Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 10

Speakers: MaÃ¯tÃ© Dupuis

Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

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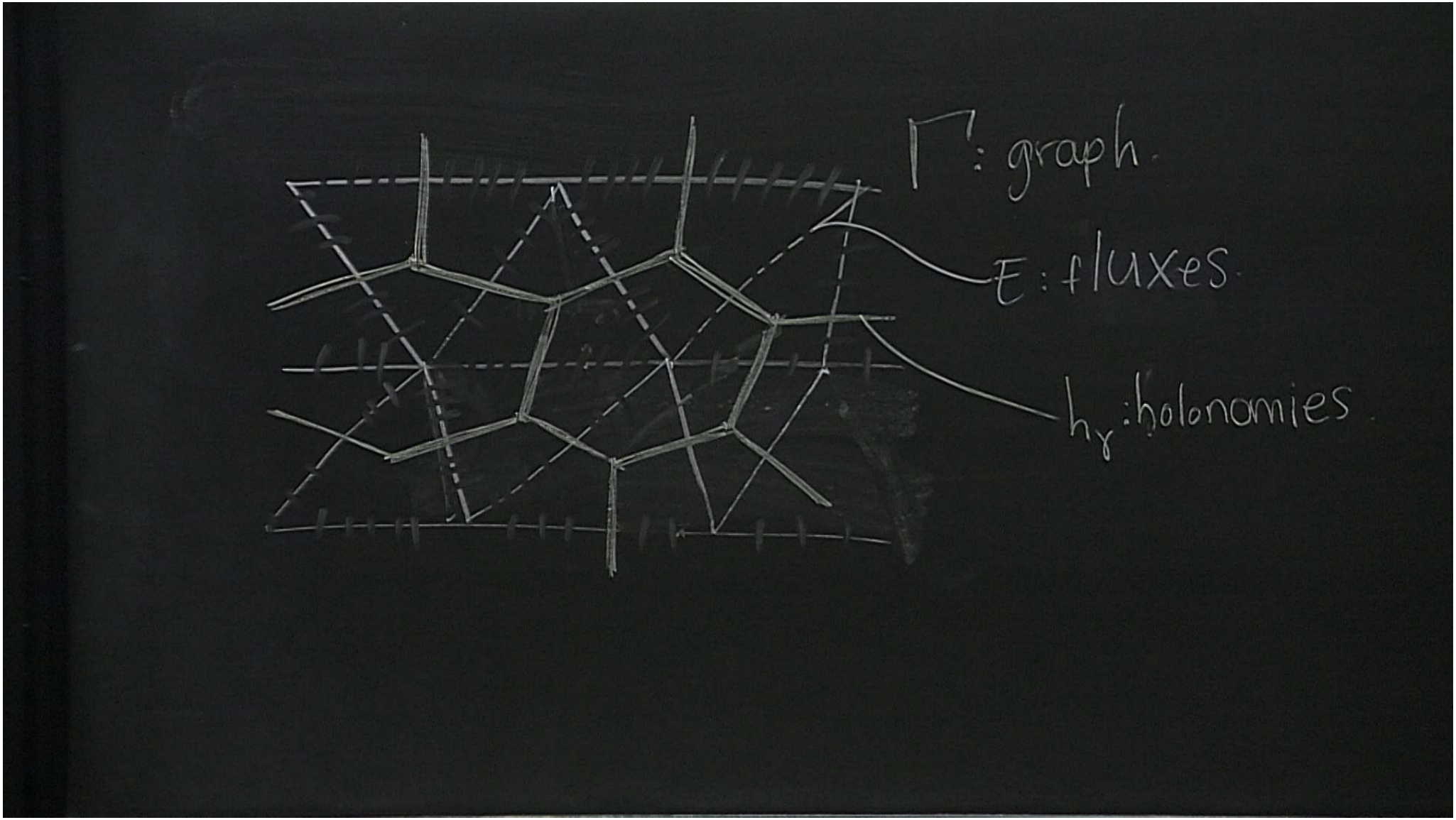
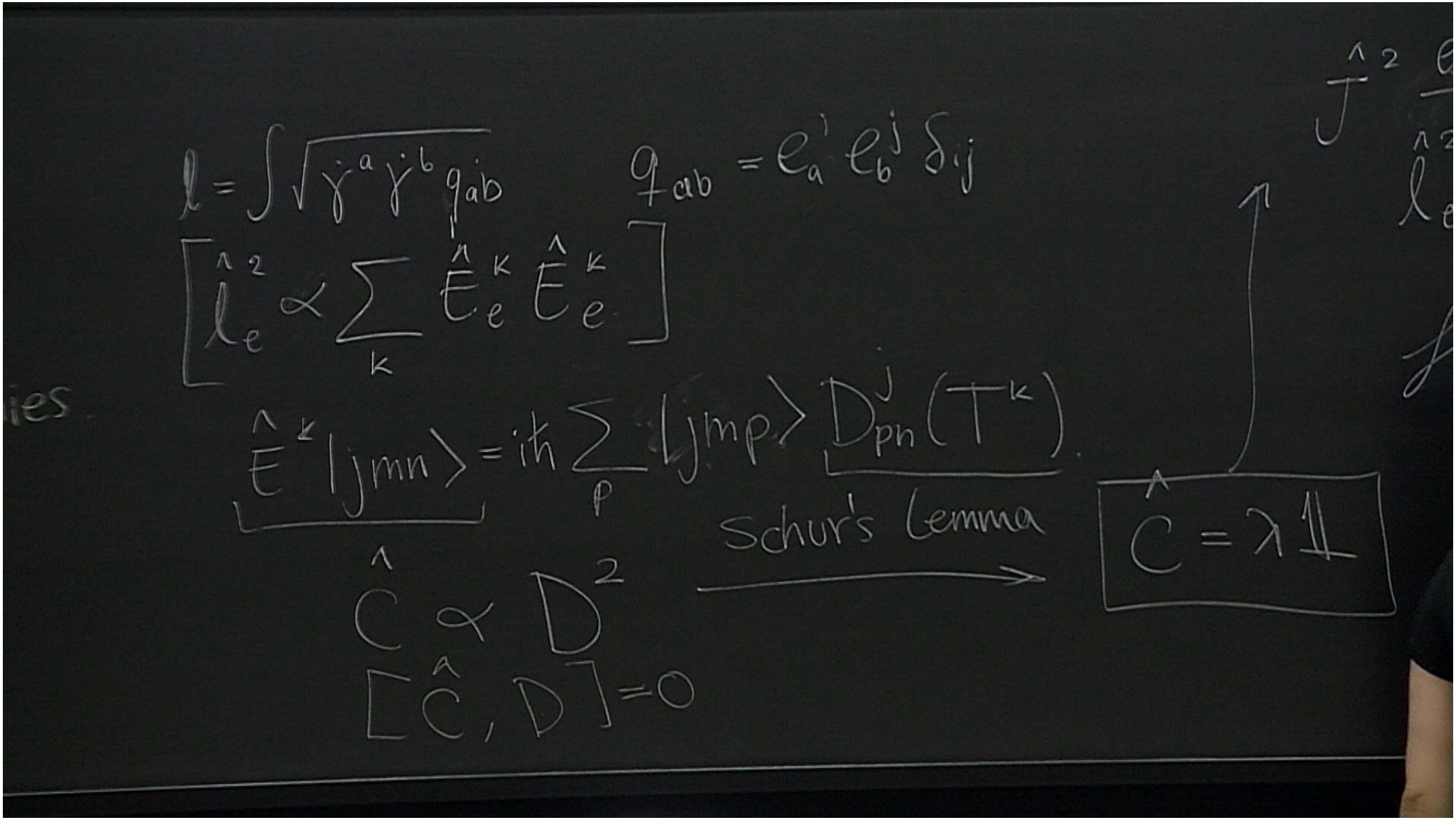
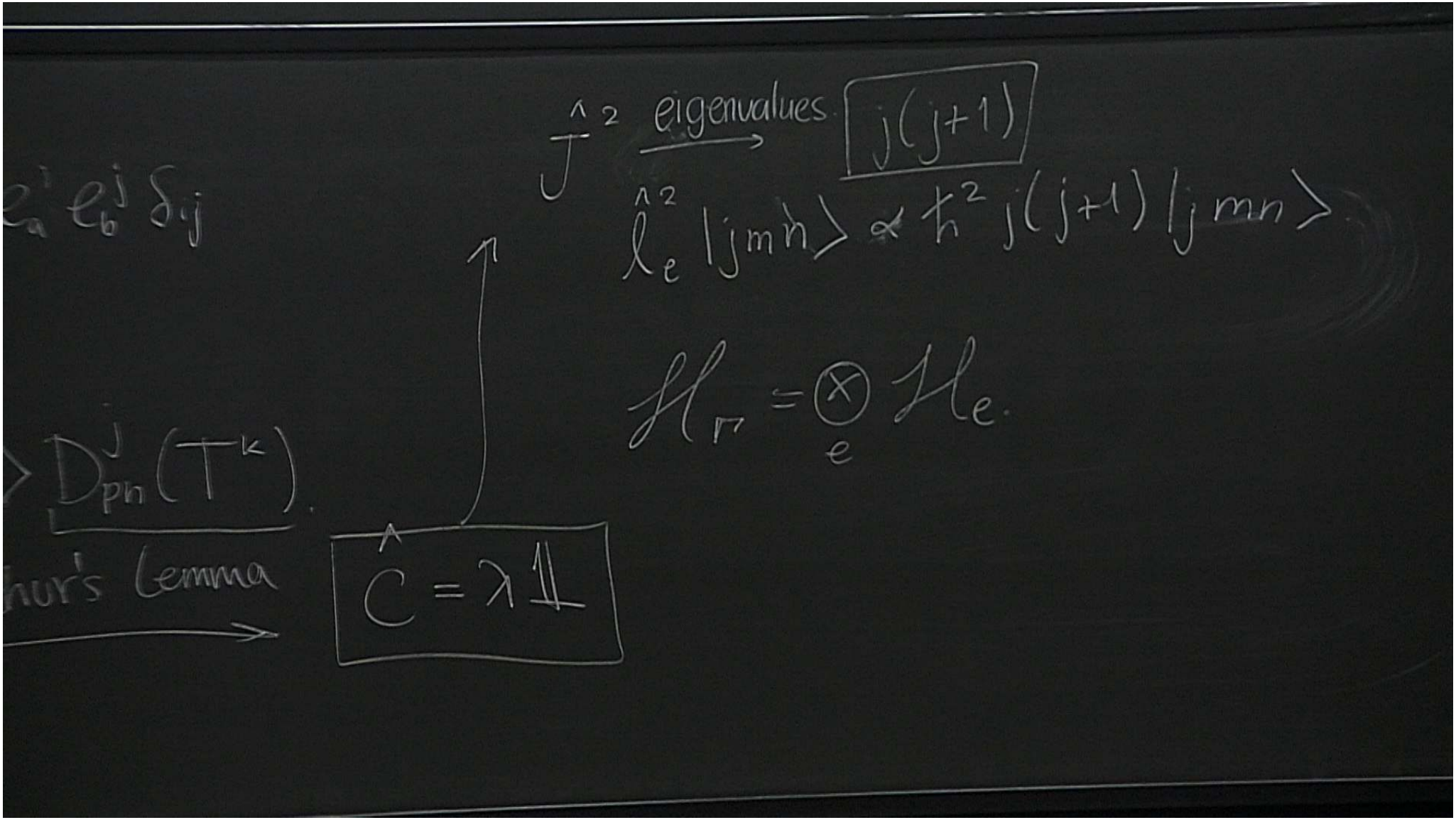
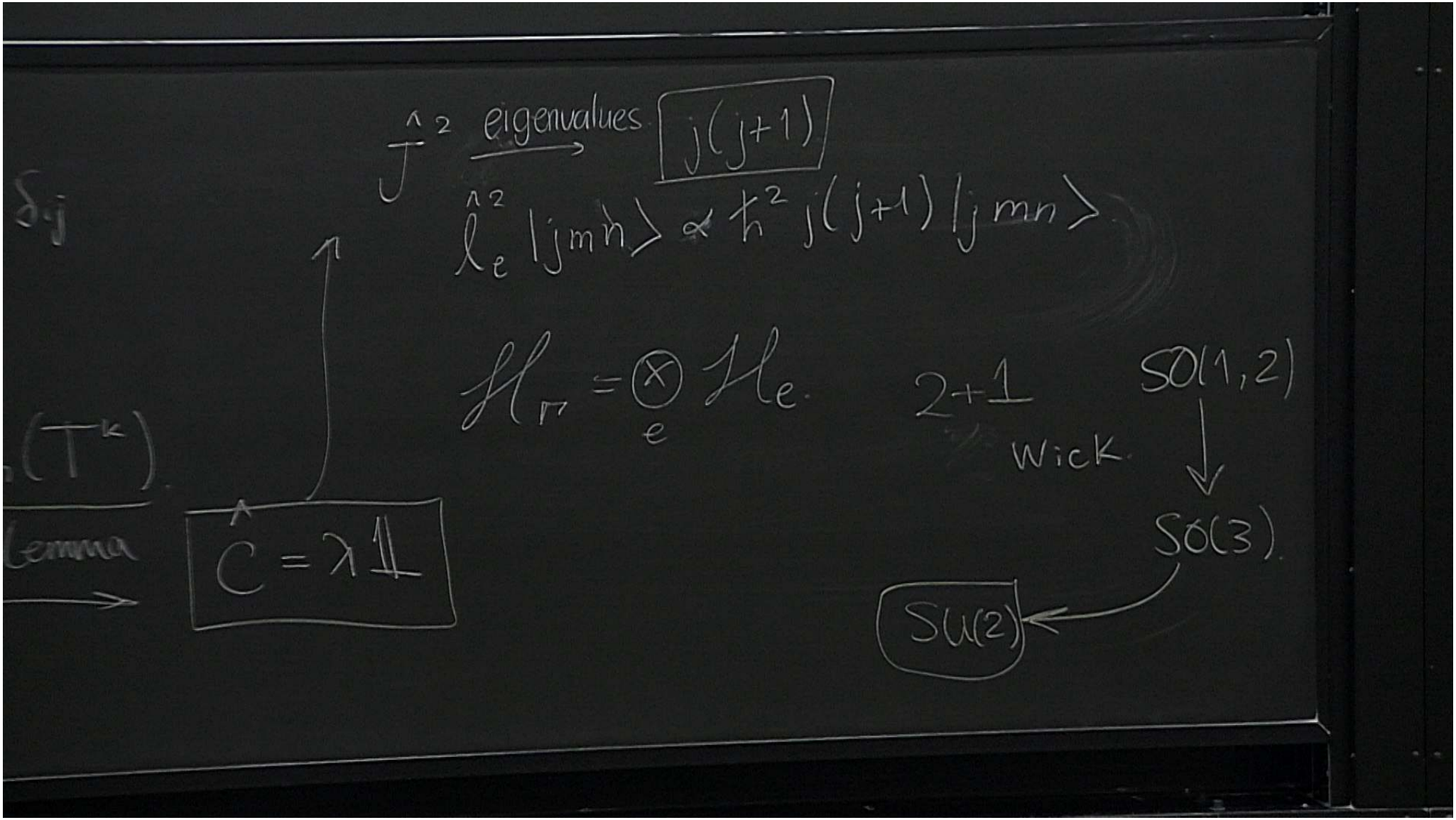
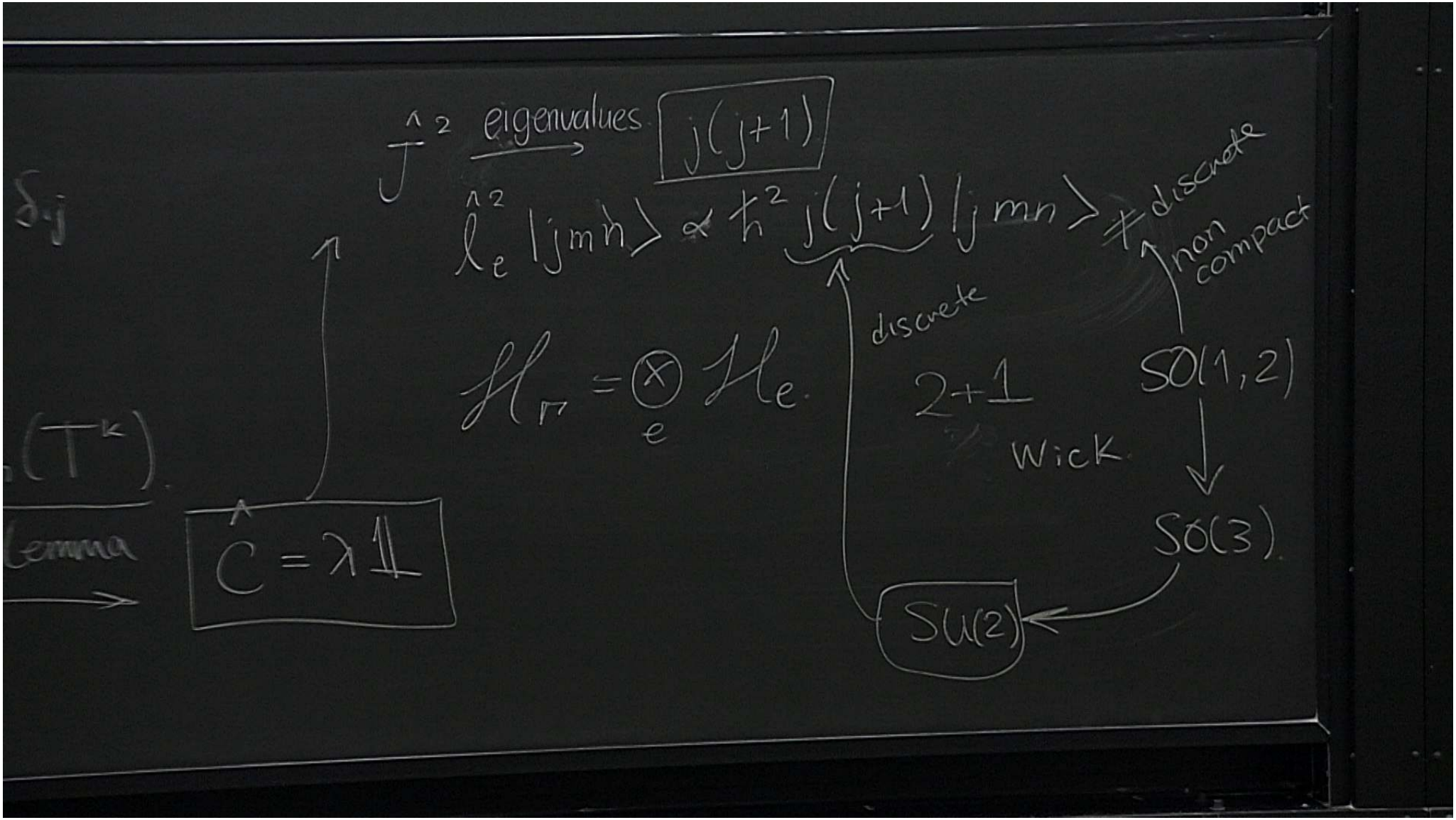
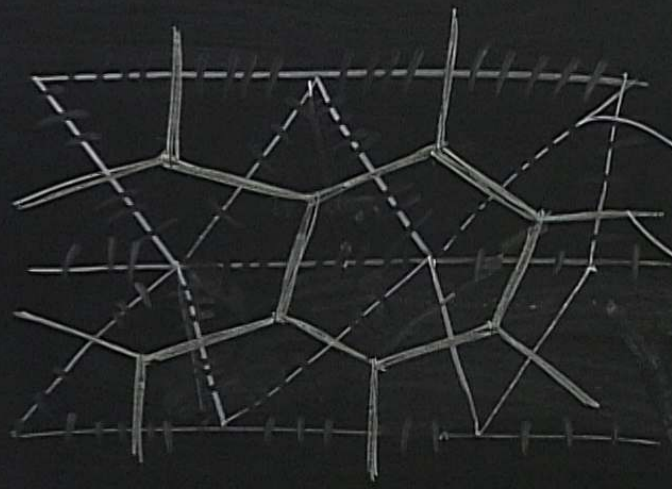

Γ : graph.
E : fluxes.
h_γ : holonomies.


$l = \int \sqrt{\dot{\gamma}^a \dot{\gamma}^b q_{ab}}$
$q_{ab} = e_a^i e_b^j \delta_{ij}$
$\hat{J}^2$
$\left[\hat{l}_e^2 \propto \sum_k \hat{E}_e^k \hat{E}_e^k\right]$
$\hat{E}^k |jmn\rangle = i\hbar \sum_p |jmp\rangle D^j_{pn}(T^k)$
$\hat{C} \propto D^2$
$[\hat{C}, D] = 0$
Schur's Lemma
$\hat{C} = \lambda \mathbb{1}$


$\hat{J}^2$ eigenvalues $j(j+1)$
$\hat{l}_e^2 |jmn\rangle \propto \hbar^2 j(j+1) |jmn\rangle$
$\mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e$
$D^j_{pn}(T^k)$
Schur's Lemma
$\hat{C} = \lambda \mathbb{1}$


$\hat{J}^2$ eigenvalues $j(j+1)$
$\hat{l}_e^2 |jmn\rangle \propto \hbar^2 j(j+1) |jmn\rangle$
$\mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e$
$2+1$
Wick
$SO(1,2)$
$SO(3)$
$SU(2)$
$\hat{C} = \lambda \mathbb{1}$
lemma
(T^k)
δ_{ij}


$\hat{J}^2$ eigenvalues $j(j+1)$
$\hat{l}_e^2 |jmn\rangle \propto \hbar^2 j(j+1) |jmn\rangle$
$\mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e$
discrete
non compact
2+1
Wick
SO(1,2)
SO(3)
SU(2)
$\hat{C} = \lambda \mathbb{1}$
lemma
(T^k)



Γ : graph

E : fluxes

h_γ : holonomies

$$l = \int \sqrt{\dot{\gamma}^a \dot{\gamma}^b q_{ab}} \qquad q_{ab} =$$

$$\left[\hat{l}_e^2 \propto \sum_k \hat{E}_e^k \hat{E}_e^k \right]$$

$$\underline{\hat{E}^k |jmn\rangle} = i\hbar \sum_p \; |jmp$$

$$\hat{C} \propto \hat{D}^2$$

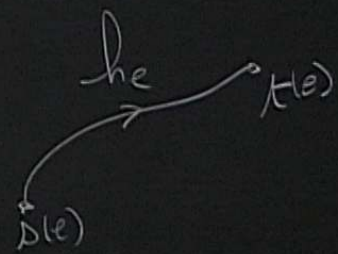
$$[\hat{C}, \hat{D}] = 0$$

The physical Hilbert space

→ Gauss constraints ($\propto \partial \underline{E} + \underline{\varepsilon} A E$)

↳ generate internal rotation

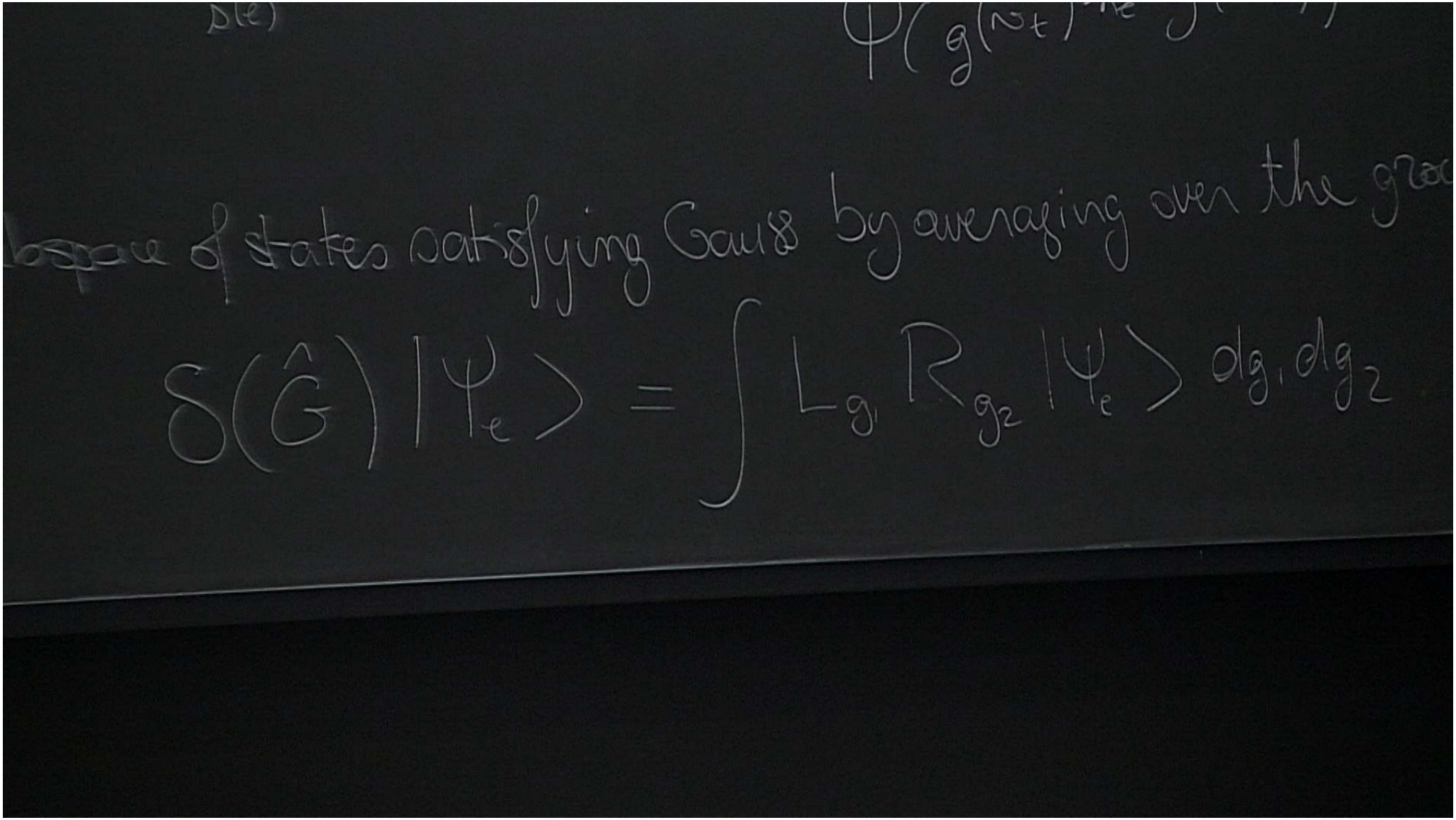
(Finite) Gauge transfo on holonomy:

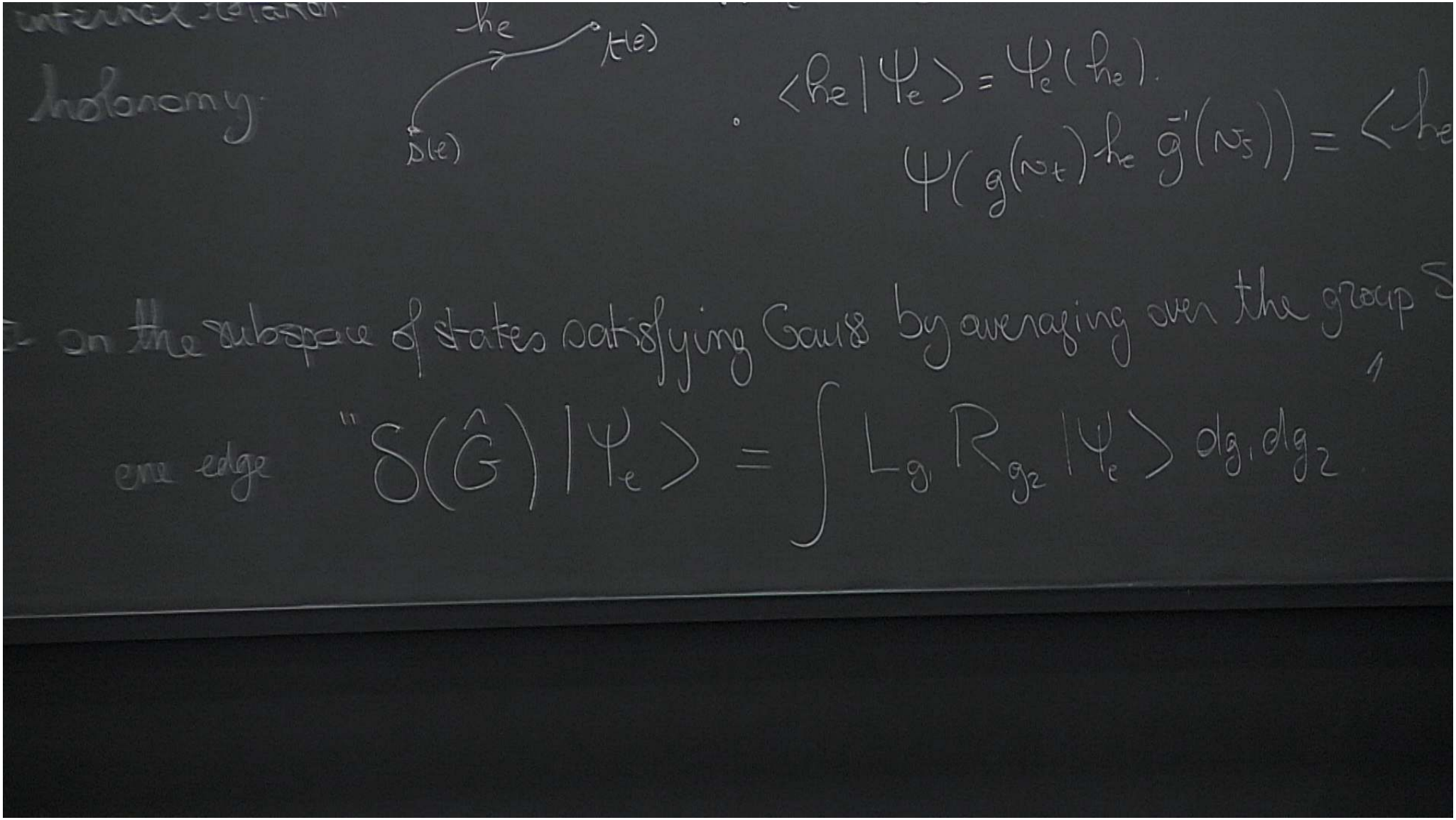


- $h_e \longrightarrow g(v_t)\, h_e\, g^{-1}(v_s)$.
- $\langle h_e | \psi_e \rangle = \psi_e(h_e)$.
- $\psi(g(v_t)\, h_e\, g^{-1}(v_s)) = \langle h_e | L_{g(v_t)^{-1}} R_{g^{-1}(v_s)} | \psi_e \rangle$.

$$(L_h f)(g) = f(h^{-1} g).$$
$$(R_h f)(g) = f(gh).$$

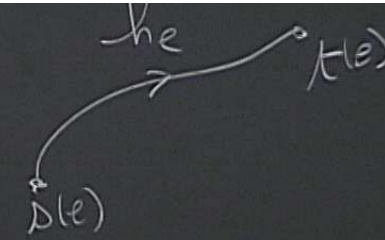
$$= \langle \psi_e | L_{g(\vec{N}_t)^{-1}} R_{g^{-1}(\vec{N}_s)} | \psi_e \rangle$$


space of states satisfying Gauss by averaging over the gro
$\delta(\hat{G})|\psi_e\rangle = \int L_{g_1} R_{g_2} |\psi_e\rangle \, dg_1 \, dg_2$


holonomy
h_e
$t(e)$
$s(e)$
$\langle h_e | \psi_e \rangle = \psi_e(h_e)$
$\psi(g(v_t) h_e \bar{g}^{-1}(v_s)) = \langle h_e$
on the subspace of states satisfying Gauss by averaging over the group
one edge
$\hat{\delta}(\hat{G}) |\psi_e\rangle = \int L_{g_1} R_{g_2} |\psi_e\rangle \, dg_1 dg_2$

↳ generate internal rotation

(Finite) Gauge transfo on holonomy:



→ Implement a projector on the subspace of states satisfying

one edge "$\delta(\hat{G})\,|\psi_e\rangle =$

$h_e \longrightarrow g(v_t) h_e g^{-1}(v_s)$.

$\langle h_e | \psi_e \rangle = \psi_e(h_e)$.

$\psi(g(v_t) h_e g^{-1}(v_s)) = \langle h_e | L_{g(v_t)^{-1}} R_{g^{-1}(v_s)} | \psi_e \rangle$

Gauge by averaging over the group $SU(2)$ for both ends of the edge.

$= \int L_{g_1} R_{g_2} | \psi_e \rangle \, dg_1 \, dg_2$

Ex: do it explicitly for the parametrized particle:

$$\hookrightarrow \delta(\hat{C})\,\Psi(t,q) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}}$$

$$\int e^{\frac{i}{\hbar} S\left(\hat{p}_{\alpha}+\hat{h}\right)} \psi(t,q)\, dS \overset{?}{=} \frac{1}{2\pi\hbar} \int_{\mathbb{R}^3}$$

$\left(\frac{\hat{p}_q^2}{2m}\right)$

some work to do!

$$\frac{1}{2\pi\hbar}\int_{\mathbb{R}} \underbrace{\psi(t,q)}_{q}\, dq$$

ψ: solution of Schrödinger eq.

Ex: do it explicitly for the parametrized particle.

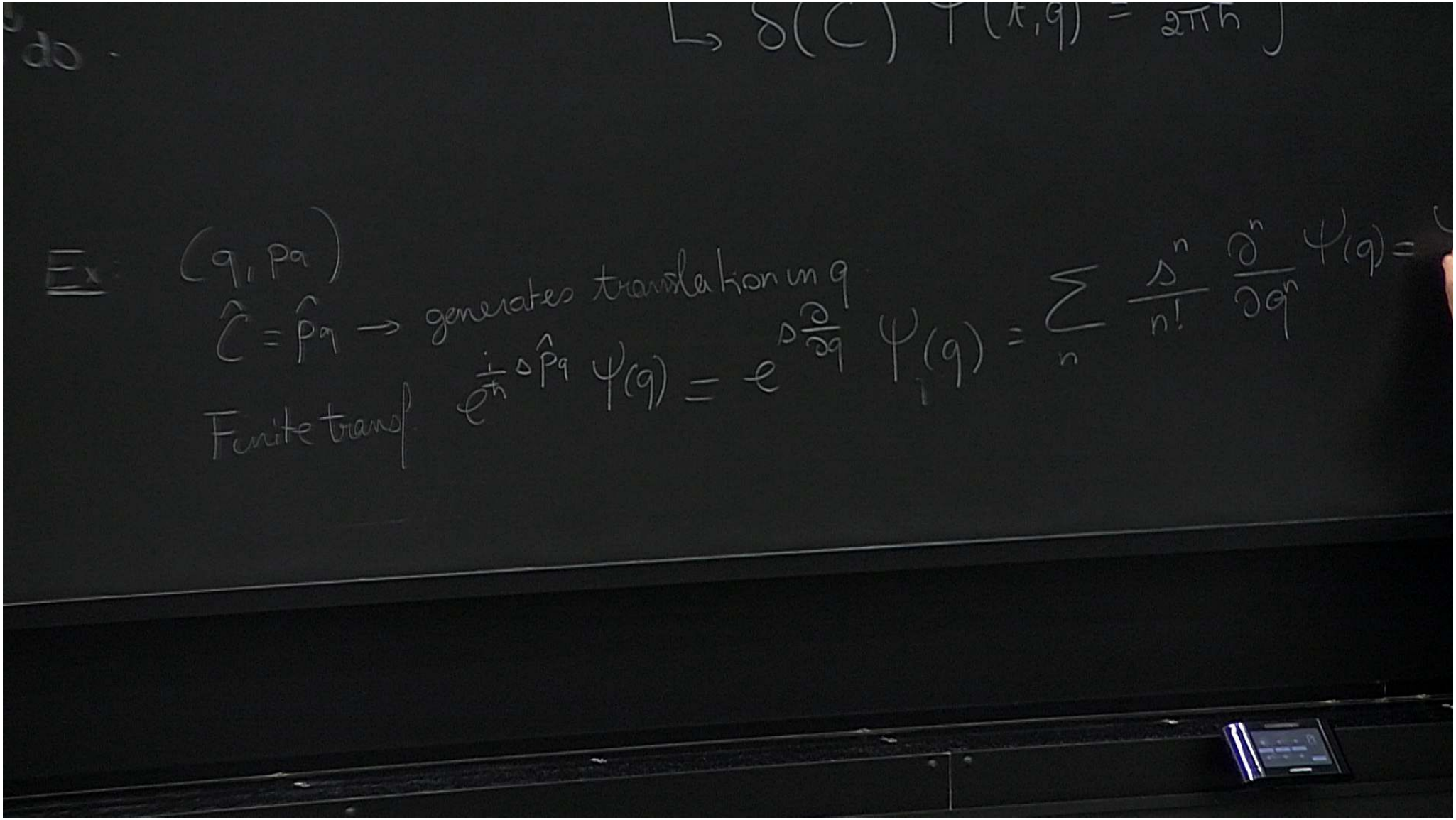
↓

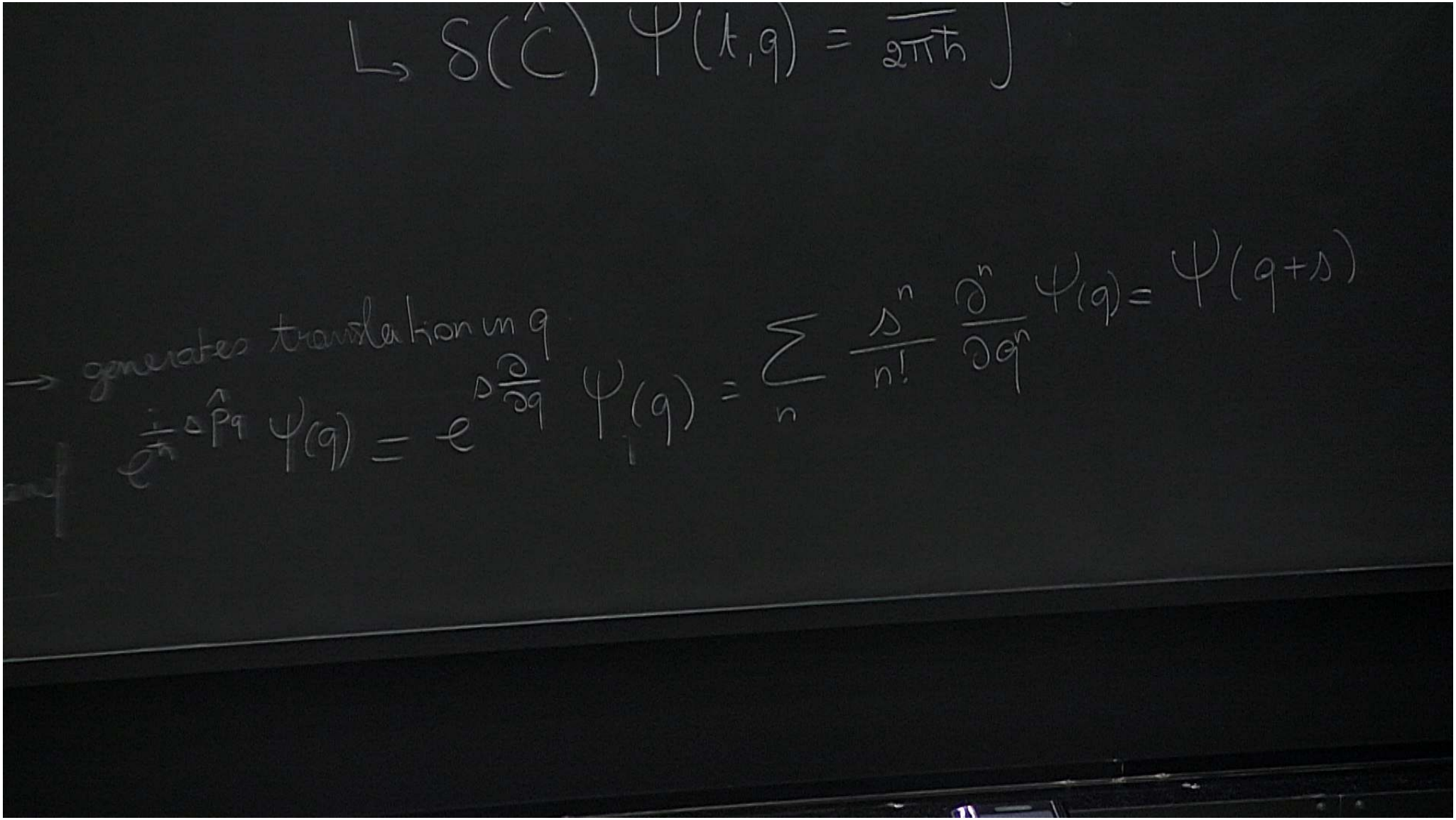
to do.

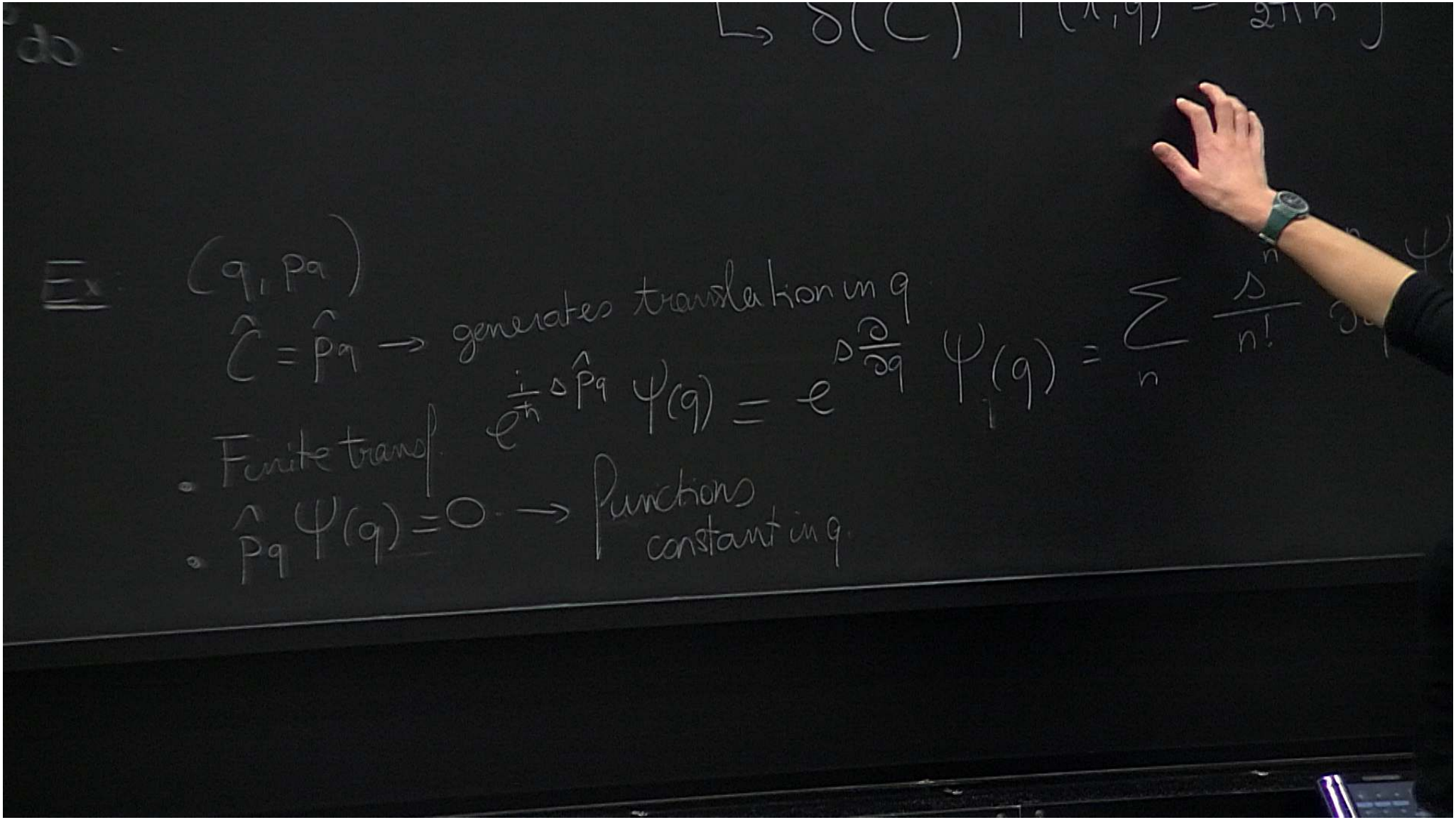
$\hookrightarrow \delta(\hat{C})\,\Psi(t,q) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}s(\hat{p}t+h$

Ex: (q, p_q)

$\hat{C} = \hat{p}_q \rightarrow$ generates translation in q


do
$\hookrightarrow \delta(C)\,\psi(x,q) = \frac{}{2\pi\hbar}$
Ex: (q, p_q)
$\hat{C} = \hat{p}_q \rightarrow$ generates translation in q
Finite transl $e^{\frac{i}{\hbar}\delta\hat{p}_q}\psi(q) = e^{\delta\frac{\partial}{\partial q}}\psi(q) = \sum_n \frac{\delta^n}{n!}\frac{\partial^n}{\partial q^n}\psi(q) =$


$\hat{C}\,\psi(t,q) = \frac{}{2\pi\hbar}$
$\rightarrow$ generates translation in q
$e^{\frac{i}{\hbar}\Delta\hat{p}_q}\psi(q) = e^{\Delta\frac{\partial}{\partial q}}\psi(q) = \sum_n \frac{\Delta^n}{n!}\frac{\partial^n}{\partial q^n}\psi(q) = \psi(q+\Delta)$


Ex: (q, p_q)
$\hat{C} = \hat{p}_q \rightarrow$ generates translation in q
Finite transf: $e^{\frac{i}{\hbar} s \hat{p}_q} \psi(q) = e^{s \frac{\partial}{\partial q}} \psi(q) = \sum_n \frac{s^n}{n!}$
$\hat{p}_q \psi(q) = 0 \rightarrow$ functions constant in q

$\left(\frac{\hat{p}_q^2}{2m}\right)$

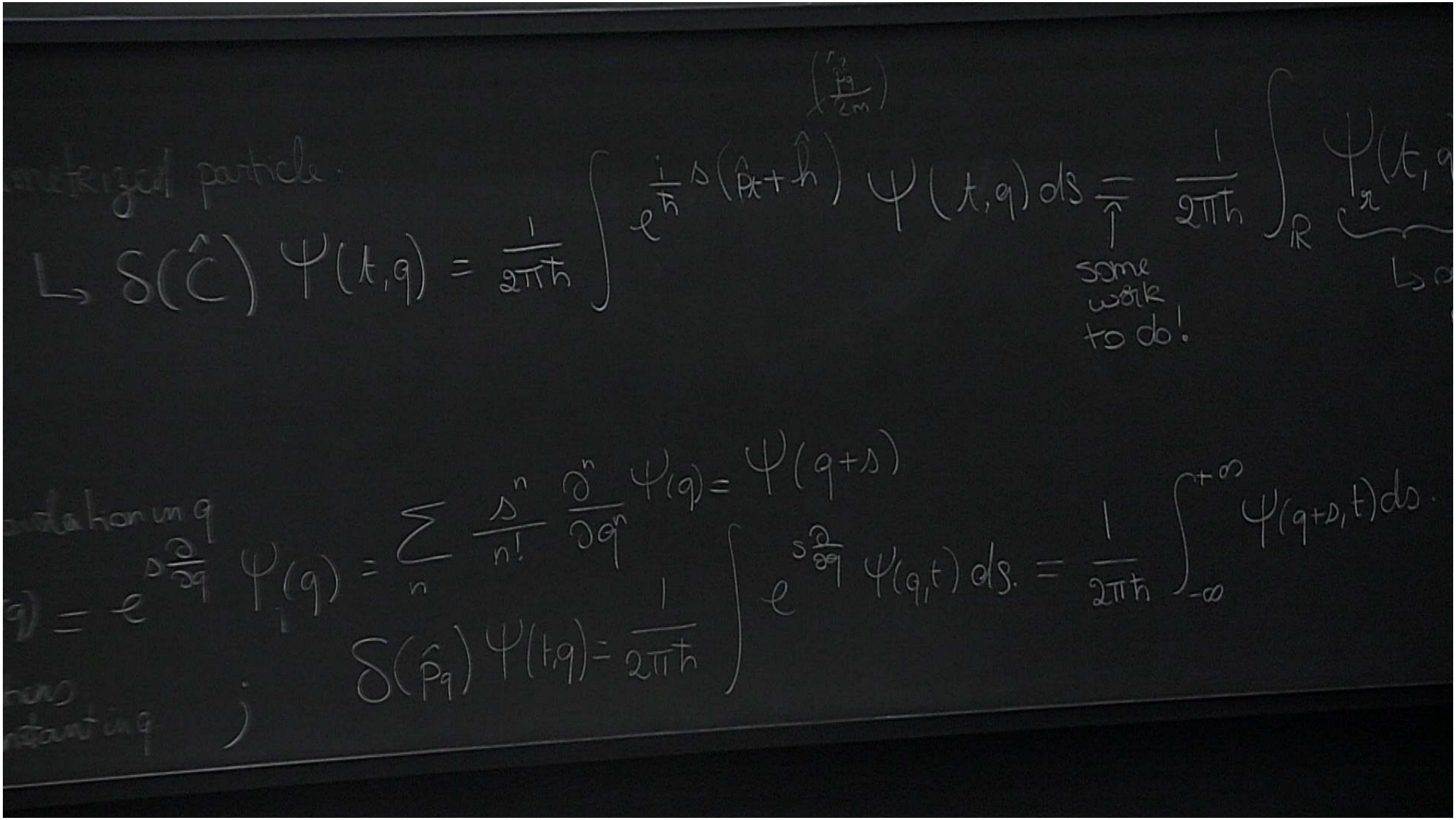
$\hookrightarrow \delta(\hat{C})\,\psi(t,q) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}s(\hat{p}t+\hat{h})}\,\psi(t,q)\,ds \overset{\uparrow}{=} \frac{1}{2\pi\hbar}\int_R \underbrace{\psi_x(t,q}_{\hookrightarrow}$

same work to do!

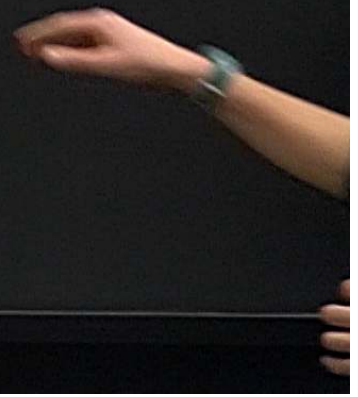
$$\sum_n \frac{s^n}{n!}\frac{\partial^n}{\partial q^n}\psi(q) = \psi(q+s)$$

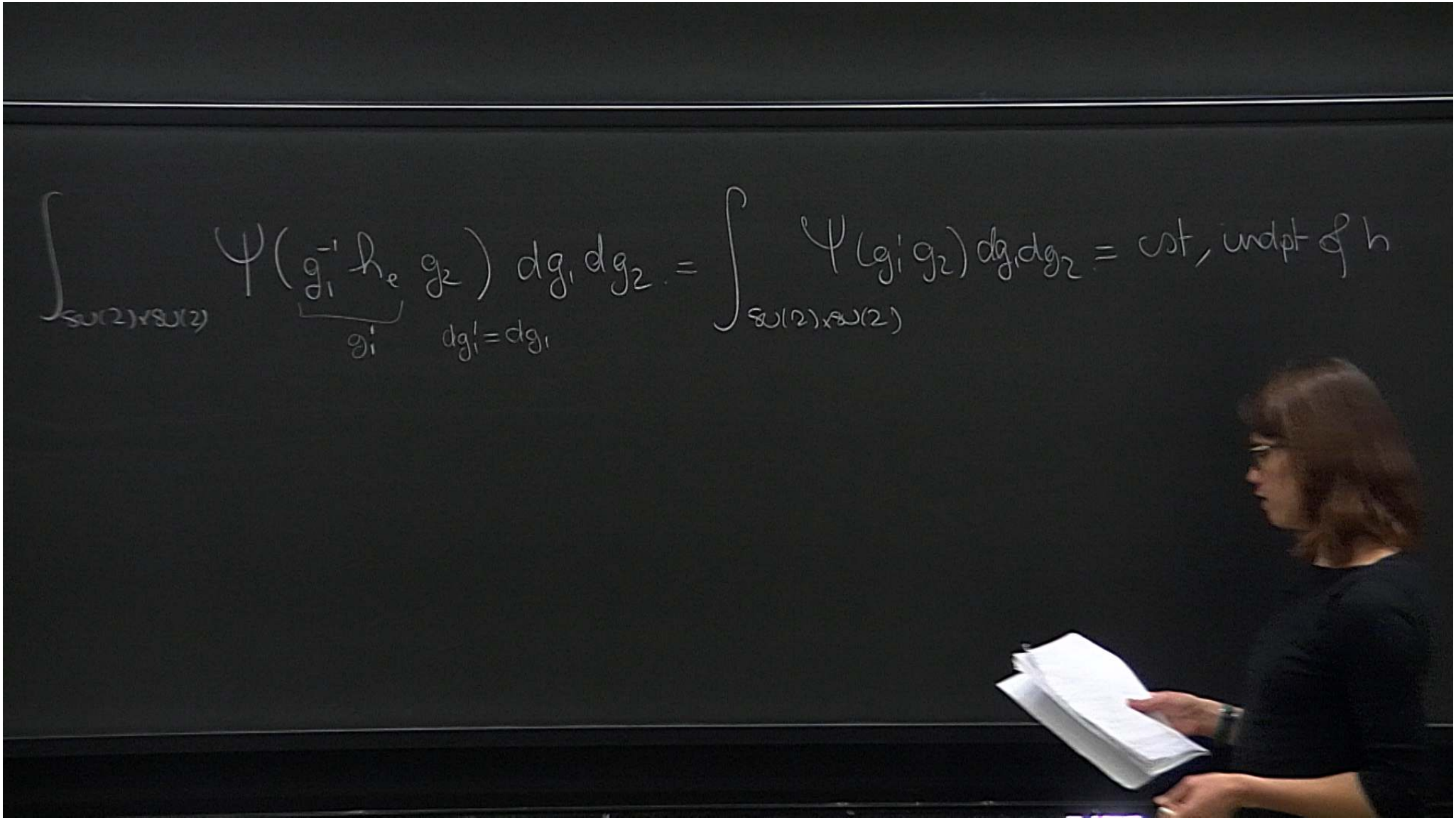
$$e^{s\frac{\partial}{\partial q}}\psi(q) = \psi(q+s)$$

$$\delta(\hat{p}_q)\,\psi(t,q) = \frac{1}{2\pi\hbar}\int e^{s\frac{\partial}{\partial q}}\psi(q,t)\,ds.$$


$\left(\frac{\hat{p}_q^2}{2m}\right)$
$\hat{L} \to \delta(\hat{C})\,\psi(t,q) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}s(\hat{p}t+\hat{h})}\,\psi(t,q)\,ds \overset{?}{=} \frac{1}{2\pi\hbar}\int_R \psi_x(t,q$
some work to do!
$\to$
$\delta(\hat{p}_q)\,\psi(t,q) = \frac{1}{2\pi\hbar}\int e^{s\frac{\partial}{\partial q}}\,\psi(q,t)\,ds. = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\psi(q+s,t)\,ds.$
$e^{s\frac{\partial}{\partial q}}\,\psi(q) = \sum_n \frac{s^n}{n!}\frac{\partial^n}{\partial q^n}\,\psi(q) = \psi(q+s)$

$$\int_{SU(2)\times SU(2)} \Psi(g_1^{-1} h_e g_2)\, dg_1\, dg_2 .$$

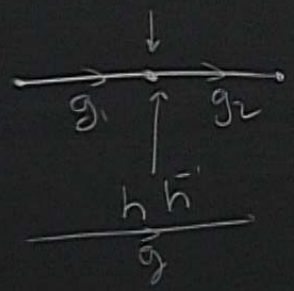



$\int_{SU(2)\times SU(2)} \Psi(g_1^{-1} h_e g_2)\, dg_1\, dg_2 = \int_{SU(2)\times SU(2)} \Psi(g_1' g_2)\, dg_1' dg_2 = \text{cst}$, indpt of h
g_1'
$dg_1' = dg_1$

→ space of solutions = trivial

$$\int_{SU(2)\times SU(2)} \Psi(\underbrace{g_1^{-1} h_e}_{g_1'}\, g_2)\, dg_1\, dg_2 = \int_{SU(2)\times SU(2)} \Psi(g_1, g_2)\, dg_1 dg_2 = \text{cst, indpt of } h$$

$dg_1' = dg_1$

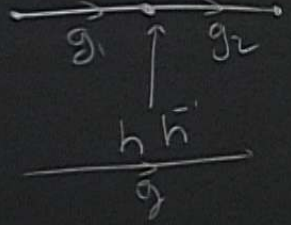


$\longrightarrow \Psi(\underbrace{g_1 g_2}_{g})$

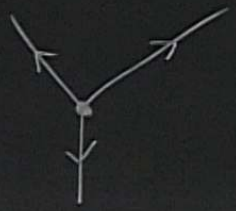
$$\int_{SU(2)\times SU(2)} \psi(g_1^{-1} h_e g_2)\, dg_1\, dg_2 = \int_{SU(2)\times SU(2)} \psi(g_1 g_2)\, dg_1\, dg_2 = \text{cst, indpt of } h$$

g_1 $\quad dg_1' = dg_1$

2-valent vertex

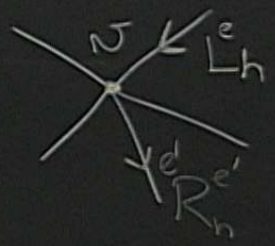


$\rightarrow \psi(\underbrace{g_1 g_2}_{g})$



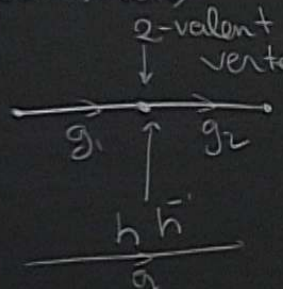
= cst, indpt of h → space of solutions = trivial

$$\delta(\hat{G}_N) = \int_{SU(2)} \left(\bigotimes_{e / N = N_{t(e)}} L_h^e \right) \otimes \left(\bigotimes_{e' / N = N_{s(e)}} R_h^{e'} \right)$$

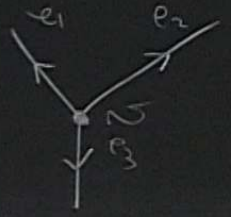


$$\int_{SU(2)\times SU(2)} \Psi(g_1^{-1} h_e g_2)\, dg_1\, dg_2 = \int_{SU(2)\times SU(2)} \Psi(g_1, g_2)\, dg_1 dg_2 = \text{cst, indpt of } h$$

g_1' $\quad dg_1' = dg_1$



$$\Psi(g_1, g_2)$$
g



$$S(\hat{G}_\Gamma)\left(\bigotimes_{i=1}^{3} |j_i, m_i, n_i\rangle\right) = \int dh\, R_h^{e_1} R_h^{e_2} R_h^{e_3} \left(\bigotimes_{i=1}^{3} |j_i, m_i, n_i\rangle\right)$$

$$\int_{SU(2)\times SU(2)} \Psi(g_1^{-1} h_e g_2)\, dg_1\, dg_2 = \int_{SU(2)\times SU(2)} \Psi(g_1 g_2)\, dg_1 dg_2 = \text{cst, indpt of } h \quad \rightarrow \text{space of}$$

g_1 $dg_1' = dg_1$

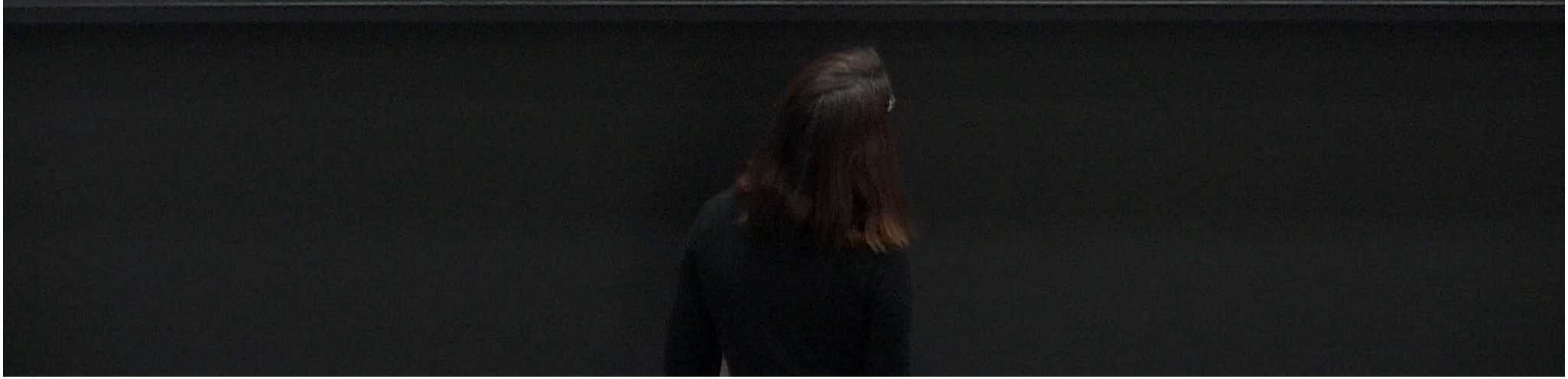
$$\rightarrow \Psi(g_1, g_2)$$

$$\hat{S}(\hat{G}_{\Gamma})\left(\bigotimes_{i=1}^{3} |j_i, m_i, n_i\rangle\right) = \int dh\, R_h^{e_1} R_h^{e_2} R_h^{e_3} \left(\bigotimes_{i=1}^{3} |j_i, m_i, n_i\rangle\right)$$

look at yesterday lecture notes

$$= \int dh \sum_{p_1 p_2 p_3} D^{j_1}_{p_1 n_1}(h) D^{j_2}_{p_2 n_2}(h) D^{j_3}_{p_3 n_3}(h) \left(\bigotimes_{i=1}^{3} |j_i, m_i, p_i\rangle\right)$$

$$\hat{S}(\hat{G}_{\Gamma}) =$$



look at yesterday lecture notes

$$= \int dh \sum_{p_1 p_2 p_3} D^{j_1}_{p_1 n_1}(h) D^{j_2}_{p_2 n_2}(h) D^{j_3}_{p_3 n_3}(h) \left(\bigotimes_{i=1} |j_i, m_i, p_i\rangle \right)$$

From recoupling theory

$$\int dh\, D^{j_1}_{m_1 n_1}(h)\, D^{j_2}_{m_2 n_2}(h)\, D^{j_3}_{m_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

↳ Wigner 3jm symbol

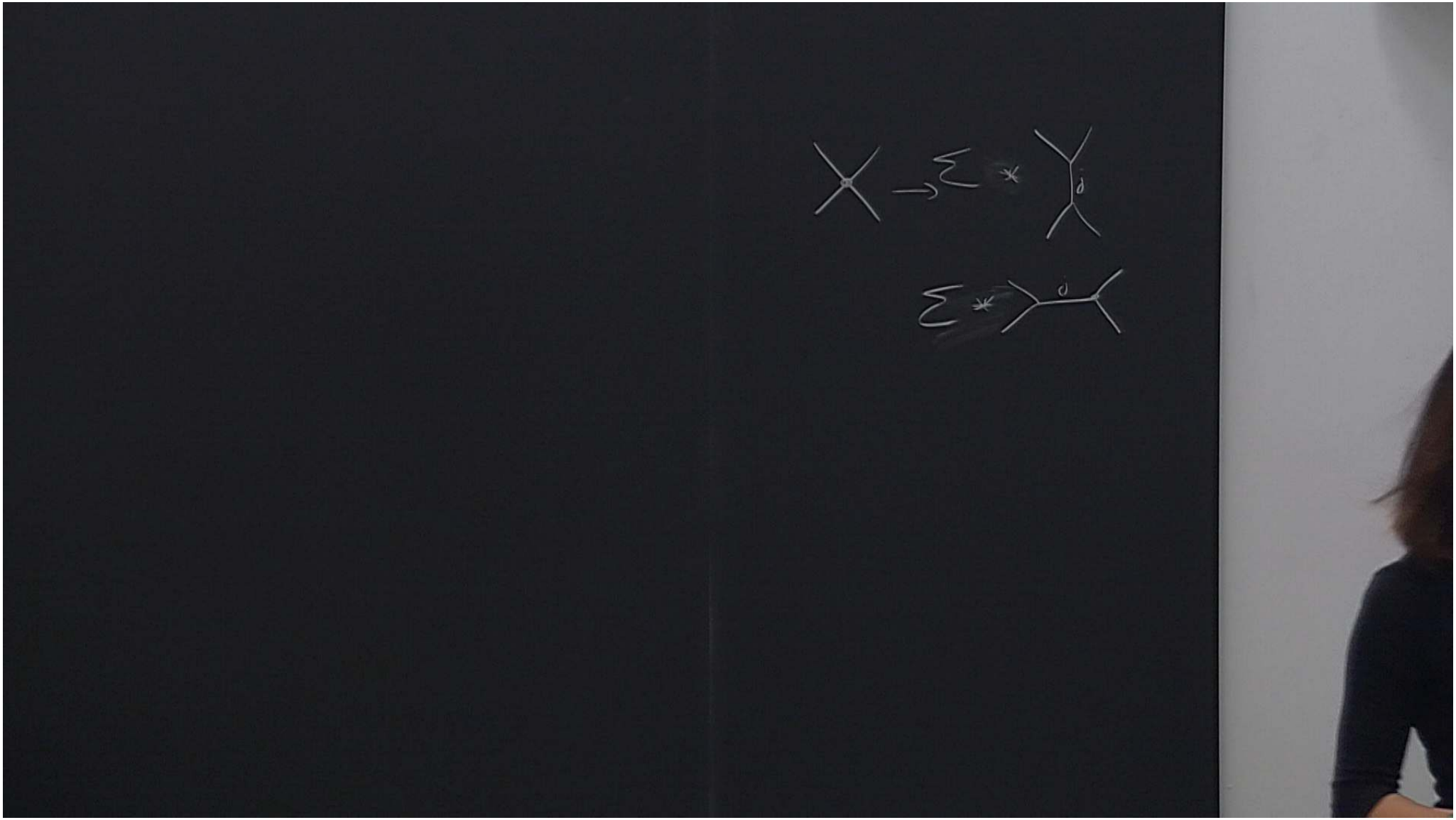
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}}$$

$$= \int dh \sum_{p_1 p_2 p_3} D^{\cdots}_{p_1 n_1}(h)\, D^{\cdots}_{p_2 n_2}(h)\, D^{\cdots}_{p_3 n_3}(h) \left(\bigotimes_{i=1} \cdots \right) \qquad R^{\cdots}_{n}$$

$$\int dh\, D^{d_1}_{m_1 n_1}(h)\, D^{d_2}_{m_2 n_2}(h)\, D^{d_3}_{m_3 n_3}(h) = \begin{pmatrix} d_1 & d_2 & d_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} d_1 & d_2 & d_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

↳ Wigner 3jm symbol

$$\begin{pmatrix} d_1 & d_2 & d_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{d_1 - d_2 - m_3}}{\sqrt{2 j_1 + 1}} \underbrace{\langle d_1 m_1 ; d_2 m_2 \mid d_3, -m_3 \rangle}_{\text{C-G coeff}}$$



look at yesterday lecture notes

$$= \int dh \sum_{p_1 p_2 p_3} D^{\cdots}_{p_1 n_1}(h) D^{\cdots}_{p_2 m_2}(h) D^{\cdots}_{p_3 m_3}(h) \left(\prod_{i=1} g_i \right)$$

From recoupling theory

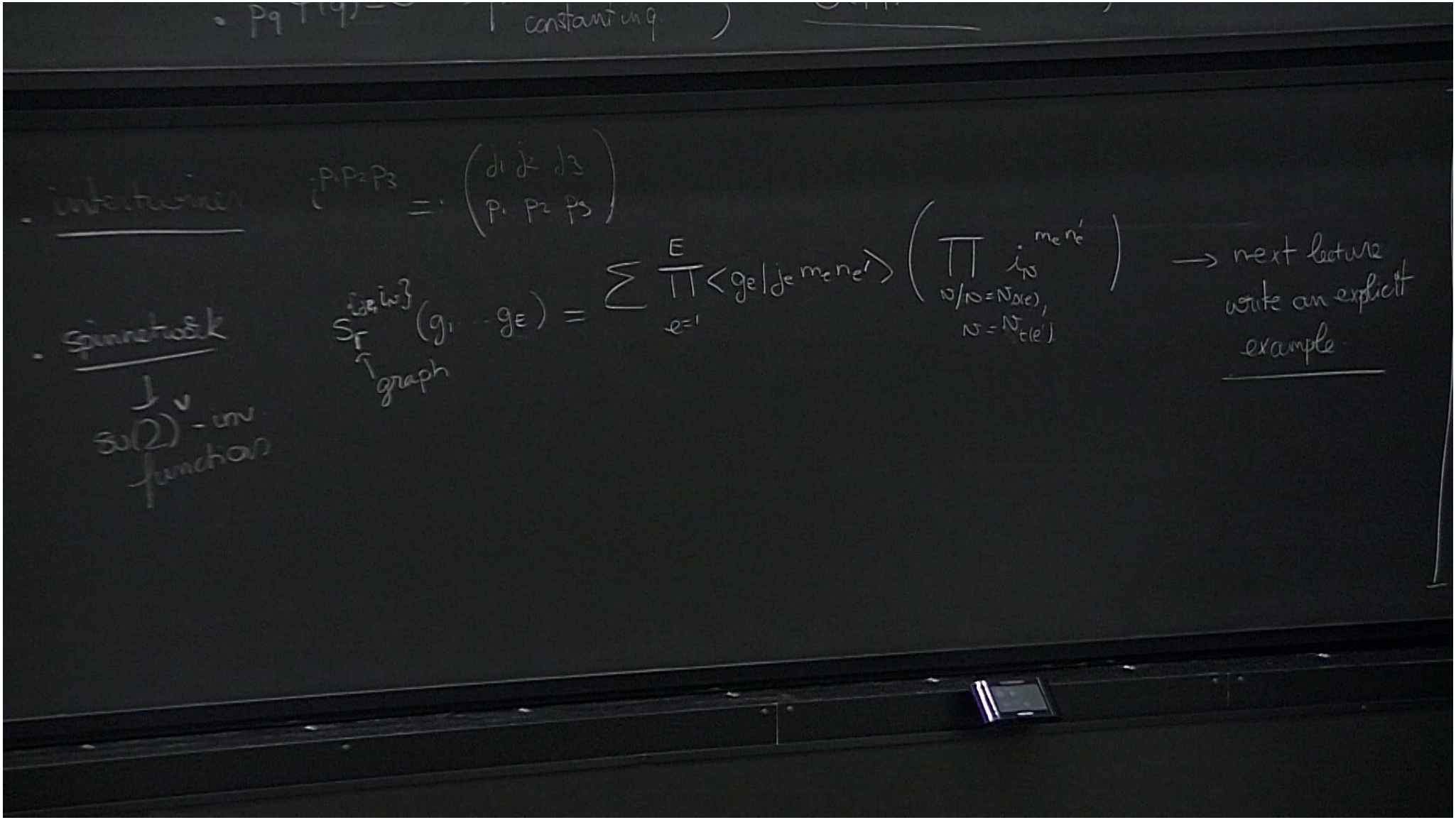
$$\int dh\, D^{d_1}_{m_1 n_1}(h_1)\, D^{d_2}_{m_2 n_2}(h_1)\, D^{d_3}_{m_3 n_3}(h) = \begin{pmatrix} d_1 & d_2 & d_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} d_1 & \\ n_1 & n \end{pmatrix}$$

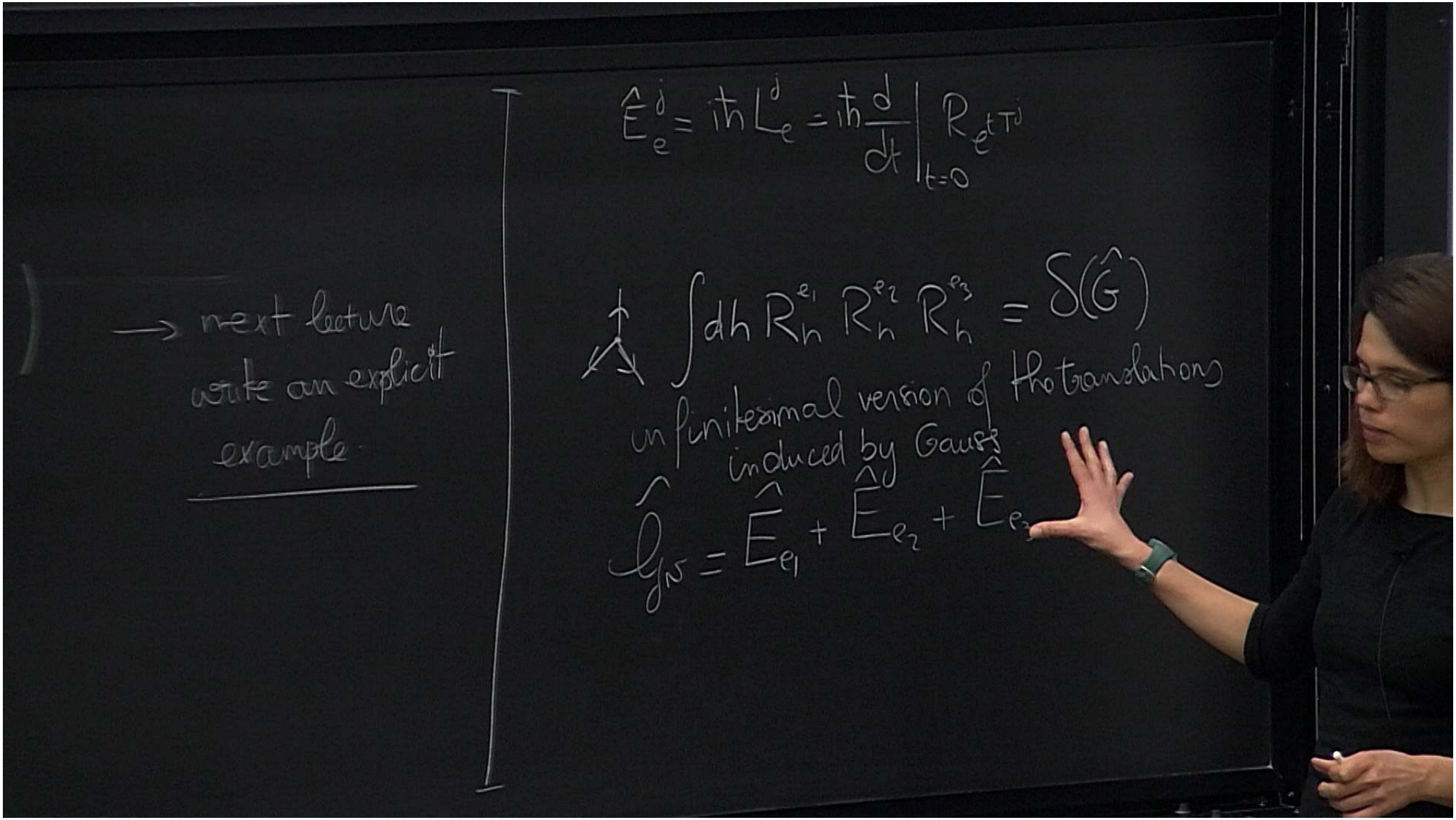
↳ Wigner 3jm

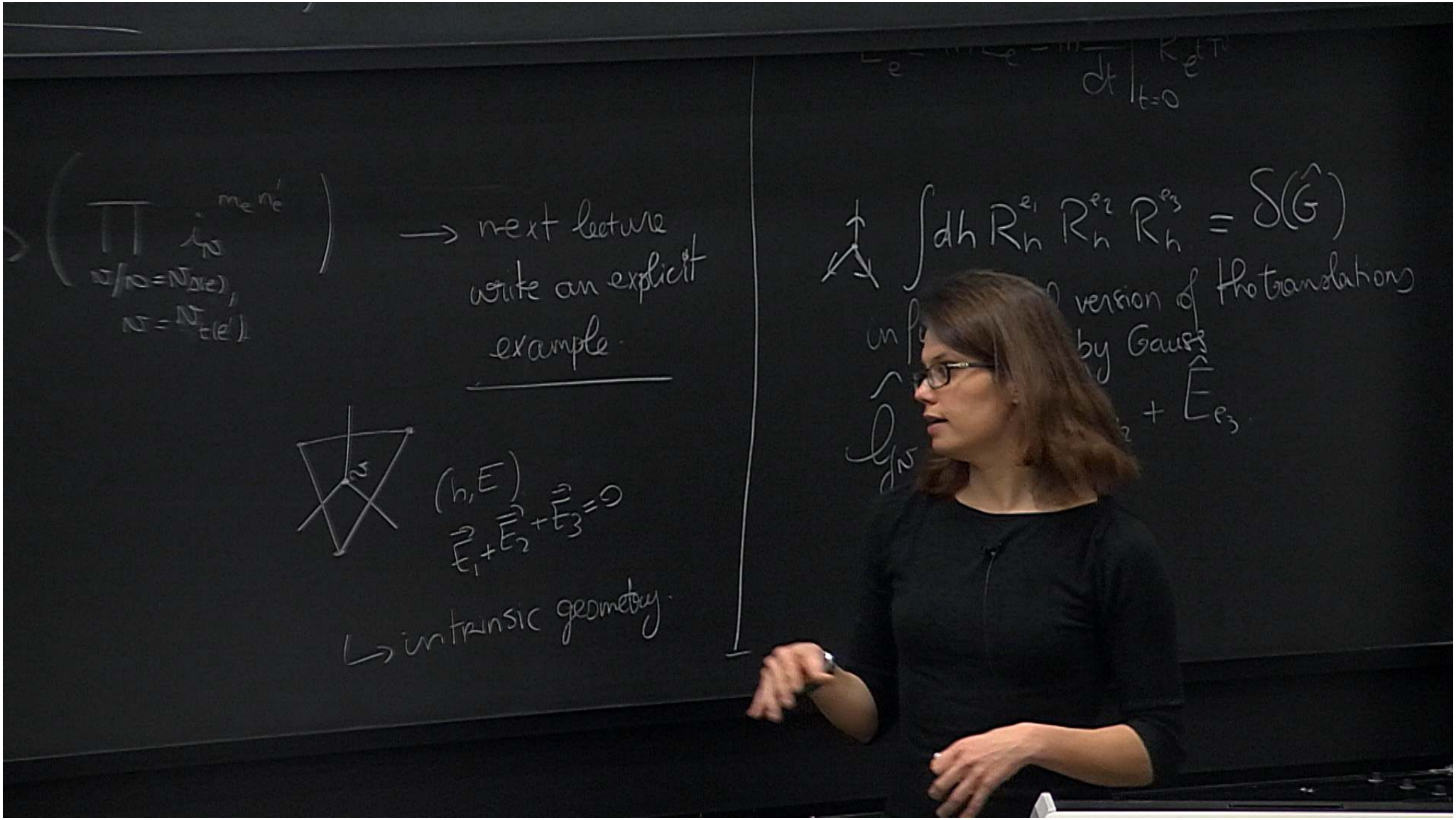
$$\begin{pmatrix} d_1 & d_2 & d_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =$$

$$\text{normalized state} = \sum_{p_1 p_2 p_3} \begin{pmatrix} d_1 & d_2 & d_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \otimes |d_1, m_1, p_1\rangle .$$

↳ unique solution (up to normalization)


constant in q
$p_1 p_2 = p_3$
$= \begin{pmatrix} d_1 & d_2 & d_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$
$S_\Gamma^{\{i_e, i_{e'}\}}(g_1 \dots g_E) = \sum \prod_{e=1}^{E} \langle g_e | j_e m_e n_{e'} \rangle \left(\prod_{v/v=N_{s(e)}, v=N_{t(e')}} i_v^{m_e n_{e'}} \right)$
graph
→ next lecture write an explicit example
$SU(2)^V$ - inv functions


$\hat{E}^j_e = i\hbar L^j_e = i\hbar \frac{d}{dt}\Big|_{t=0} R_{e^{tT^j}}$
$\int dh\, R^{e_1}_h R^{e_2}_h R^{e_3}_h = \delta(\hat{G})$
infinitesimal version of the translations
induced by Gauss
$\hat{G}_N = \hat{E}_{e_1} + \hat{E}_{e_2} + \hat{E}_{e_3}$
→ next lecture
write an explicit
example


→ next lecture write an explicit example.
(h, E)
$\vec{E}_1 + \vec{E}_2 + \vec{E}_3 = 0$
↳ intrinsic geometry.
$\int dh\, R_n^{e_1} R_n^{e_2} R_n^{e_3} = \delta(\hat{G})$
version of the translations
by Gauss