

Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 11

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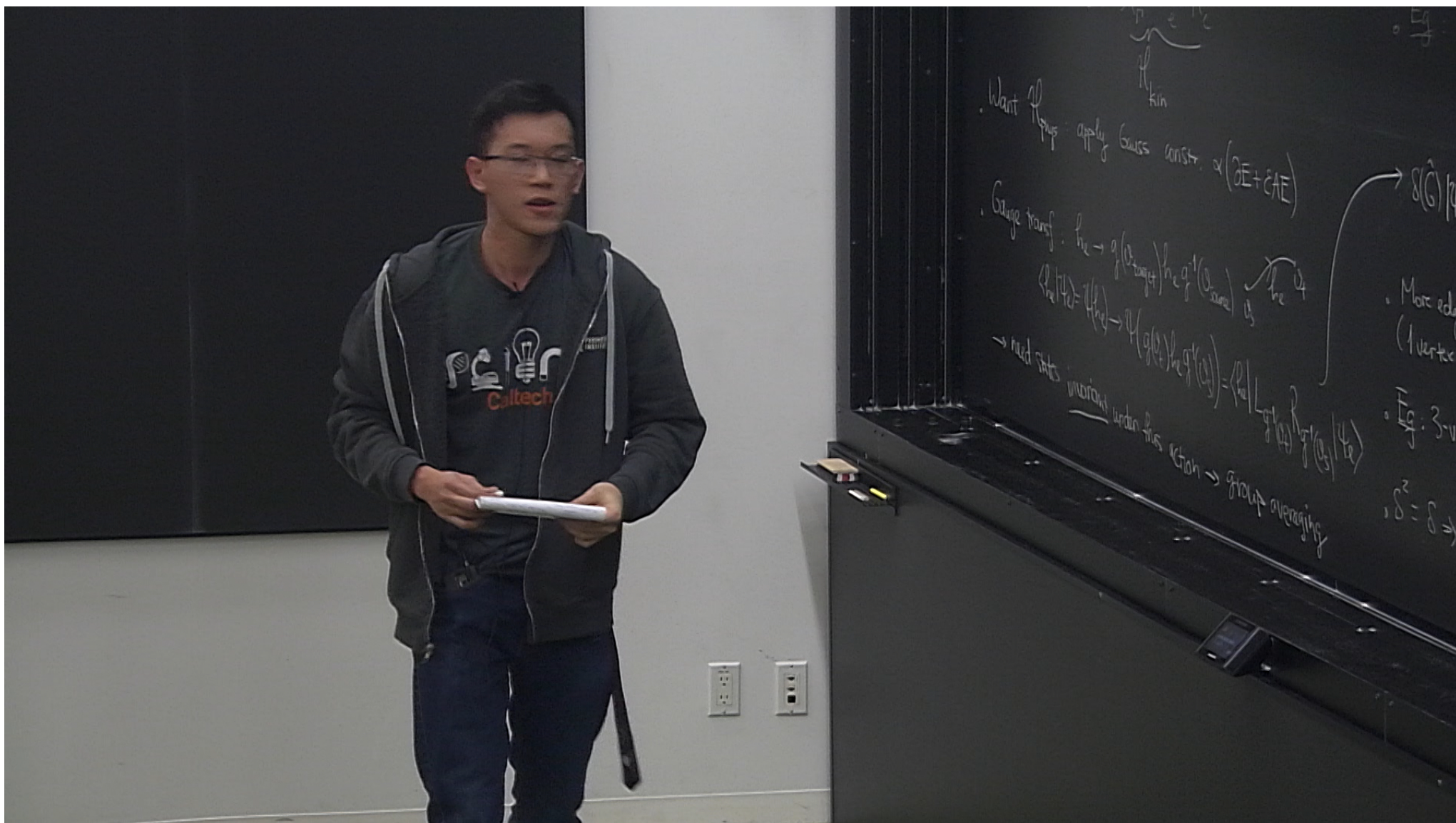
$$\mathcal{H}_e = L^2(\mathrm{SU}(2), dg) \rightarrow \underbrace{\mathcal{H}_\Gamma}_{\mathcal{H}_{\mathrm{kin}}} = \bigotimes_e \mathcal{H}_e$$

Want $\mathcal{H}_{\mathrm{phys}}$: apply Gauss constr. $\propto (\partial E + \epsilon A E)$

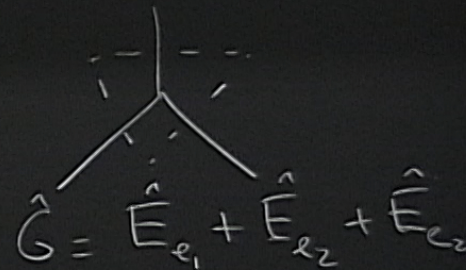
Gauge transf: $h_e \rightarrow g(\mathcal{O}_{\mathrm{target}}) h_e g^{-1}(\mathcal{O}_{\mathrm{source}})$

$\langle h_e | \psi_e \rangle = \psi(h_e) \rightarrow \psi(g(\mathcal{O}_t) h_e g^{-1}(\mathcal{O}_s)) = \langle h_e | L_{g^{-1}(\mathcal{O}_t)} R_{g^{-1}(\mathcal{O}_s)} | \psi_e \rangle$

→ need states invariant under this action → group averaging



• Eg: $\overline{f(h)} = \int R_g f(h) dg = \int f(h_g) dg$
 $\overline{f(h+1)} = \int \underbrace{f(h_g+1)}_{\tilde{f}} \underbrace{dg}_{\tilde{dg}} = \int f(h_{g'}) dg$



$$\hat{G} = \hat{E}_{e_1} + \hat{E}_{e_2} + \hat{E}_{e_3}$$

→ $S(\hat{G})|4_e\rangle = \int L_{g_1} R_{g_2} |4_e\rangle dg_1 dg_2$

• More edges: $S(\hat{G}_0) = \int \left(\bigotimes_{\text{target}} L_h^e \right) \otimes \left(\bigotimes_{\text{source}} R_h^{e'} \right)$
 (1 vertex)

• Eg: 3-vertex  = $S(\hat{G}_0) \left(\bigotimes_{i=1}^3 |j_i m_i n_i\rangle \right) = \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes | \rangle \right)$

• $S^2 = S \Rightarrow$ normalized state: $\sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1}^3 |j_i m_i p_i\rangle \right)$

Flatness constraint

$$\hookrightarrow h_{\gamma} = 1 - \int_{S(\gamma)} F + O(\epsilon^3)$$

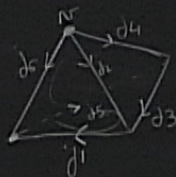
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impose flatness.

$$\Rightarrow h_{\gamma} = 1.$$

$$\hookrightarrow \text{loops: } H_1 = g_4^{-1} g_5 g_6, H_2 = g_2^{-1} g_1 g_6, H_3 = g_2 g_5^{-1} g_4^{-1}, H_4 = g_5^{-1} g_3^{-1} g_1^{-1}$$

Concrete example

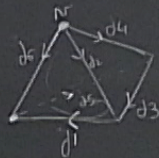


Flatness constraint

$$\hookrightarrow h_{\gamma} = 1 - \int_{S(\gamma)} \bar{F} + O(\epsilon^3) \quad \leftarrow \text{impose flatness} \Rightarrow h_{\gamma} = 1.$$

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Concrete example



$$\hookrightarrow \text{loops: } H_1 = g_4^{-1} g_5 g_6, \quad H_2 = g_2^{-1} g_3 g_6, \quad H_3 = g_2 g_5^{-1} g_4^{-1}, \quad H_4 = g_5^{-1} g_3^{-1} g_1^{-1}$$

$$\Psi \rightarrow \delta(H_1) \delta(H_2) \delta(H_3) \delta(H_4) \Psi \rightarrow \int_{\mathcal{G}} \delta(g h^{-1}) f(h) dh = f(g)$$

! redundancy! $H_4 = g_5 H_1 H_3 H_2 g_6^{-1}$, imposing $H_1 = 1 = H_2 = H_3 \Rightarrow H_4 = 1$

* Physical inner product?

→ parametrized particle

$$\rightarrow \hat{C} = \hat{p}_q$$

$$\delta(\hat{p}_q) \Psi(t, q) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \Psi(t, q+s) ds$$

$$\langle \delta(\hat{p}) \Psi_2, \delta(\hat{p}) \Psi_1 \rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\Psi_2(t, q+s_2)} \Psi_1(t, q+s_1) ds_1 ds_2 dt dq = \infty$$

$$dt \, dq = \infty$$

self-adjoint op.

$$\langle \Psi_1, \hat{C} \Psi_2 \rangle = \langle \hat{C} \Psi_1, \Psi_2 \rangle.$$

+ formally using the projector property
($P^2 = P$).

$$\langle S(\hat{C}) \Psi_2, S(\hat{C}) \Psi_1 \rangle_{\text{phys}}$$

$$:= \langle \Psi_2, S(\hat{C}) \Psi_1 \rangle_{\text{kin}}$$

* Physical inner product?

→ parametrized particle

$$\rightarrow \hat{C} = \hat{p}_q \quad \delta(\hat{p}_q) \Psi(t, q) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \Psi(t, q+s) ds$$

$$\langle \delta(\hat{p}) \Psi_2, \delta(\hat{p}) \Psi_1 \rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\Psi_2(t, q+s)} \Psi_1(t, q-s) ds_1 ds_2 dt dq = \infty$$

→ Ψ_1, Ψ_2 on the tetrahedral graph satisfying the Gauss constraint

$$\langle \underbrace{\delta(H_1) \delta(H_2) \delta(H_3) \Psi_2}_{\Psi_2^{\text{phys}}}, \underbrace{\delta(H_1) \delta(H_2) \delta(H_3) \Psi_1}_{\Psi_1^{\text{phys}}} \rangle_{\text{phys}} = \langle \Psi_2, \delta(H_1) \delta(H_2) \delta(H_3) \Psi_1 \rangle_{\text{kin}}$$

$$\Psi_1 = \text{Tetra} \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{bmatrix} = \sum_{m_i, n_i} f(d_i m_i n_i) |j_i m_i n_i\rangle$$

$$f(d_i m_i n_i) = (-1)^{\sum j_i - m_i} \begin{pmatrix} d_1 & d_2 & d_3 \\ n_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} d_1 & d_5 & d_5 \\ -m_1 & n_5 & -m_5 \end{pmatrix} \begin{pmatrix} d_2 & d_5 & d_4 \\ n_2 & n_5 & n_4 \end{pmatrix} \begin{pmatrix} d_3 & d_4 & d_5 \\ n_3 & -m_4 & -m_5 \end{pmatrix}$$

$$\Psi_2 = \text{Tetra} \begin{bmatrix} d'_1 & d'_2 & d'_3 \\ d'_4 & d'_5 & d'_6 \end{bmatrix}$$

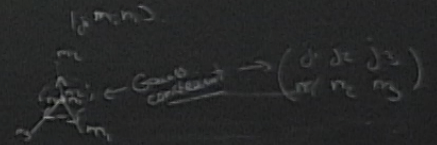
• Transition amplitude between Ψ_1, Ψ_2

$$\Psi_1 = \text{Tetra} \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{bmatrix} = \sum_{m_i, n_i} f(j_1 m_i n_i) |j_1 m_i n_i\rangle$$

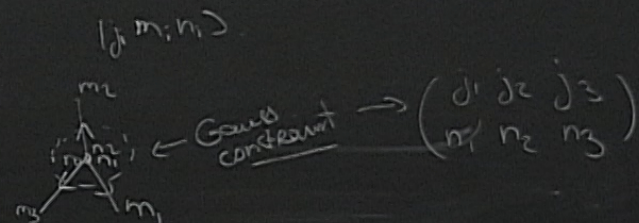
$$f(j_1 m_i n_i) = (-1)^{\sum j - m_i} \begin{pmatrix} d_1 & d_2 & d_3 \\ n_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} d_1 & d_5 & d_6 \\ -m_1 & n_5 & -m_6 \end{pmatrix} \begin{pmatrix} d_2 & d_6 & d_4 \\ n_2 & n_6 & n_4 \end{pmatrix} \begin{pmatrix} d_3 & d_4 & d_5 \\ n_3 & -m_4 & -m_5 \end{pmatrix}$$

$$\Psi_2 = \text{Tetra} \begin{bmatrix} d'_1 & d'_2 & d'_3 \\ d'_4 & d'_5 & d'_6 \end{bmatrix}$$

$$\text{Transition amplitude between } \Psi_1, \Psi_2 \rightarrow \text{Trans}(j'_1 d'_1) : \int_{6^c} \sum_{m_i, n_i} f(j'_1 m_i n_i) \prod_{i=1}^6 \overline{D_{n_i m_i}^{j'_i}(g_i)} \sum_{m_i} f(j_1 m_i n_i) \prod_{i=1}^6 D_{n_i m_i}^{j_i}(g_i)$$



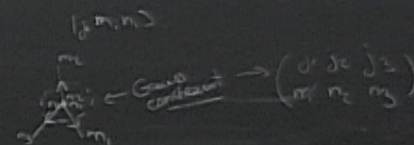
$$\begin{pmatrix} d_1 & d_5 & d_6 \\ -m_1 & n_5 & -m_6 \end{pmatrix} \begin{pmatrix} d_2 & d_6 & d_4 \\ n_2 & n_6 & n_4 \end{pmatrix} \begin{pmatrix} d_3 & d_4 & d_5 \\ n_3 & -m_4 & -m_5 \end{pmatrix}$$



$$(j', d') : \int_{G^c} \sum_{m', n'} f(j', m', n') \prod_{i=1}^6 \overline{D_{n', m'}^{d'}(g_i)} \sum_{m, n} f(j, m, n) \prod_{i=1}^6 D_{n, m}^d(g_i) \delta(H_1) \delta(H_2) \delta(H_3) \prod_{k=1}^6 dg_k$$

$$\Psi_1 = \text{Tetra} \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{bmatrix} = \sum_{m_i, n_i} f(j_1 m_1 n_1) \langle j_1 m_1 n_1 \rangle$$

$$f(j_1 m_1 n_1) = (-1)^{\sum j_i - m_i} \begin{pmatrix} d_1 & d_2 & d_3 \\ n_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} d_1 & d_5 & d_6 \\ -m_1 & n_5 & -m_6 \end{pmatrix} \begin{pmatrix} d_2 & d_5 & d_4 \\ n_2 & n_6 & n_4 \end{pmatrix} \begin{pmatrix} d_3 & d_4 & d_5 \\ n_3 & -m_4 & -m_5 \end{pmatrix}$$



$$\Psi_2 = \text{Tetra} \begin{bmatrix} d'_1 & d'_2 & d'_3 \\ d'_4 & d'_5 & d'_6 \end{bmatrix}$$

Transition amplitude between $\Psi_1, \Psi_2 \rightarrow \text{Trans}(j'_1, j_1) :$

$$= \int_{G^6} \frac{\sum_{m_i, n_i} f(j'_1 m'_1 n'_1) \prod_{i=1}^6 \overline{D_{n_i m_i}^{j_i}(g_i)} \sum_{m_i, n_i} f(j_1 m_1 n_1) \prod_{i=1}^6 D_{n_i m_i}^{j_i}(g_i)}{\prod_{i=1}^6 \sqrt{d_i d'_i} \left\{ \begin{matrix} d'_1 & d'_2 & d'_3 \\ d'_4 & d'_5 & d'_6 \end{matrix} \right\} \left\{ \begin{matrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{matrix} \right\}}$$

$\underbrace{\qquad\qquad\qquad}_{\text{Gauss constraint}} \rightarrow \begin{pmatrix} d'_1 & d'_2 & d'_3 \\ m'_1 & n'_2 & n'_3 \end{pmatrix}$
 $\underbrace{\qquad\qquad\qquad}_{\text{Gauss constraint}} \rightarrow \begin{pmatrix} d_1 & d_2 & d_3 \\ m_1 & n_2 & n_3 \end{pmatrix}$

ansf. $h_e \rightarrow g(v_{\text{target}}) h_e g^{-1}(v_{\text{source}})$

$\langle h_e | 4_e \rangle = 4(h_e) \rightarrow 4(g(v_t) h_e g^{-1}(v_s)) = \langle h_e | L_{g^{-1}(v_t)} R_{g^{-1}(v_s)} | 4_e \rangle$

states invariant under this action $\rightarrow$ group averaging

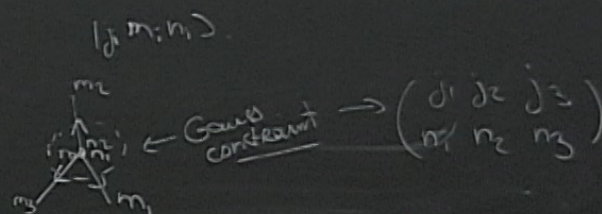
More edges (1 vertex) $S(G_0) = \int \left(\bigotimes_{\text{target}} L_a^c \right) \otimes \left(\bigotimes_{\text{source}} R_b^c \right)$

Eg. 3-vertex $\gamma = S(\hat{G}_0) \left(\begin{smallmatrix} 3 \\ 1=1 \end{smallmatrix} \middle| j_1 m_1 n_1 \right) = \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \left(\begin{smallmatrix} 3 \\ 1=1 \end{smallmatrix} \middle| \right)$

$S^z = S \rightarrow$ normalized stat. $\sum_{m_1, n_1, p_1} \left(\begin{smallmatrix} 3 \\ 1=1 \end{smallmatrix} \middle| j_1 m_1 n_1 \right) \left(\begin{smallmatrix} 3 \\ 1=1 \end{smallmatrix} \middle| j_1 m_1 p_1 \right)$

$|j, m, n_i\rangle$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ -m_1 & n_5 & -m_6 \end{pmatrix} \begin{pmatrix} j_2 & j_6 & j_4 \\ n_2 & n_6 & n_4 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ n_3 & -m_4 & -m_5 \end{pmatrix}$$



$$\begin{aligned} \text{Trans}(j_i', j_i) &= \int_{G^e} \sum_{m_i, n_i} f(j_i', m_i, n_i) \prod_{i=1}^6 \overline{D_{n_i, m_i}^{j_i'}(g_i)} \sum_{m_i, n_i} f(j_i, m_i, n_i) \prod_{i=1}^6 D_{n_i, m_i}^{j_i}(g_i) \delta(H_1) \delta(H_2) \delta(H_3) \prod_{k=1}^6 dg_k \\ &= \frac{1}{\prod_{i=1}^6} \sqrt{d_i} \sqrt{d_j} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \underbrace{\left\{ \begin{matrix} j_1' & j_2' & j_3' \\ j_4' & j_5' & j_6' \end{matrix} \right\}}_{\text{6j symbol}} \end{aligned}$$

$\langle \Psi_2 | \Psi_1 \rangle = \sqrt{\langle \Psi_2 | \Psi_2 \rangle} \sqrt{\langle \Psi_1 | \Psi_1 \rangle}$
 $\Rightarrow$ triangulated 2-sphere - one independent physical state.

$$\delta(\hat{G}_0) = \int_{\text{target}} \left(\otimes L_h^e \right) \otimes \left(\otimes R_h^e \right)$$

$$\hat{Y} = \delta(\hat{G}_0) \left(\otimes_{i=1}^3 |j_i, m_i, n_i\rangle \right) = \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \sum_{p_1, p_2, p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\otimes | \right)$$

$$= \prod_{i=1}^4 \prod_{j=1}^4 d_{ij} \left(d_0 d_5 d_6 \right) \underbrace{\left(d_4 d_5 d_6 \right)}_{\rightarrow \text{6j symbol}} \rightarrow \langle \Psi_2 | \Psi_1 \rangle = \Rightarrow \text{triangulated independent pl}$$

$$\Psi_{\text{Phy}}(j_i) = \langle \text{Tetra} \left[\begin{smallmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{smallmatrix} \right] | \delta(H_1) \delta(H_2) \delta(H_3) \rangle$$

$$= \sqrt{d_{d_1} \dots d_{d_6}} \left\{ \begin{smallmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{smallmatrix} \right\}$$

$\Rightarrow$ Physical Hilbert is one dimensional

reflecting that the reduce phase space for a spherical spatial hypersurface is zero dimensional

in this action $\rightarrow$ group averaging

$$\text{Tr} P_1 P_2 P_3 \dots$$

$$\begin{aligned}
 \Psi_2 &= \text{Tetra} \begin{bmatrix} d^1 & d^2 & d^3 \\ d^4 & d^5 & d^6 \end{bmatrix} \\
 \text{Transition amplitude between } \Psi_1, \Psi_2 &\rightarrow \text{Trans}(j'_1, j_1) = \int \sum_{m_1, n_1} f(j'_1, m_1, n_1) \prod_{i=1}^6 \overline{D_{n_i, m_i}^{\sigma_i}(g_i)} \sum_{m_2, n_2} f(j_2, m_2, n_2) \prod_{i=1}^6 D_{n_i, m_i}^{\sigma_i}(g_i) \delta(H_1) \delta(H_2) \delta(H_3) \\
 &= \frac{1}{6!} \prod_{i=1}^6 \sqrt{d_{j'_i} d_{j_i}} \left\{ \begin{bmatrix} d^1 & d^2 & d^3 \\ d^4 & d^5 & d^6 \end{bmatrix} \right\} \left\{ \begin{bmatrix} d^1 & d^2 & d^3 \\ d^4 & d^5 & d^6 \end{bmatrix} \right\} \xrightarrow{\text{6}_g \text{ symbol}} \langle \Psi_2 | \Psi_1 \rangle = \overline{\langle \Psi_1 | \Psi_2 \rangle} \\
 &\quad \xrightarrow{\text{triangulated 2-sphere on independent physical states}}
 \end{aligned}$$

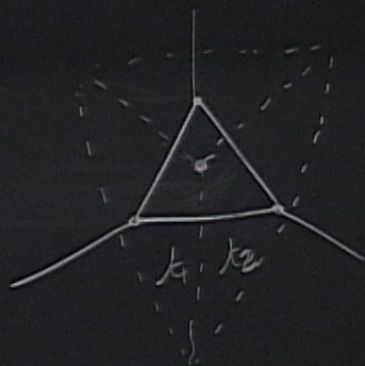
$$\begin{aligned}
 \Psi_{\text{phys}}(j_i) &= \langle \text{Tetra} \begin{bmatrix} d^1 & d^2 & d^3 \\ d^4 & d^5 & d^6 \end{bmatrix} | \delta(H_1) \delta(H_2) \delta(H_3) \rangle \\
 &= \sqrt{d_{j_1} \cdots d_{j_6}} \left\{ \begin{bmatrix} d^1 & d^2 & d^3 \\ d^4 & d^5 & d^6 \end{bmatrix} \right\}
 \end{aligned}$$

$\Rightarrow$ Physical Hilbert is one dimensional
 reflecting that the reduce phase space for
 a spherical spatial hypersurface
 is zero dimensional

$$\hat{H}|\Psi\rangle = 0 \Leftrightarrow \text{Biedenharn-Elliott relation} \rightarrow \{G\}$$



under this action $\rightarrow$ group averaging



Flatness:

Σ with its geometry

$\Rightarrow$ it can be embedded locally
into flat 3-space.

Transition amplitude between $\Psi_1, \Psi_2 \rightarrow \text{Trans}(j'_1 j_1) : \int \sum_{m_i n_i} \prod_{i=1}^6 \int d^{m_i n_i} j_i \overline{d^{m_i n_i} j_i} \rightarrow \langle \Psi_2 | \Psi_1 \rangle$
 $= \frac{1}{\prod_{i=1}^6 \sqrt{d_{j_i} d_{j'_i}}} \left\{ \begin{matrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{matrix} \right\} \underbrace{\left\{ \begin{matrix} d'_1 & d'_2 & d'_3 \\ d'_4 & d'_5 & d'_6 \end{matrix} \right\}}_{\text{6j symbol}} \rightarrow \langle \Psi_2 | \Psi_1 \rangle$
 $\Rightarrow$ triangle independent

$\Psi_{\text{Phy}}(j_i) = \langle \text{Tetra} \left[\begin{matrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{matrix} \right] | \delta(H_1) \delta(H_2) \delta(H_3) \rangle$
 $= \sqrt{d_{j_1} \cdots d_{j_6}} \left\{ \begin{matrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{matrix} \right\} \Rightarrow \text{Physical Hilbert is one dimensional}$
 reflecting that the reduce phase space for a spherical spatial hypersurface is zero dimensional



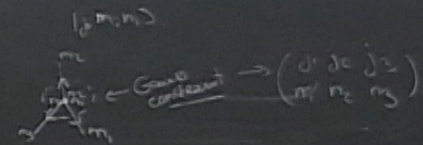
$\hat{H}|\Psi\rangle = 0 \Leftrightarrow \text{Bidenhoorn-Elliot relation}$
 $\hookrightarrow \{G\}$

under this action $\rightarrow$ group averaging

$\text{M.P.}, \text{P.P.}, \text{P.P.}, \text{P.P.}$

$$\Psi_1 = \text{Tetra} \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{bmatrix} = \sum_{m_i, n_i} f(j_i, m_i, n_i) |j_i, m_i, n_i\rangle$$

$$f(j_i, m_i, n_i) = (-1)^{\sum j_i - m_i} \begin{pmatrix} d_1 & d_2 & d_3 \\ n_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} d_1 & d_5 & d_6 \\ -m_1 & n_5 & -m_6 \end{pmatrix} \begin{pmatrix} d_2 & d_6 & d_4 \\ n_2 & n_6 & n_4 \end{pmatrix} \begin{pmatrix} d_3 & d_4 & d_5 \\ n_3 & -m_4 & -m_5 \end{pmatrix}$$



$$\Psi_2 = \text{Tetra} \begin{bmatrix} d'_1 & d'_2 & d'_3 \\ d'_4 & d'_5 & d'_6 \end{bmatrix}$$

Transition amplitude between $\Psi_1, \Psi_2 \rightarrow \text{Trans}(j'_i, j_i) = \int_{\mathcal{G}^c} \sum_{m_i, n_i} f(j'_i, m_i, n_i) \prod_{i=1}^6 \overline{D_{n_i, m_i}^{j_i}(g_i)} \sum_{m_i, n_i} f(j_i, m_i, n_i) \prod_{i=1}^6 D_{n_i, m_i}^{j_i}(g_i) \delta(H_1) \delta(H_2) \delta(H_3) \prod_{i=1}^6 dg_i$

$$= \prod_{i=1}^6 \frac{1}{\sqrt{d_i}} \sqrt{d_{j_i}} \left\{ \begin{matrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{matrix} \right\} \underbrace{\left\{ \begin{matrix} d'_1 & d'_2 & d'_3 \\ d'_4 & d'_5 & d'_6 \end{matrix} \right\}}_{\text{6j symbol}} \rightarrow \langle \Psi_2 | \Psi_1 \rangle = \sqrt{K_{\Psi_2} | \Psi_2} \sqrt{K_{\Psi_1} | \Psi_1}$$

$\Rightarrow$ triangulated 2-sphere - one independent physical state

$$\Psi_{\text{phys}}(j_i) = \langle \text{Tetra} \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{bmatrix} | \delta(H_1) \delta(H_2) \delta(H_3) \rangle$$

$$= \sqrt{d_{j_1} \cdots d_{j_6}} \left\{ \begin{matrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{matrix} \right\}$$

$\Rightarrow$ Physical Hilbert is one dimensional reflecting that the reduce phase space for a spherical spatial hypersurface is zero dimensional



$$\hat{H}|\Psi\rangle = 0 \Leftrightarrow \text{Biedenharn Elliot relation}$$

$\hookrightarrow \{6j\}$