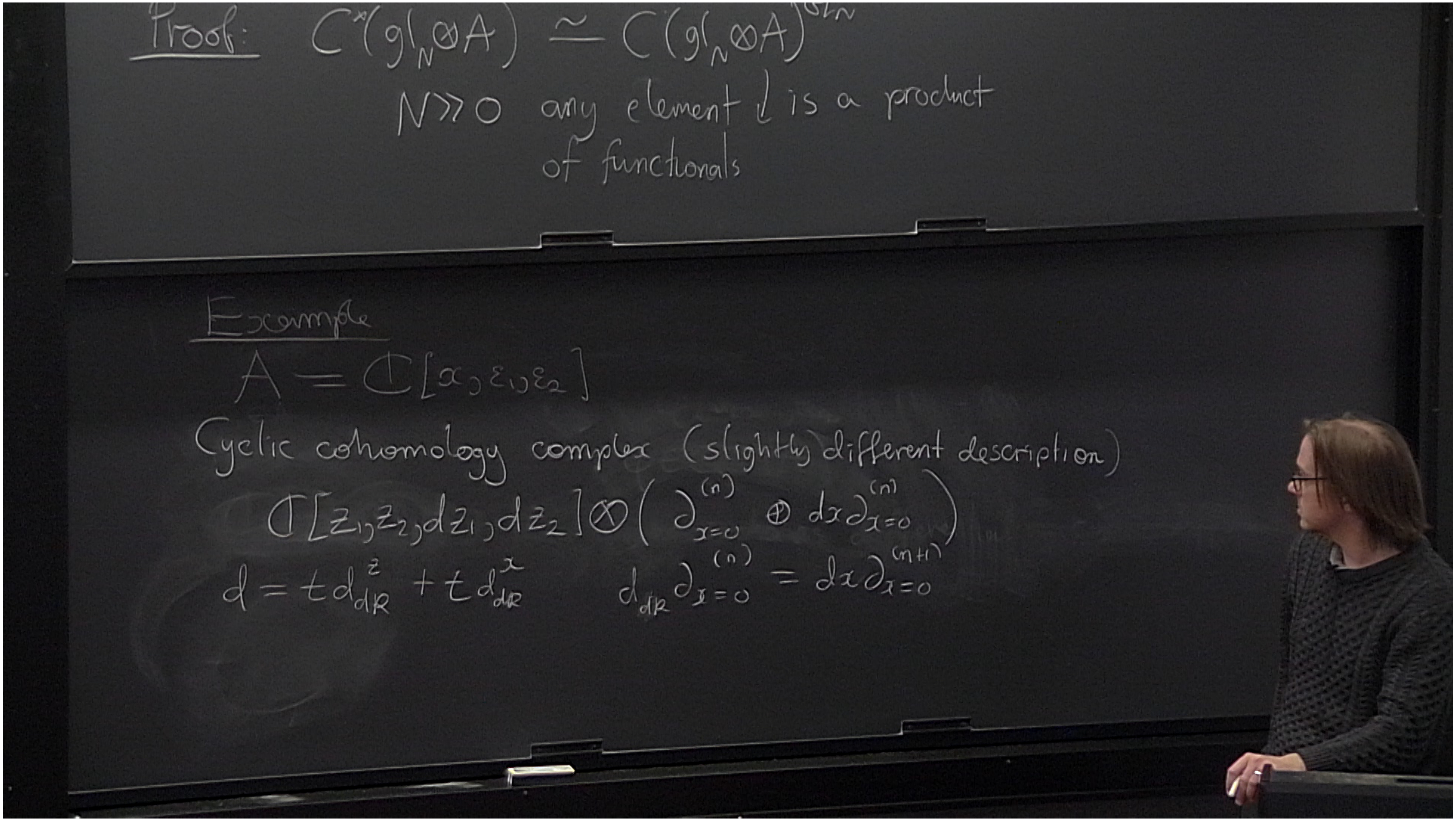
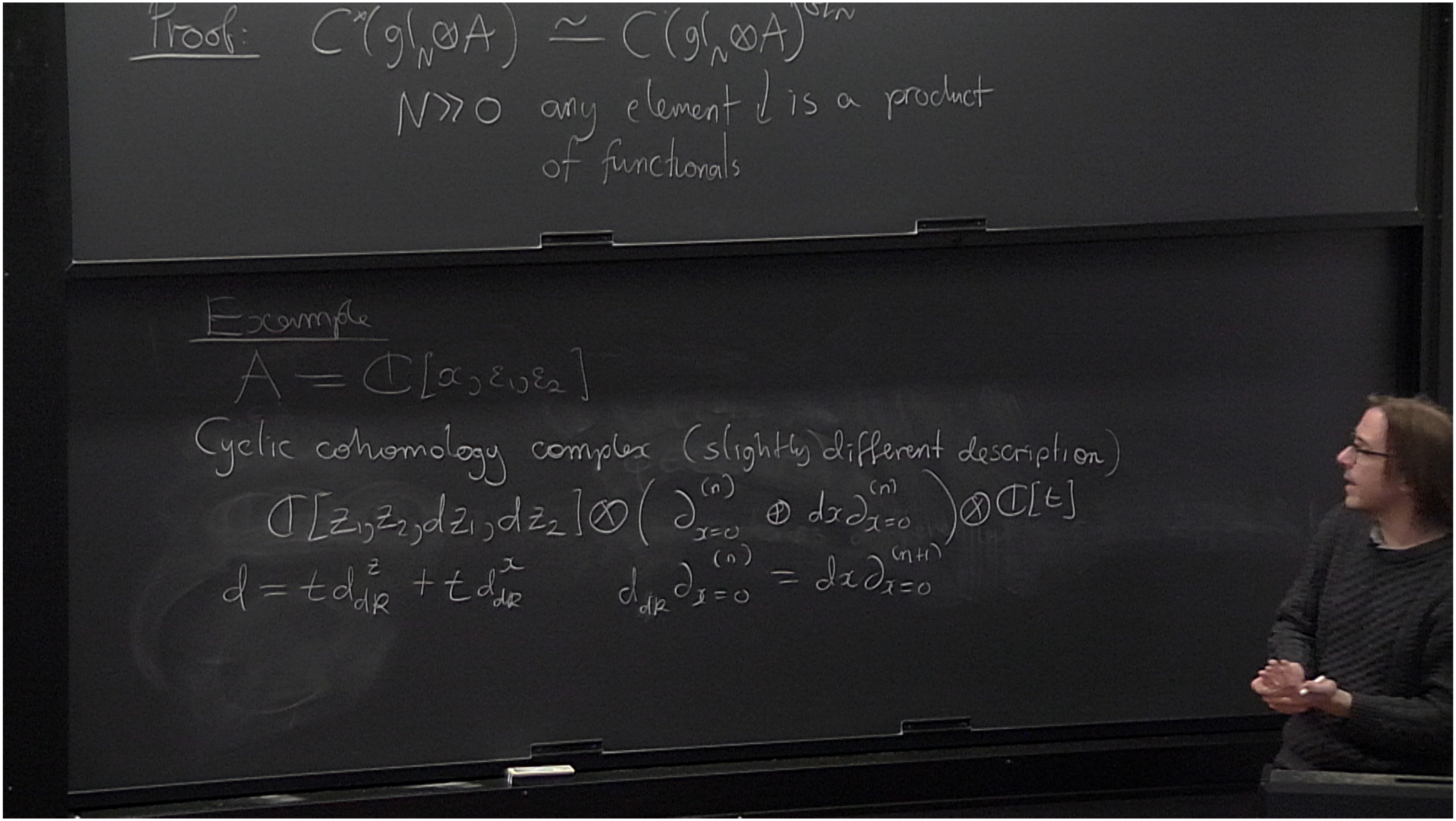
$\partial^{i_1} \cdots \partial^{i_n} \longrightarrow$ linear functional

$f(x_i)\, m \longmapsto (\partial^{i_1} \cdots \partial^{i_n} f)(0)\, \mathrm{Tr}\, m$

Something with 1 $\eta \longrightarrow$ quadratic functional

$\partial^I \eta^i \longrightarrow \{(f_1 \otimes m_1, f_2 \otimes m_2) \mapsto \mathrm{Tr}(m_1 m_2)(f_1 D f_2)(0)\}$


Proof: $C^*(gl_N \otimes A) \simeq C^*(gl_N \otimes A)^{GL_N}$
$N \gg 0$ any element is a product of functionals
Example
$A = \mathbb{C}[x, \varepsilon_1, \varepsilon_2]$
Cyclic cohomology complex (slightly different description)
$\mathbb{C}[z_1, z_2, dz_1, dz_2] \otimes \left(\partial^{(n)}_{x=0} \oplus dx\, \partial^{(n)}_{x=0} \right)$
$d = t\, d^{z}_{dR} + t\, d^{x}_{dR}$
$d_{dR} \partial^{(n)}_{x=0} = dx\, \partial^{(n+1)}_{x=0}$



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Closed things:

$$F(z_i)\, dz_1 dz_2\, dx\, \partial^{(n)}_{x=0}$$

$\mathbb{C}[z_1, z_2, dz_1, dz_2] \otimes (\partial_{x=0} \oplus dx\,\partial_{x=0}) \otimes \mathbb{C}[\varepsilon]$

$d = t\,d^z_{dR} + t\,d^x_{dR} \qquad d_{dR}\,\partial^{(n)}_{x=0} = dx\,\partial^{(n+1)}_{x=0}$

Closed things.

$F(z_i)\,dz_1\,dz_2\;dx\,\partial^{(n)}_{x=0}$

Conceptual Understanding

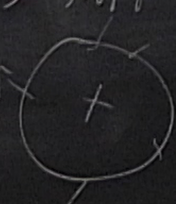
Closed string field theory for B-model on $\mathbb{C}^3$

has fields $\Omega^{*,*}(\mathbb{C}^3)[t] \qquad \bar{\partial} + t\partial$

Brane wrapping $\mathbb{C} \subseteq \mathbb{C}^3$

Any closed string field which is supported at $0 \in \mathbb{C}$

$\Longrightarrow$ single trace operator



differential is $d = \left(\sum \partial_{x_i} \frac{\partial}{\partial \eta^i} + \frac{\partial}{\partial \varepsilon_\alpha} \frac{\partial}{\partial \gamma^\alpha} \right) t$

LQT Theorem

BRST coho. of single trace operators

$\cong$ Coho. of closed string fields supported at a point.

Chiral Alg

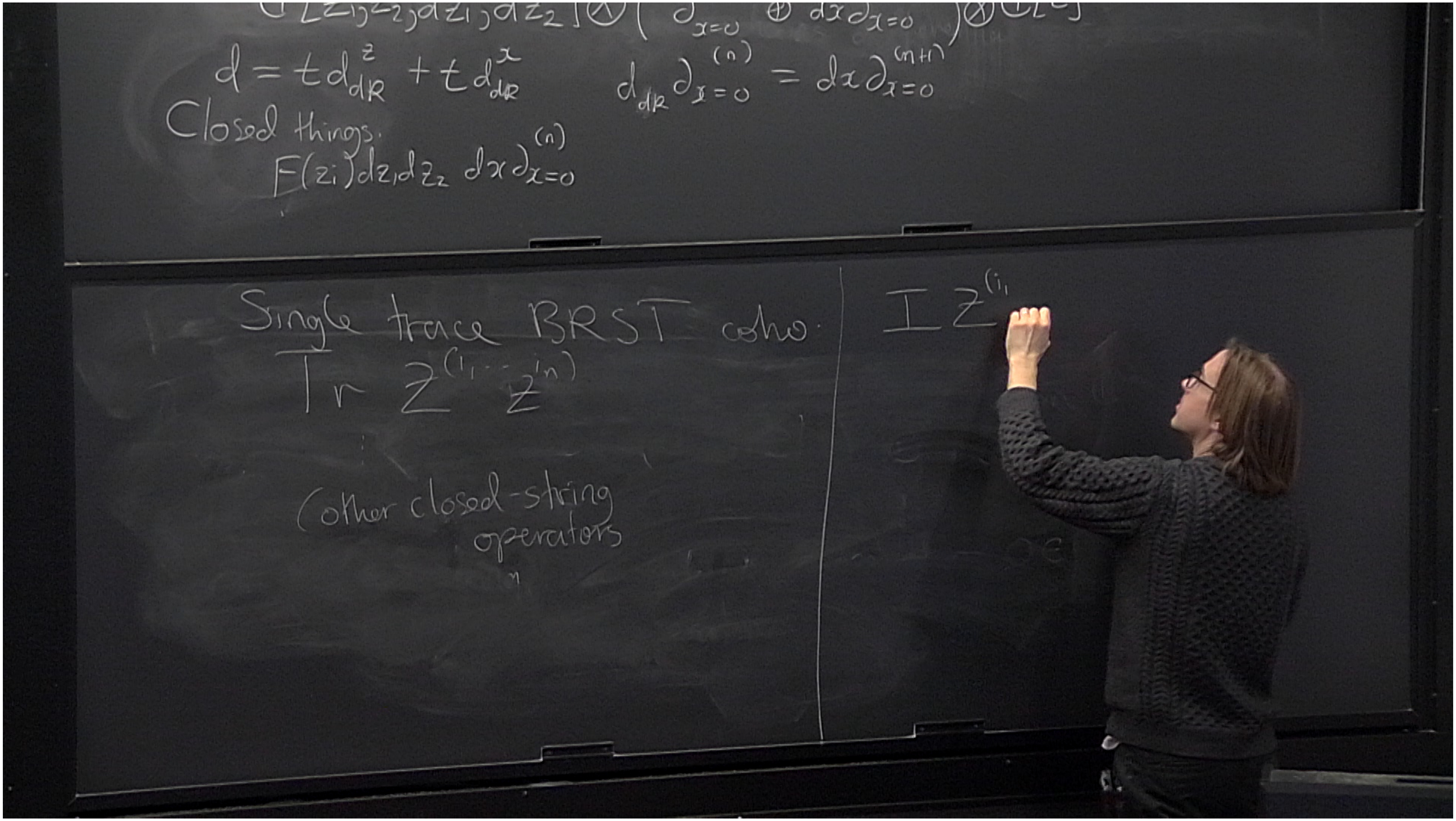
BRST reduction of Z_1, adj valued symplectic bosons
(use b-c ghosts)

$\mathbb{C} \subseteq \mathbb{C}^3$

Include a space filling brane of dim^n $k|k$

$\Longrightarrow$ matter fields $I \in \mathrm{Hom}(\mathbb{C}^{k|k}, \mathbb{C}^N)$ $J \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{k|k})$

$$IJ \sim \frac{1}{z}$$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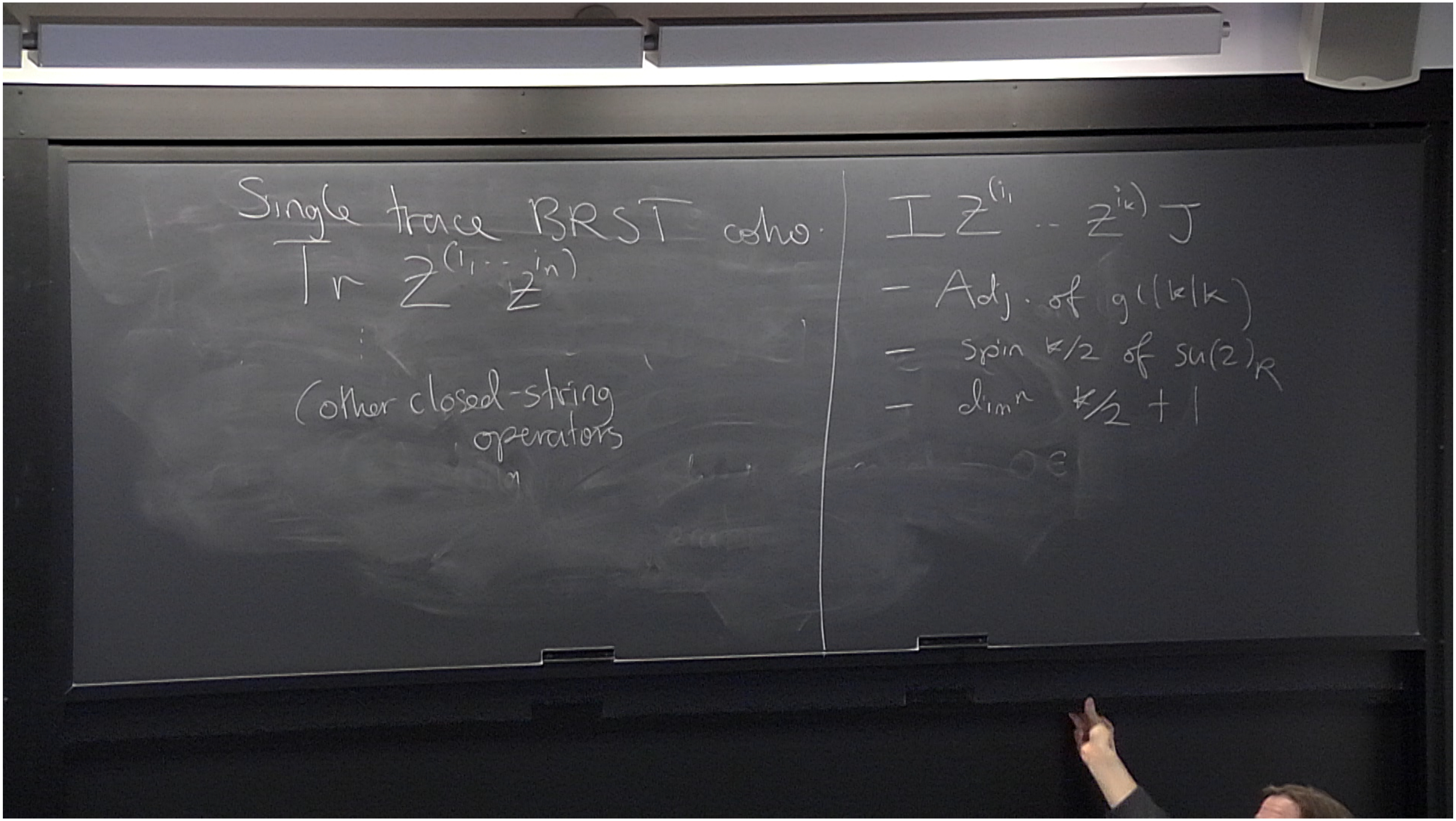
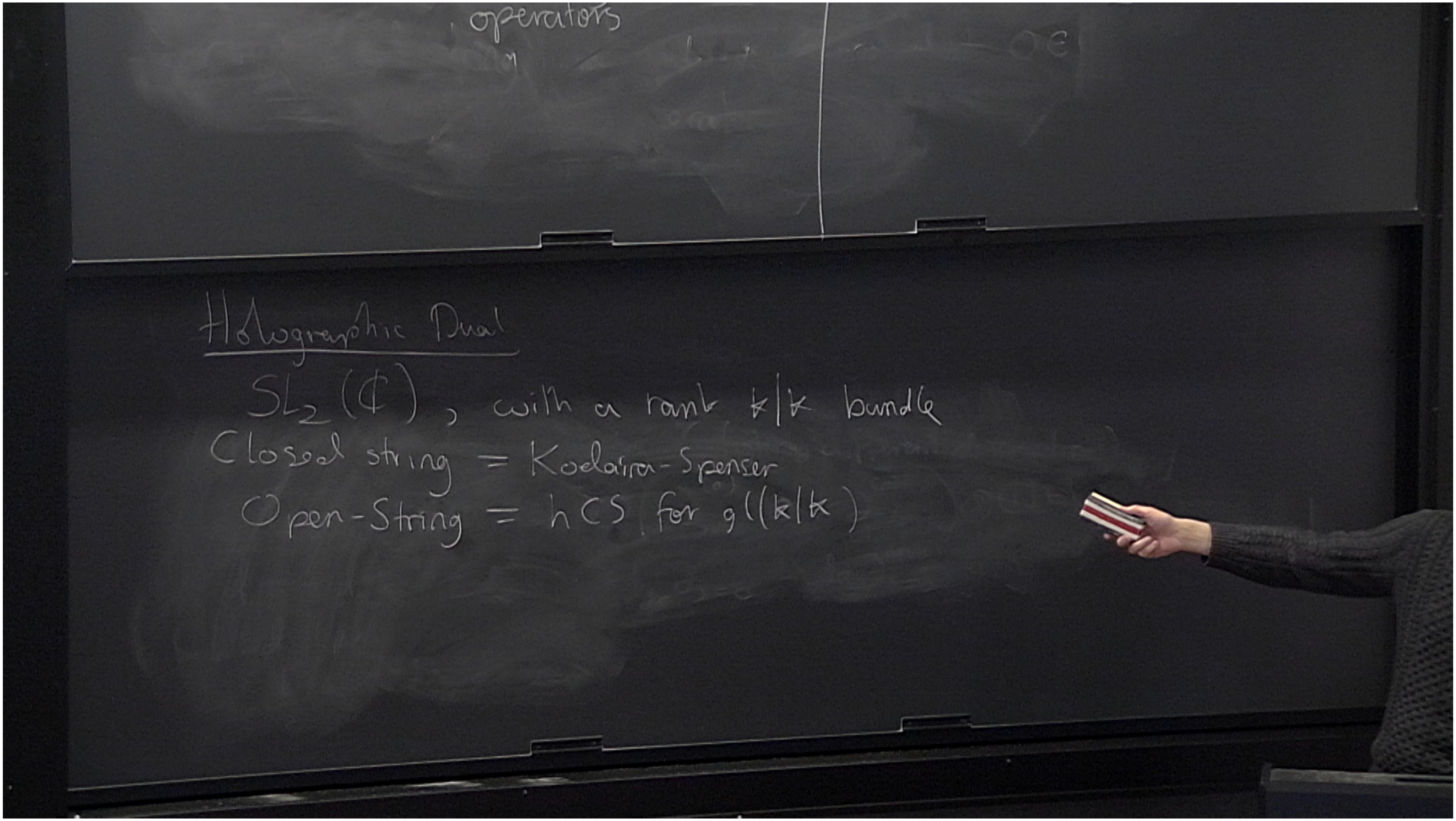
Closed things:
$F(z_i)\,dz_1 dz_2\,dx\,\partial^{(n)}_{x=0}$
Single trace BRST coho.
$\mathrm{Tr}\, Z^{(i_1} \cdots Z^{i_n)}$
(other closed-string operators
$I\, Z^{(i_1}$


Single trace BRST coho.
$\mathrm{Tr}\, Z^{(i_1 \cdots} Z^{i_n)}$
⋮
(other closed-string operators
$I Z^{(i_1} \cdots Z^{i_k)} J$
– Adj. of $gl(k|k)$
= spin $k/2$ of $su(2)_R$
– dim^n $k/2+1$


operators
Holographic Dual
$SL_2(\mathbb{C})$, with a rank $k|k$ bundle
Closed string = Kodaira-Spenser
Open-String = hCS for $gl(k|k)$

operators

Holographic Dual

$SL_2(\mathbb{C})$, with a rank $k|k$ bundle

Closed string = Kodaira-Spenser

Open-String = hCS for $gl(k|k)$

$\overline{SL_2(\mathbb{C})} \setminus SL_2(\mathbb{C}) = \mathbb{P}^1_z \times \mathbb{P}^1_w$

$A \in \Omega^{0,1}(SL_2\mathbb{C}, gl(k|k))$ vanish on the boundary

Use coordinates n, z, w near $\mathbb{C}_z \times \mathbb{P}^1_w$ in boundary

- Boundary is $n = 0$
- Beltrami diff. $\quad N \dfrac{n^2 \, d\bar{w}}{(1+|w|^2)^2} \partial_z$

$n \sim dw^{1/2}$

Use coordinates n, z, w near $\mathbb{C}_z \times \mathbb{P}^1_w$ in boundary

- Boundary is $n = 0$
- Beltrami diff. $\sim \dfrac{n^2 d\bar{w}}{(1+|w|^2)^2} \partial_z$

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Sol'ns to EOM

$$A = \delta_{z=0} t^a$$

$t^a \in gl(K|K)$

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$n \sim dw^{1/2}$

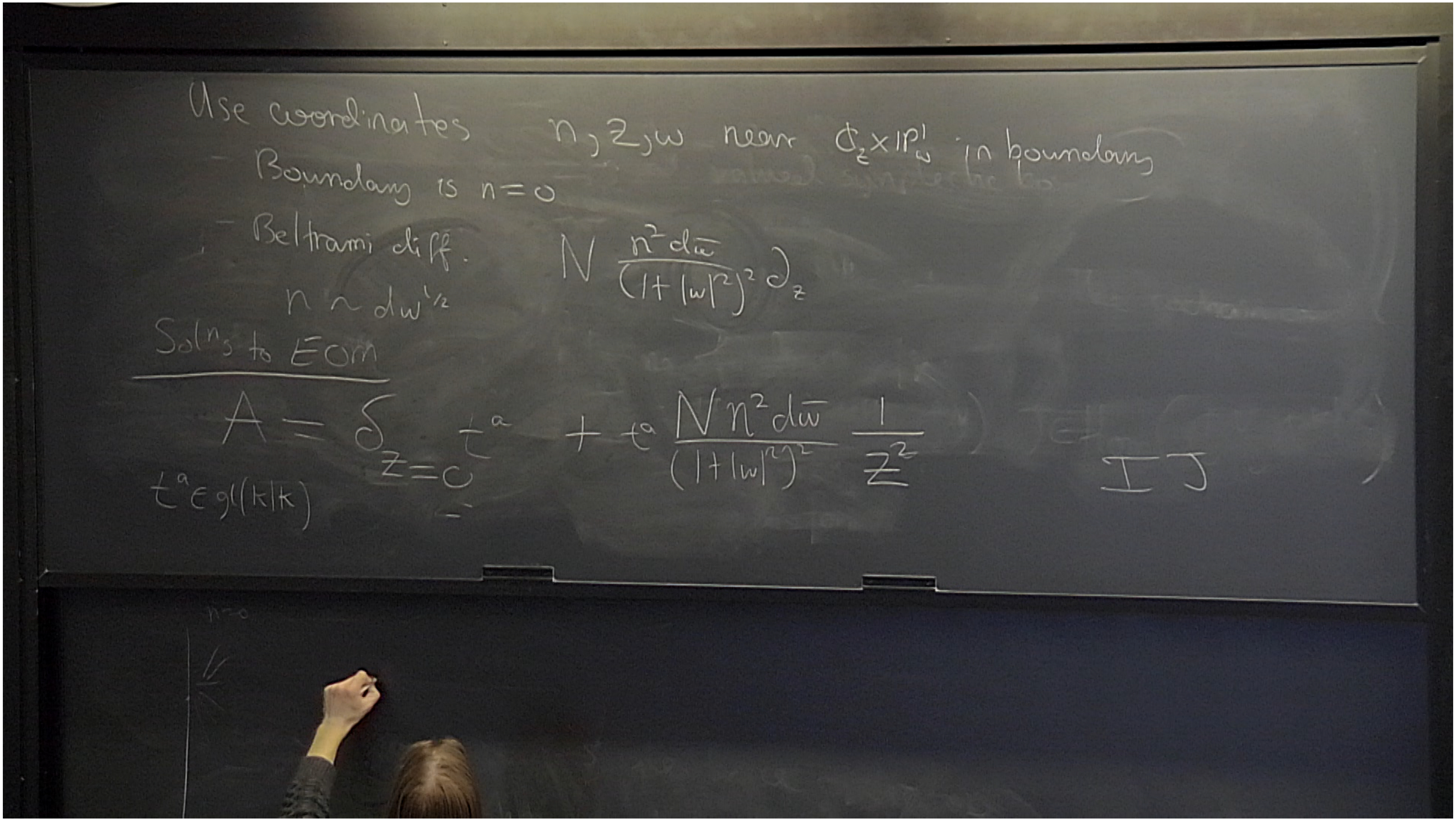
Sol'ns to EOM

$$A = \delta_{z=0}\, t^a + \frac{N n^2 d\bar{w}}{(1+|w|^2)^2} \frac{1}{z^2}$$

$t^a \in gl(k|k)$

LQT Theorem

BRST coho. of single trace ...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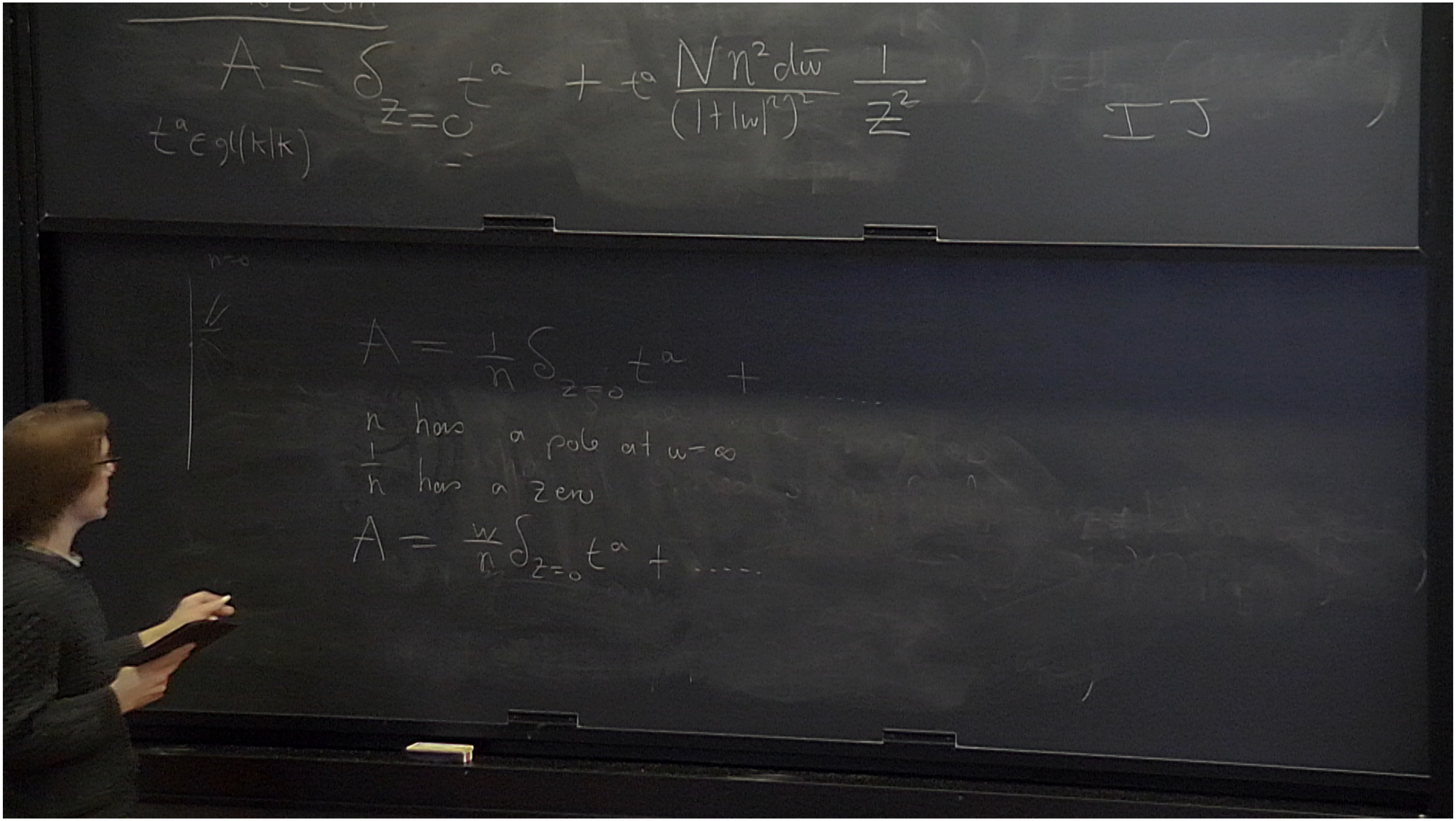
$n \sim dw^{1/2}$
Solns to EOM
$A = \delta_{z=0}\, t^a + t^a \frac{N n^2 d\bar{w}}{(1+|w|^2)^2}\frac{1}{z^2}$
$I\ J$
$t^a \in gl(k|k)$


A = δ_{z=0} t^a + t^a N n² dw̄ / (1+|w|²)² 1/z²
t^a ∈ gl(K|K)
I J
A = 1/n δ_{z=0} t^a + ...
n has a pole at w=∞
1/n has a zero
A = w/n δ_{z=0} t^a + ...

$$A = \delta_{z=0}\, t^a + t^a \frac{N \eta^2 d\bar{w}}{(1+|w|^2)^2}\frac{1}{z^2}$$

$I \quad J$

$t^a \in gl(K|K)$

$$A = \frac{1}{\eta}\delta_{z=0}\, t^a + \ldots\ldots$$

η has a pole at $w=\infty$

$\frac{1}{\eta}$ has a zero

$$A = \frac{w}{\eta}\delta_{z=0}\, t^a + \ldots$$

$$A = \frac{w^l}{\eta^k}\delta_{z=0}\, t^a + \ldots \qquad l \leq K$$

$$A = \delta_{z=0}\, t^a + t^a \frac{N \eta^2 d\bar{w}}{(1+|w|^2)^2}\frac{1}{z^2}$$

$I \quad J$

$t^a \in gl(k|k)$

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Under $SL(2)$

$\frac{w^l}{\eta^k}\delta_{z=0}$ is in rep of spin $k/2$

It has dimension

$$A = \delta_{z=0}\, t^a + t^a \frac{N\eta^2 d\bar{w}}{(1+|w|^2)^2} \frac{1}{z^2}$$

$t^a \in gl(k|k)$

$I \quad J$

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$$A = \frac{w}{\eta}\delta_{z=0}\, t^a + \ldots\ldots$$

$$A = \frac{w^i}{\eta^k}\delta_{z=0}\, t^a + \ldots \qquad (i \leq k$$

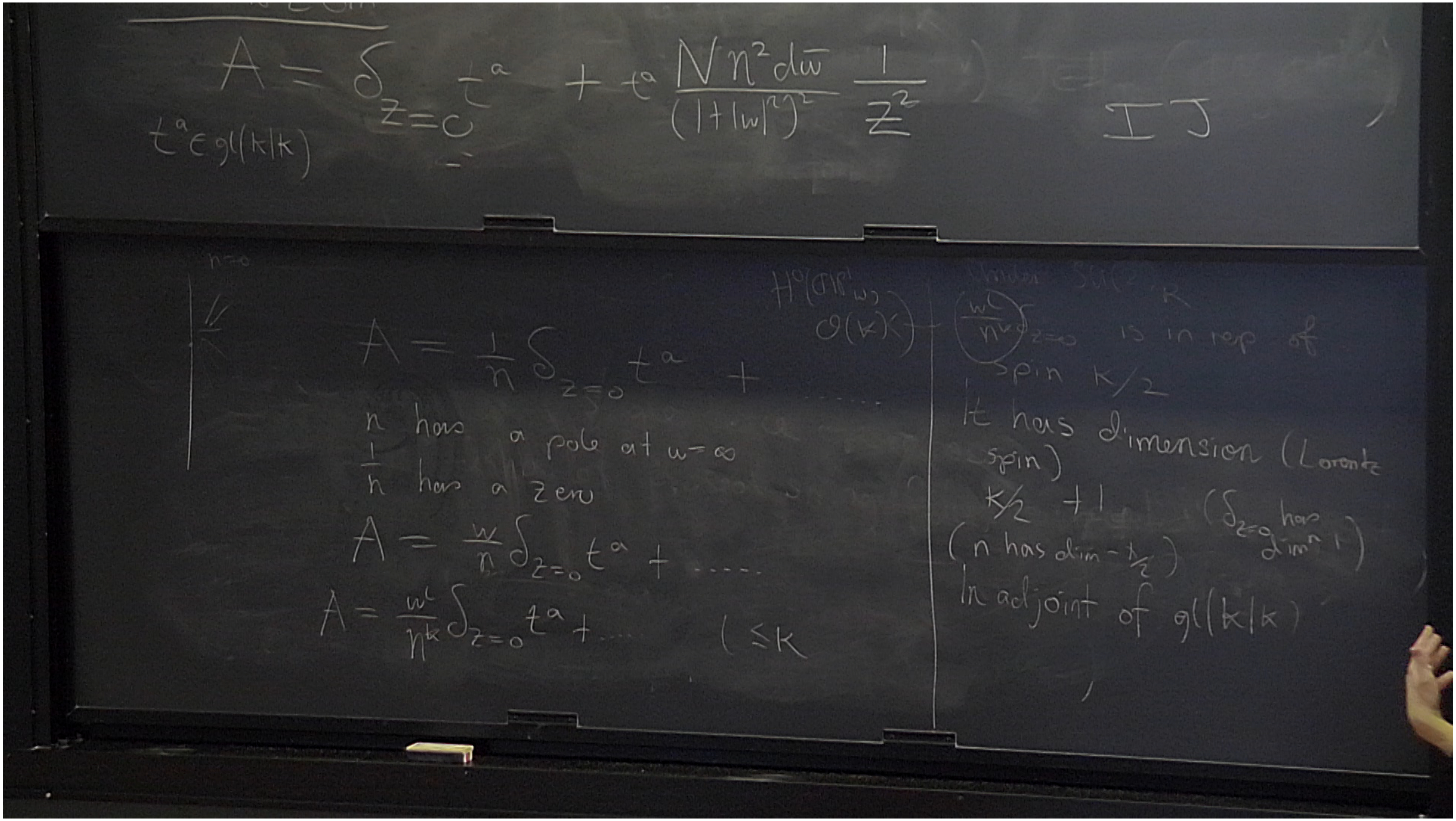
Under $SU(2)_R$

$\frac{w^i}{\eta^k}\delta_{z=0}$ is in rep of spin $k/2$

It has dimension (Lorentz spin) $k/2 + 1$ ($\delta_{z=0}$ has dim^n 1)

(η has dim $-\frac{1}{2}$)

In adjoint of $gl(k|k)$



$A = \delta_{z=0}\, t^a + t^a \dfrac{N \eta^2 d\bar{w}}{(1+|w|^2)^2} \dfrac{1}{z^2}$
$t^a \in gl(k|k)$
$I \; J$
$A = \dfrac{1}{\eta} \delta_{z=0}\, t^a + \ldots\ldots$
η has a pole at $w = \infty$
$\dfrac{1}{\eta}$ has a zero
$A = \dfrac{w}{\eta} \delta_{z=0}\, t^a + \ldots\ldots$
$A = \dfrac{w^l}{\eta^k} \delta_{z=0}\, t^a + \ldots$ $(l \leq k$
$H^0(\mathbb{CP}^1_w, \mathcal{O}(k))$
Under $SU(2)_R$
$\dfrac{w^l}{\eta^k} \delta_{z=0}$ is in rep of spin $k/2$
It has dimension (Lorentz spin)
$k/2 + 1$ ($\delta_{z=0}$ has dim^n 1)
(η has dim $-\frac{1}{2}$)
In adjoint of $gl(k|k)$

operators

Formula

$$\frac{n^{-k}\int_{z=0} t^{a}}{\text{Tr}\, I Z_1^{k} J} + \underbrace{\cdots\cdots}_{\text{local}} + n^{k+2} N^{k+1} \frac{\bar{w}^{k} d\bar{w}}{(1+|w|^2)^{k+2}} \frac{1}{z^{k+2}}$$

Two point fn
between two operators
is of order $N^{k+1} z^{-k-2}$