Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 10

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Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

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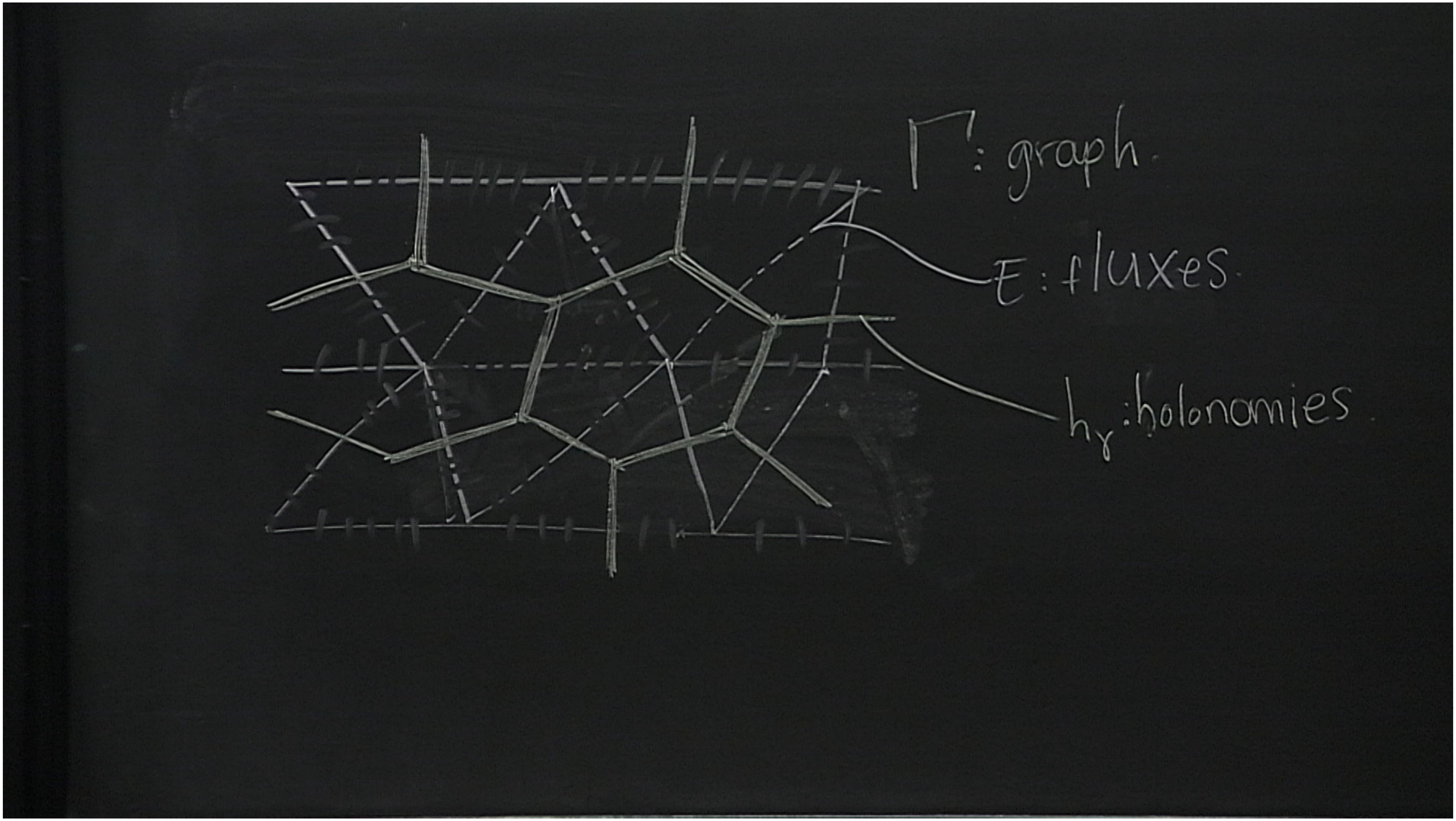
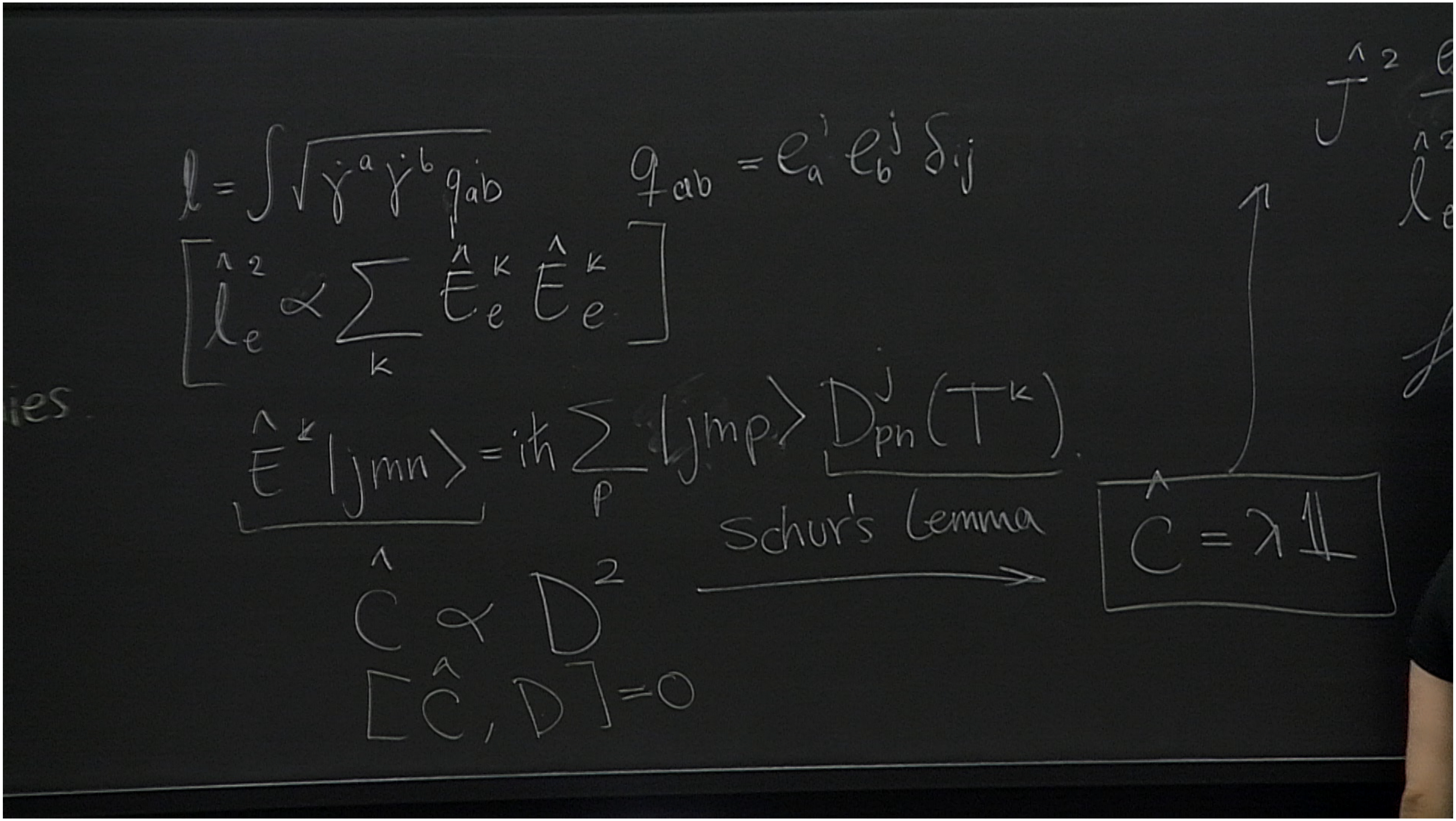
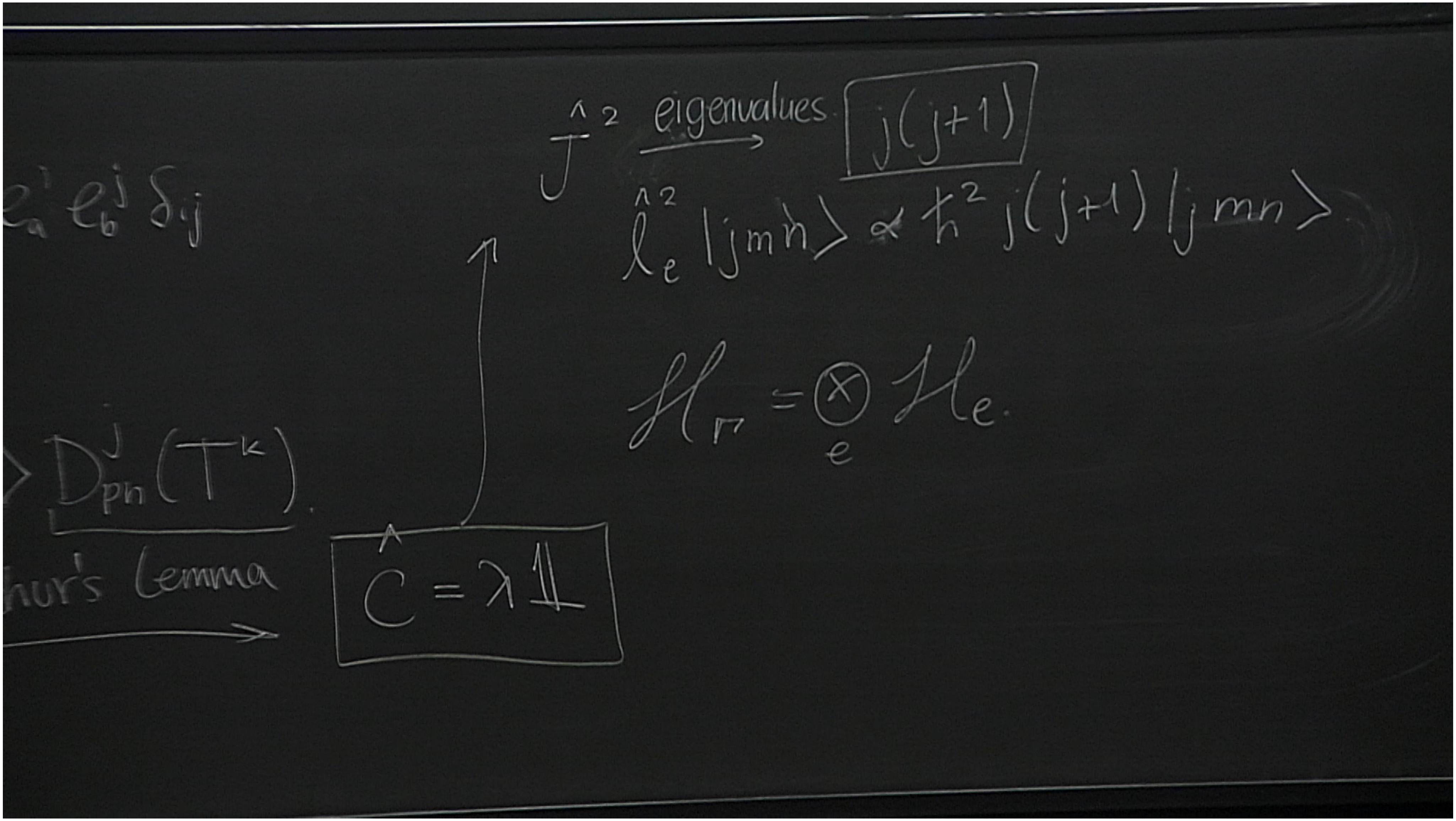
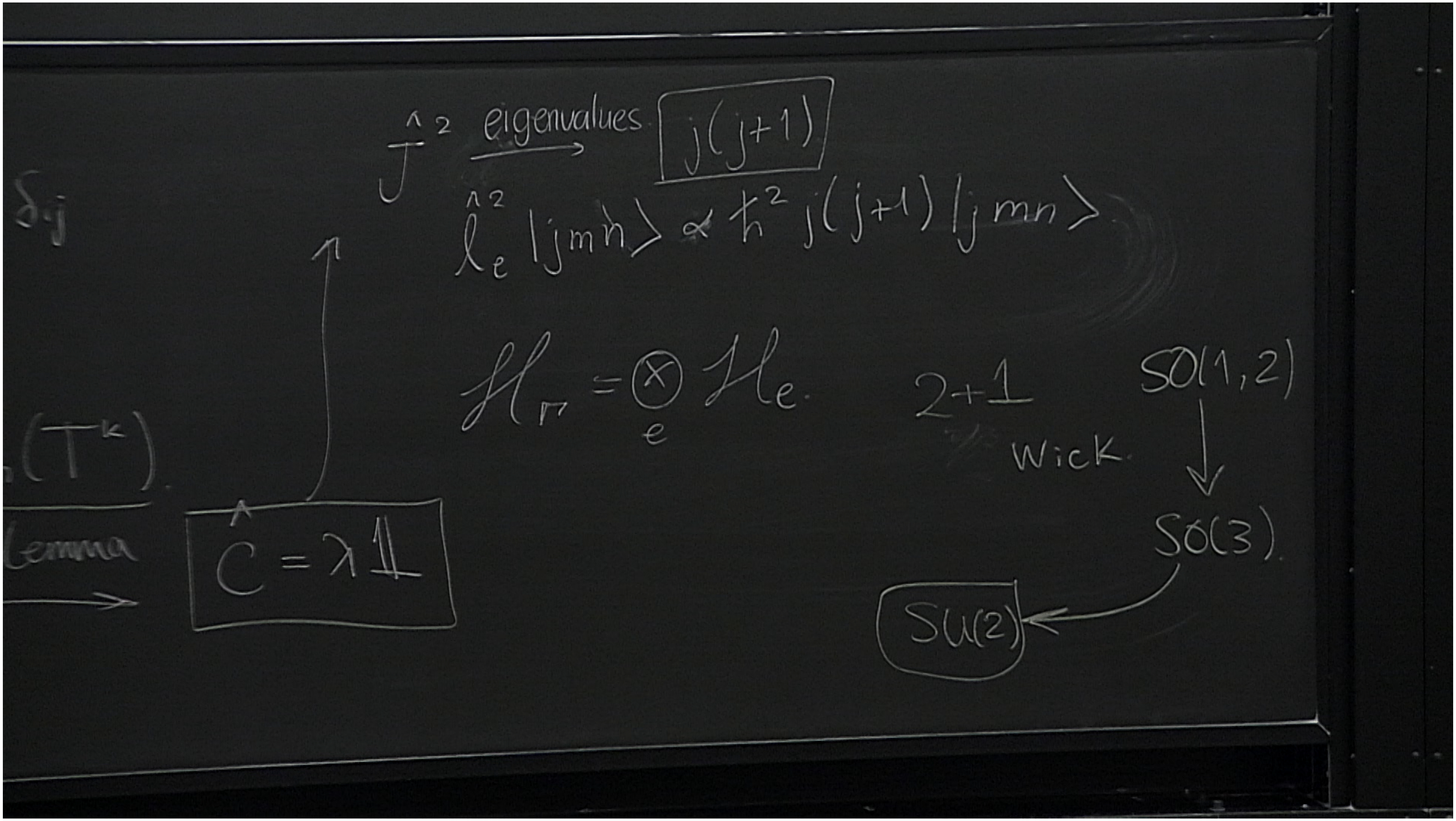

Γ: graph.
E: fluxes.
h_γ: holonomies.


$l = \int \sqrt{\dot\gamma^a \dot\gamma^b q_{ab}}$
$q_{ab} = e_a^i e_b^j \delta_{ij}$
$\left[\hat{l}_e^2 \propto \sum_k \hat{E}_e^k \hat{E}_e^k \right]$
$\hat{E}^k |jmn\rangle = i\hbar \sum_p |jmp\rangle D^j_{pn}(T^k)$
Schur's Lemma
$\hat{C} \propto D^2$
$[\hat{C}, D] = 0$
$\hat{C} = \lambda \mathbb{1}$
$\hat{J}^2$


$\hat{J}^2$ eigenvalues $j(j+1)$
$\hat{l}_e^2 |jmn\rangle \propto \hbar^2 j(j+1) |jmn\rangle$
$\mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e$.
$e_a^i e_b^j \delta_{ij}$
$D^j_{pn}(T^k)$
hur's Lemma
$\hat{C} = \lambda \mathbb{1}$



$\hat{J}^2$ eigenvalues $j(j+1)$
$\hat{l}_e^2 |jmn\rangle \propto \hbar^2 j(j+1) |jmn\rangle$
$\mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e$
2+1
Wick
SO(1,2)
SO(3)
SU(2)
(T^k)
Lemma
$\hat{C} = \lambda \mathbb{1}$

δ_{ij}

$\hat{J}^2 \xrightarrow{\text{eigenvalues}} \boxed{j(j+1)}$

$\hat{l}_e^2 \, |jmn\rangle \propto \hbar^2 \underbrace{j(j+1)} \, |jmn\rangle \neq$ discrete

(T^k).

lemma

$\boxed{\hat{C} = \lambda \mathbb{1}}$

$\mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e$.

discrete

2+1

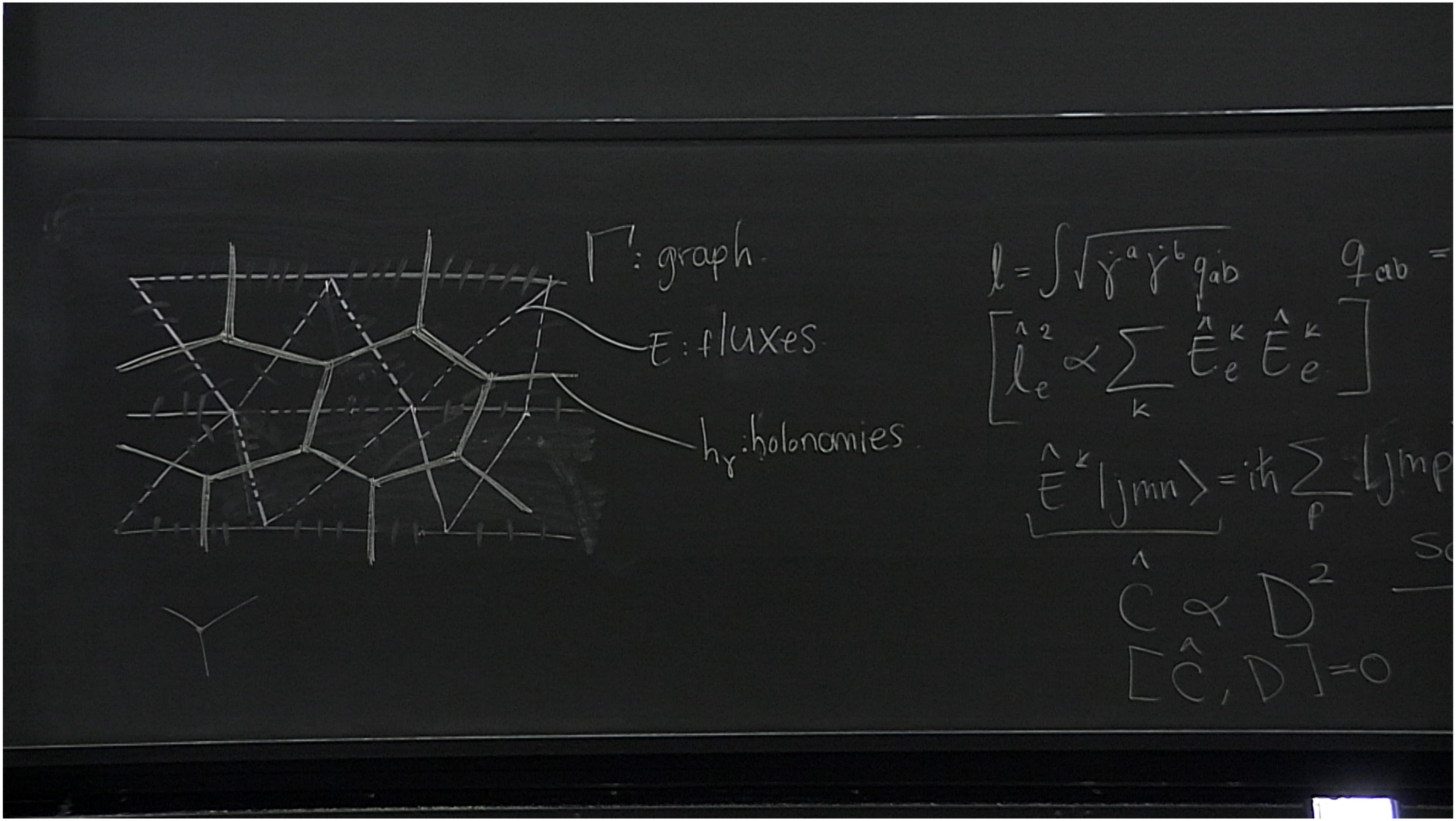
Wick.

non compact

SO(1,2)

SO(3).

SU(2)

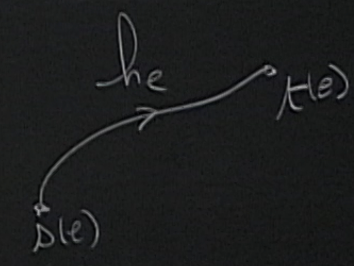

Γ: graph
E: fluxes
hγ: holonomies
l = ∫√(γ̇ᵃ γ̇ᵇ qab)
qab =
[l̂e² ∝ Σk Êe^k Êe^k]
Ê^k |jmn⟩ = iħ Σp |jmp
Ĉ ∝ D²
[Ĉ, D] = 0

The physical Hilbert space

→ Gauss constraints ($\propto \partial E + \varepsilon A E$)

↳ generates internal rotation

(Finite) Gauge transfo on holonomy.



- $h_e \longrightarrow g(v_t)\, h_e\, g^{-1}(v_s)$.
- $\langle h_e | \Psi_e \rangle = \Psi_e(h_e)$.
- $\Psi\big(g(v_t)\, h_e\, g^{-1}(v_s)\big) = \langle h_e | L_{g(v_t)^{-1}} R_{g^{-1}(v_s)} | \Psi_e \rangle$.

$$(L_h f)(g) = f(h^{-1} g).$$

$$(R_h f)(g) = f(g h).$$

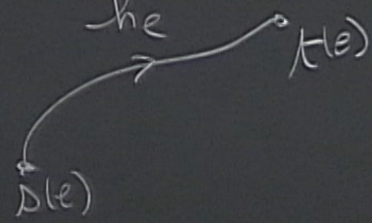
$$= \langle h_e | L_{g(v_t)^{-1}} R_{g^{-1}(v_s)} | \Psi_e \rangle$$

...space of states satisfying Gauss by averaging over the grou...

$$\delta(\hat{G})\,|\Psi_e\rangle = \int L_{g_1} R_{g_2} |\Psi_e\rangle \, dg_1 \, dg_2$$

internal rotation:

holonomy:



$\langle h_e | \Psi_e \rangle = \Psi_e(h_e)$.

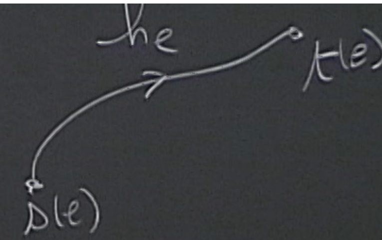
$\Psi(g(v_t)\, h_e\, g^{-1}(v_s)) = \langle h_e$

on the subspace of states satisfying Gauss by averaging over the group S

one edge "$\delta(\hat{G}) | \Psi_e \rangle = \int L_{g_1} R_{g_2} | \Psi_e \rangle \, dg_1 \, dg_2$

↳ generate internal rotation.

(Finite) Gauge transfo on holonomy:



→ Implement a projector on the subspace of states satisfying

one edge "$\delta(\hat{G})|\Psi_e\rangle =$

$$h_e \longrightarrow g(v_t)\, h_e\, g^{-1}(v_s).$$

$$\langle h_e | \Psi_e \rangle = \Psi_e(h_e).$$

$$\Psi(g(v_t)\, h_e\, g^{-1}(v_s)) = \langle h_e | L_{g(v_t)^{-1}} R_{g^{-1}(v_s)} | \Psi_e \rangle.$$

Gauss by averaging over the group SU(2) for both ends of the edge.

$$= \int L_{g_1} R_{g_2} | \Psi_e \rangle \, dg_1 \, dg_2$$

Ex: do it explicitly for the parametrized particle.

$$\hookrightarrow \delta(\hat{C})\,\Psi(t,q) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}}$$

$$\left(\frac{\hat{p}_q^2}{2m}\right)$$

$$\int e^{\frac{i}{\hbar} s \left(\hat{p}_t + \hat{h}\right)} \psi(t, q)\, ds \underset{\uparrow}{=} \frac{1}{2\pi\hbar} \int_{\mathbb{R}^3}$$

some work to do!

$$\frac{1}{2\pi\hbar}\int_{\mathbb{R}} \underbrace{\psi_r(t,q)}\,dr$$

↳ solution of Schrödinger eq.

Ex: do it explicitly for the parametrized particle.

↓

to do.

$$\hookrightarrow \delta(\hat{C})\,\Psi(t,q) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}s(\hat{p}_t + h}$$

Ex: (q, p_q)

$\hat{C} = \hat{p}_q \longrightarrow$ generates translation in q

do ·

$\hookrightarrow \delta(C)\ \Psi(t,q) = \ldots 2\pi\hbar$

Ex: (q, p_q)

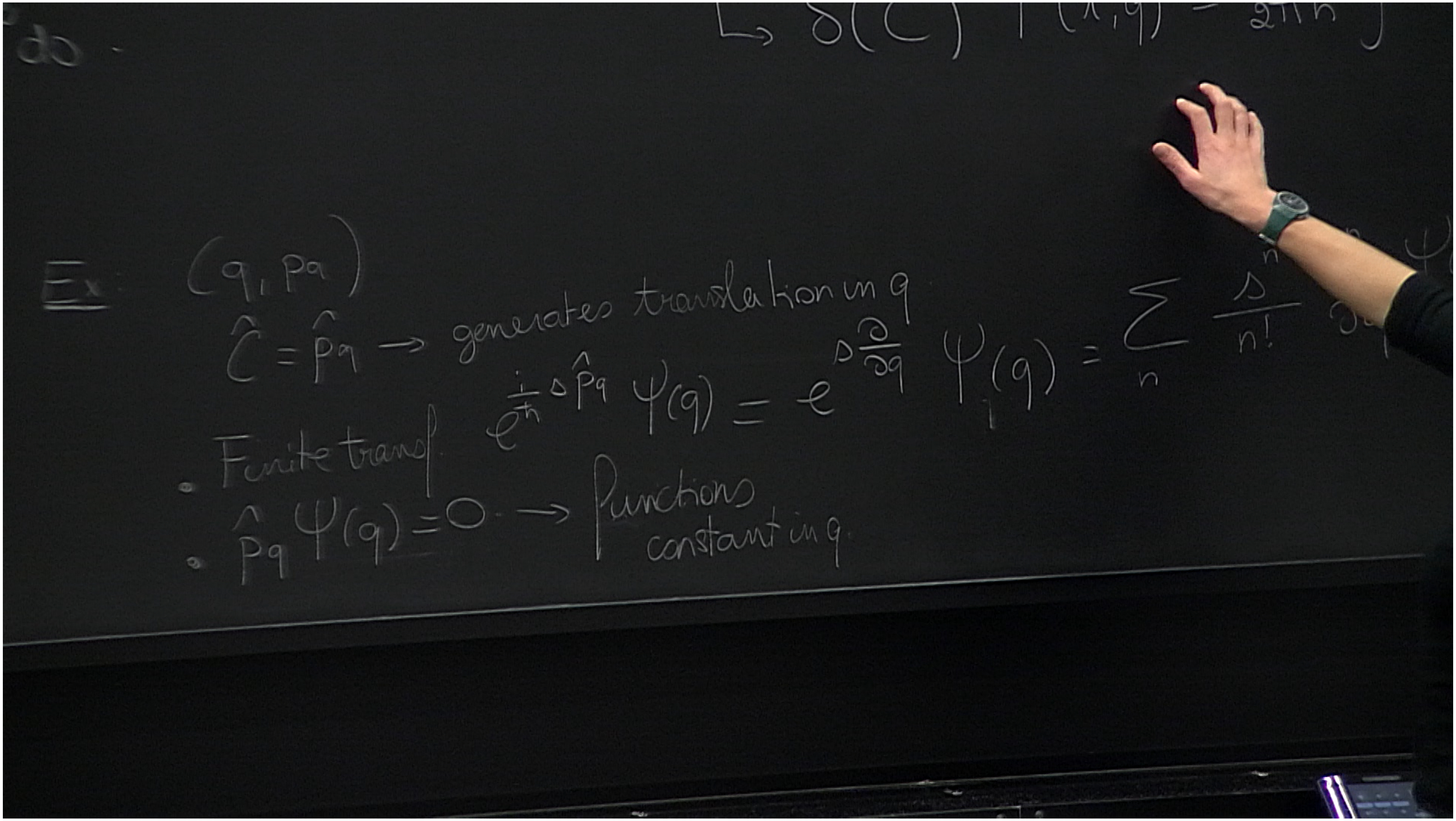
$\hat{C} = \hat{p}_q \rightarrow$ generates translation in q

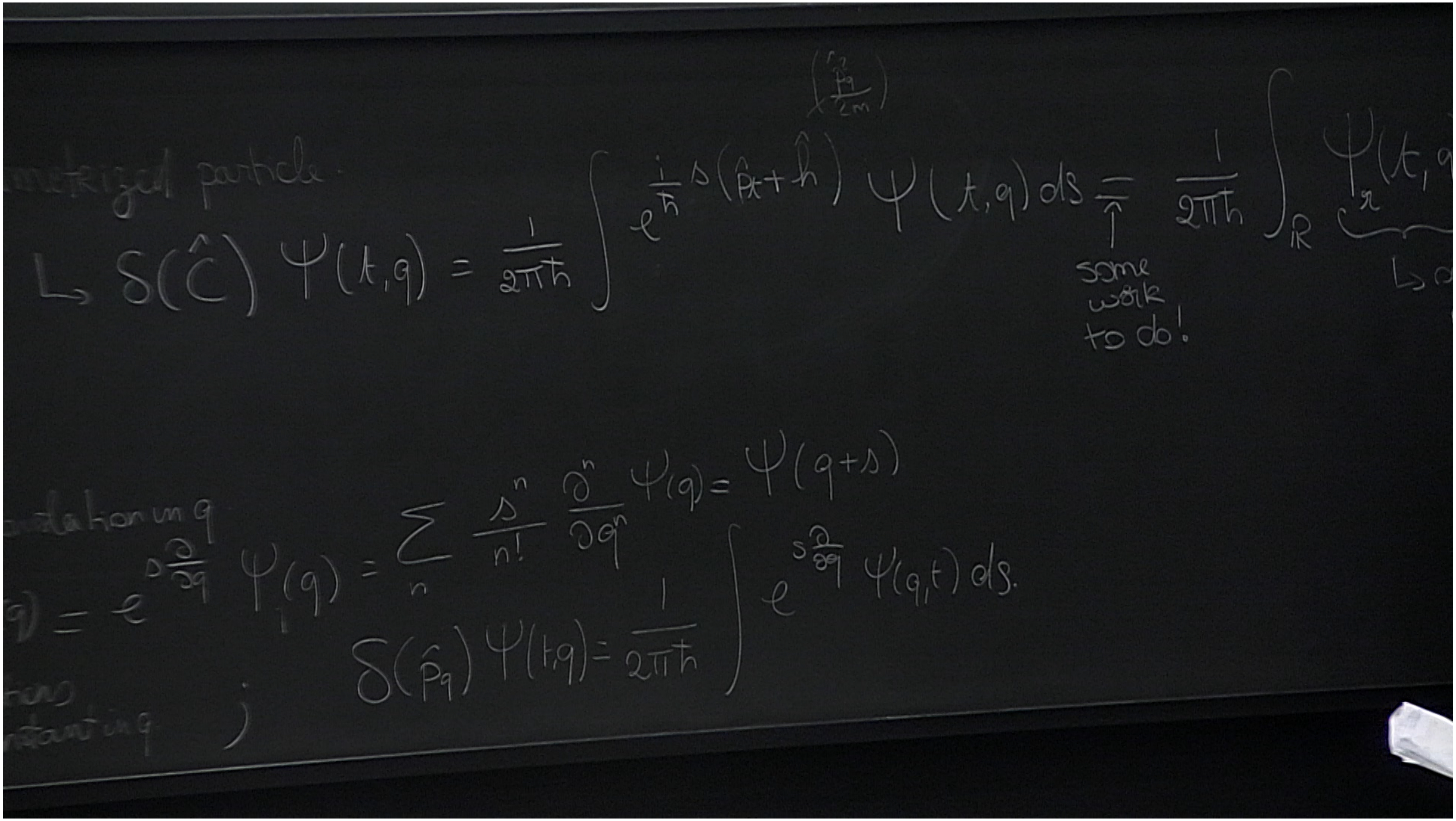
Finite transf. $e^{\frac{i}{\hbar} s \hat{p}_q}\, \psi(q) = e^{s \frac{\partial}{\partial q}}\, \psi(q) = \sum_n \frac{s^n}{n!} \frac{\partial^n}{\partial q^n} \psi(q) = \ldots$

$$\hookrightarrow \delta(\hat{C})\,\Psi(t,q) = \frac{\quad}{2\pi\hbar}\int$$

→ generates translation in q

$$e^{\frac{i}{\hbar} s \hat{p}_q}\,\Psi(q) = e^{s\frac{\partial}{\partial q}}\,\Psi(q) = \sum_n \frac{s^n}{n!}\frac{\partial^n}{\partial q^n}\Psi(q) = \Psi(q+s)$$


Ex: (q, pq)
Ĉ = p̂q → generates translation in q
Finite transf. e^(i/ħ s p̂q) ψ(q) = e^(s ∂/∂q) ψ(q) = Σn s^n/n!
p̂q ψ(q) = 0 → functions constant in q.


...metrized particle.
$\hookrightarrow \delta(\hat{C})\Psi(t,q) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}s(\hat{p}_t+\hat{h})}\Psi(t,q)\,ds = \frac{1}{2\pi\hbar}\int_{\mathbb{R}}\Psi_r(t,q$
$\left(\frac{\hat{p}_q^2}{2m}\right)$
some work to do!
...lation in q
$(q) = e^{s\frac{\partial}{\partial q}}\Psi(q) = \sum_n \frac{s^n}{n!}\frac{\partial^n}{\partial q^n}\Psi(q) = \Psi(q+s)$
$\delta(\hat{p}_q)\Psi(t,q) = \frac{1}{2\pi\hbar}\int e^{s\frac{\partial}{\partial q}}\Psi(q,t)\,ds.$

...metrized particle.

$$\hookrightarrow \delta(\hat{C})\,\Psi(t,q) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}s\left(\hat{p}_t + \hat{h}\right)}\,\Psi(t,q)\,ds \underset{\uparrow}{=} \frac{1}{2\pi\hbar}\int_{\mathbb{R}} \Psi_r(t,q\ldots$$

$$\left(\hat{h} = \frac{\hat{p}_q^2}{2m}\right)$$

some work to do!

↳ ...

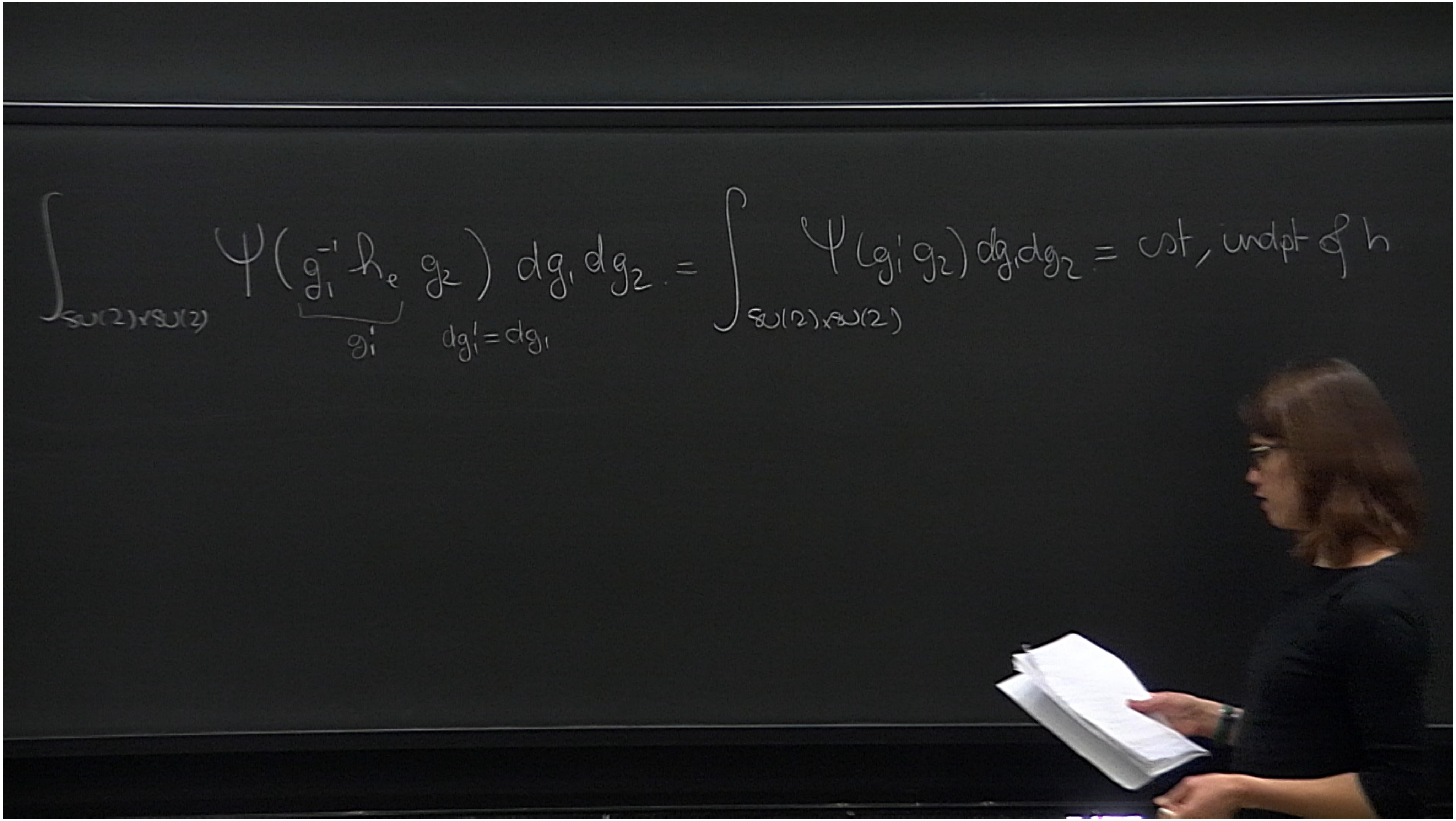
...slation in q

$$\Psi(q) = e^{s\frac{\partial}{\partial q}}\,\Psi(q) = \sum_n \frac{s^n}{n!}\frac{\partial^n}{\partial q^n}\Psi(q) = \Psi(q+s)$$

$$\delta(\hat{p}_q)\,\Psi(t,q) = \frac{1}{2\pi\hbar}\int e^{s\frac{\partial}{\partial q}}\,\Psi(q,t)\,ds = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\Psi(q+s,t)\,ds.$$

...hens
...stanting q ;

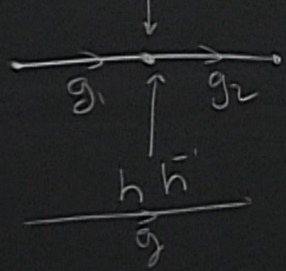
$$\int_{SU(2)\times SU(2)} \Psi(g_1^{-1} h_e \, g_2) \, dg_1 \, dg_2 .$$


∫_{SU(2)×SU(2)} Ψ(g_1^{-1} h_e g_2) dg_1 dg_2 = ∫_{SU(2)×SU(2)} Ψ(g_1' g_2) dg_1 dg_2 = cst, indpt of h
g_1'
dg_1' = dg_1

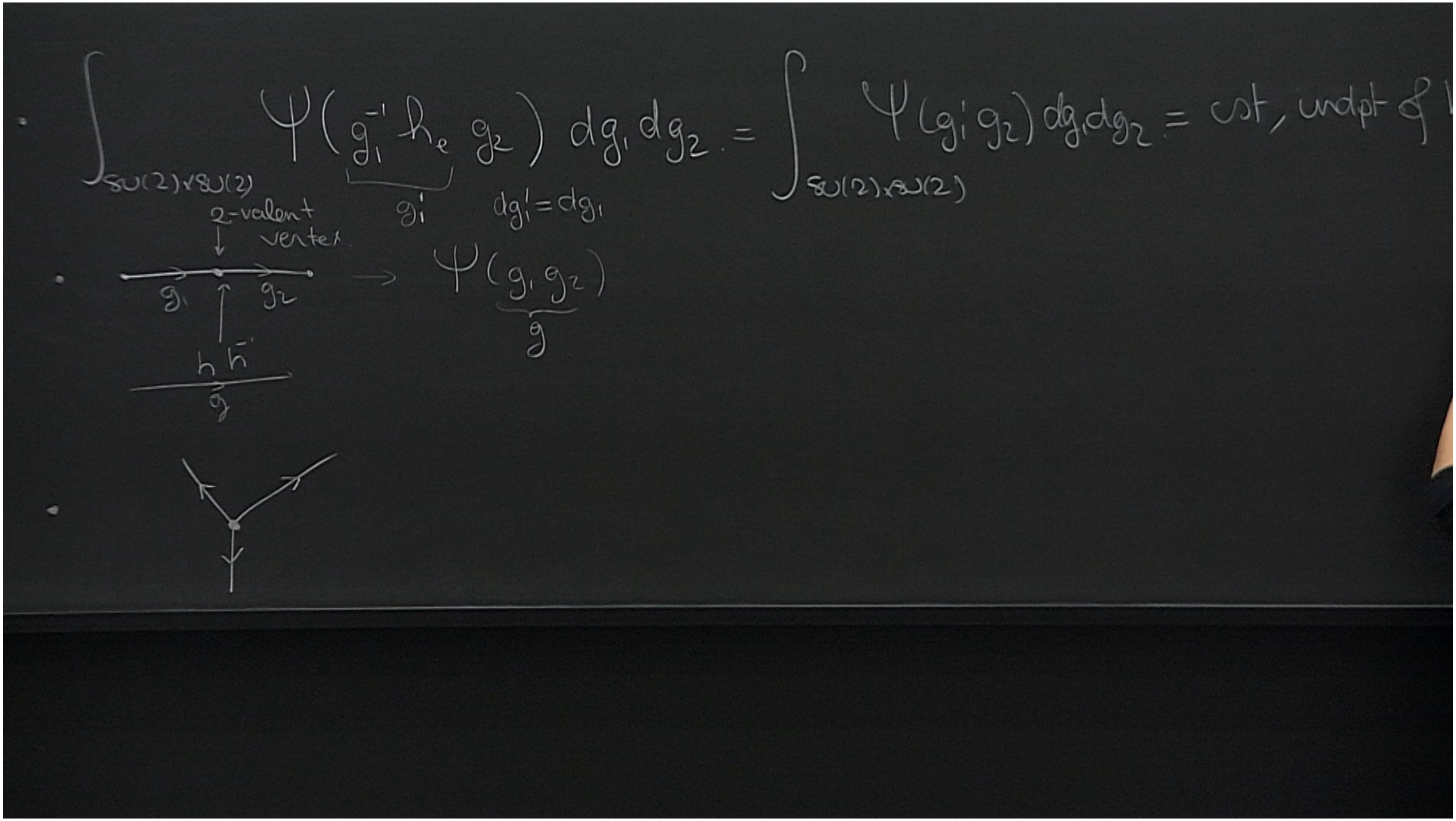
→ space of solutions = trivial

- $$\int_{SU(2)\times SU(2)} \Psi(\underbrace{g_1^{-1} h_e}_{g_1'}\, g_2)\, dg_1\, dg_2 = \int_{SU(2)\times SU(2)} \Psi(g_1' g_2)\, dg_1' dg_2 = \text{cst, indpt of } h$$

$dg_1' = dg_1$

- 

$\longrightarrow \Psi(\underbrace{g_1 g_2}_{g})$

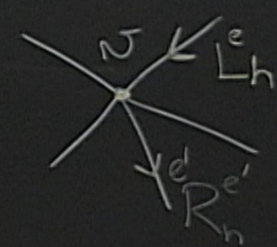


$\int_{SU(2)\times SU(2)} \Psi(g_1^{-1} h_e g_2)\, dg_1 dg_2 = \int_{SU(2)\times SU(2)} \Psi(g_1' g_2)\, dg_1 dg_2 = cst$, undpt of
g_1'
$dg_1' = dg_1$
2-valent vertex
g_1 g_2
$\Psi(g_1 g_2)$
g
$h\ h^{-1}$
g

= cst, undpt of h

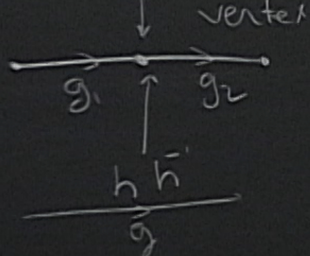
→ space of solutions = trivial

$$\delta(\hat{G}_v) = \int_{SU(2)} \left(\bigotimes_{e / v = v_{t(e)}} L^{e}_{h} \right) \otimes \left(\bigotimes_{e' / v = v_{s(e)}} R^{e'}_{h} \right)$$

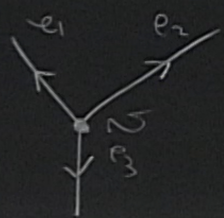


$$\int_{SU(2)\times SU(2)} \Psi(\underbrace{g_1^{-1} h_e}_{g_1'}\, g_2)\, dg_1\, dg_2 = \int_{SU(2)\times SU(2)} \Psi(g_1' g_2)\, dg_1\, dg_2 = \text{cst, indpt of } h$$

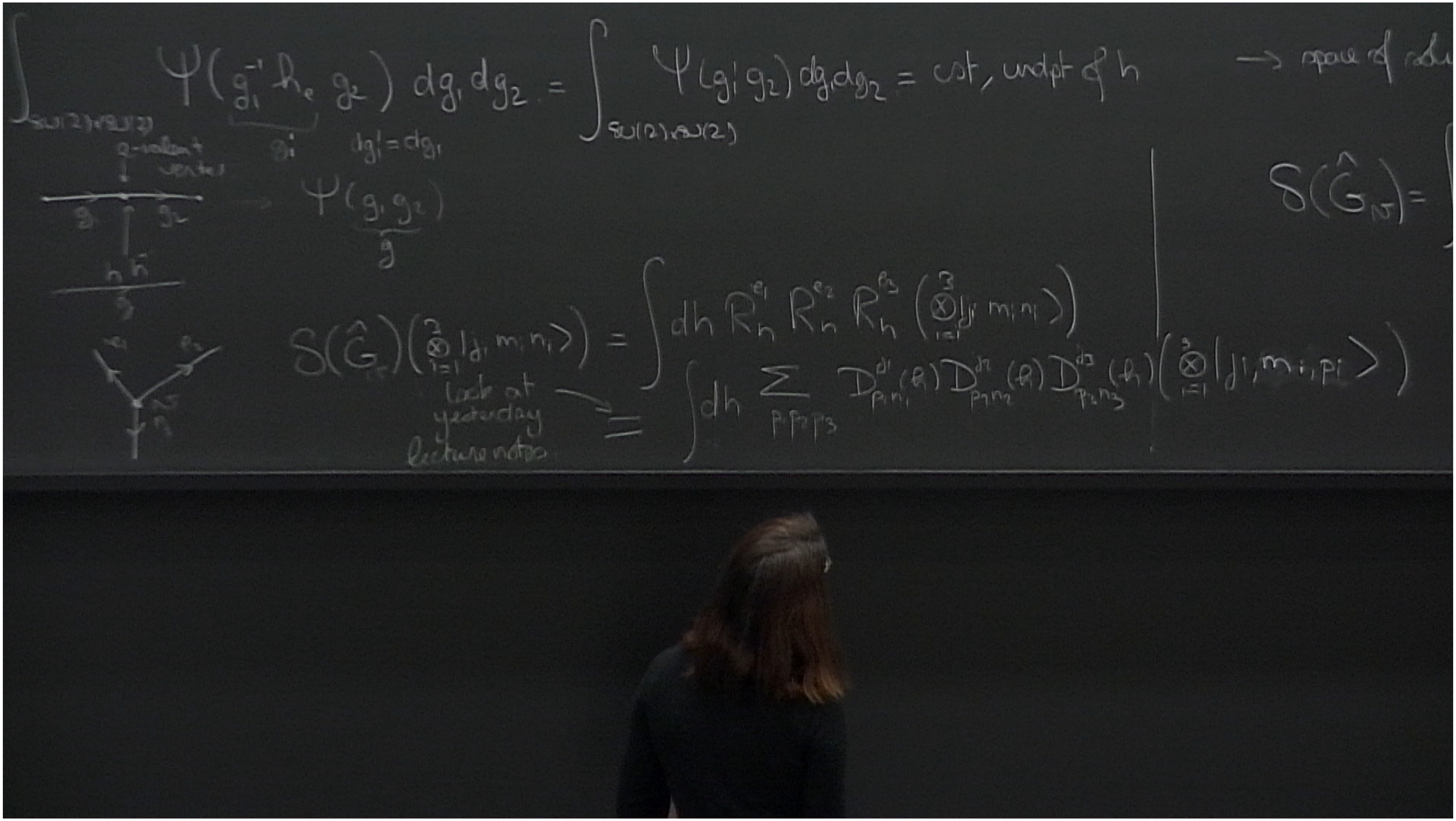
$dg_1' = dg_1$



$\longrightarrow \Psi(\underbrace{g_1 g_2}_{g})$



$$S(\hat{G})\left(\bigotimes_{i=1}^{3} |j_i\, m_i\, n_i\rangle\right) = \int dh\, R_h^{e_1} R_h^{e_2} R_h^{e_3} \left(\bigotimes_{i=1}^{3} |j_i\, m_i\, n_i\rangle\right)$$


Ψ(g1^-1 h_e g2) dg1 dg2 = ∫ Ψ(g1' g2) dg1 dg2 = cst, undpt of h
SU(2)×SU(2)
2-valent vertex
dg1' = dg1
Ψ(g1 g2)
S(Ĝ_Γ)(⊗_{i=1}^{3} |ji mi ni>) = ∫dh R_h^{e1} R_h^{e2} R_h^{e3} (⊗_{i=1}^{3} |ji mi ni>)
look at yesterday lecture notes
= ∫dh Σ_{p1p2p3} D^{j1}_{p1n1}(h) D^{j2}_{p2n2}(h) D^{j3}_{p3n3}(h) (⊗_{i=1}^{3} |ji, mi, pi>)
→ space of solu
S(Ĝ_Γ) =

look at yesterday lecture notes.

$$= \int dh \sum_{p_1 p_2 p_3} D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h) \left(\bigotimes_{i=1} |j_i, m_i, p_i\rangle \right)$$

From recoupling theory.

$$\int dh\, D^{j_1}_{m_1 n_1}(h)\, D^{j_2}_{m_2 n_2}(h)\, D^{j_3}_{m_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

↳ Wigner 3jm symbol.

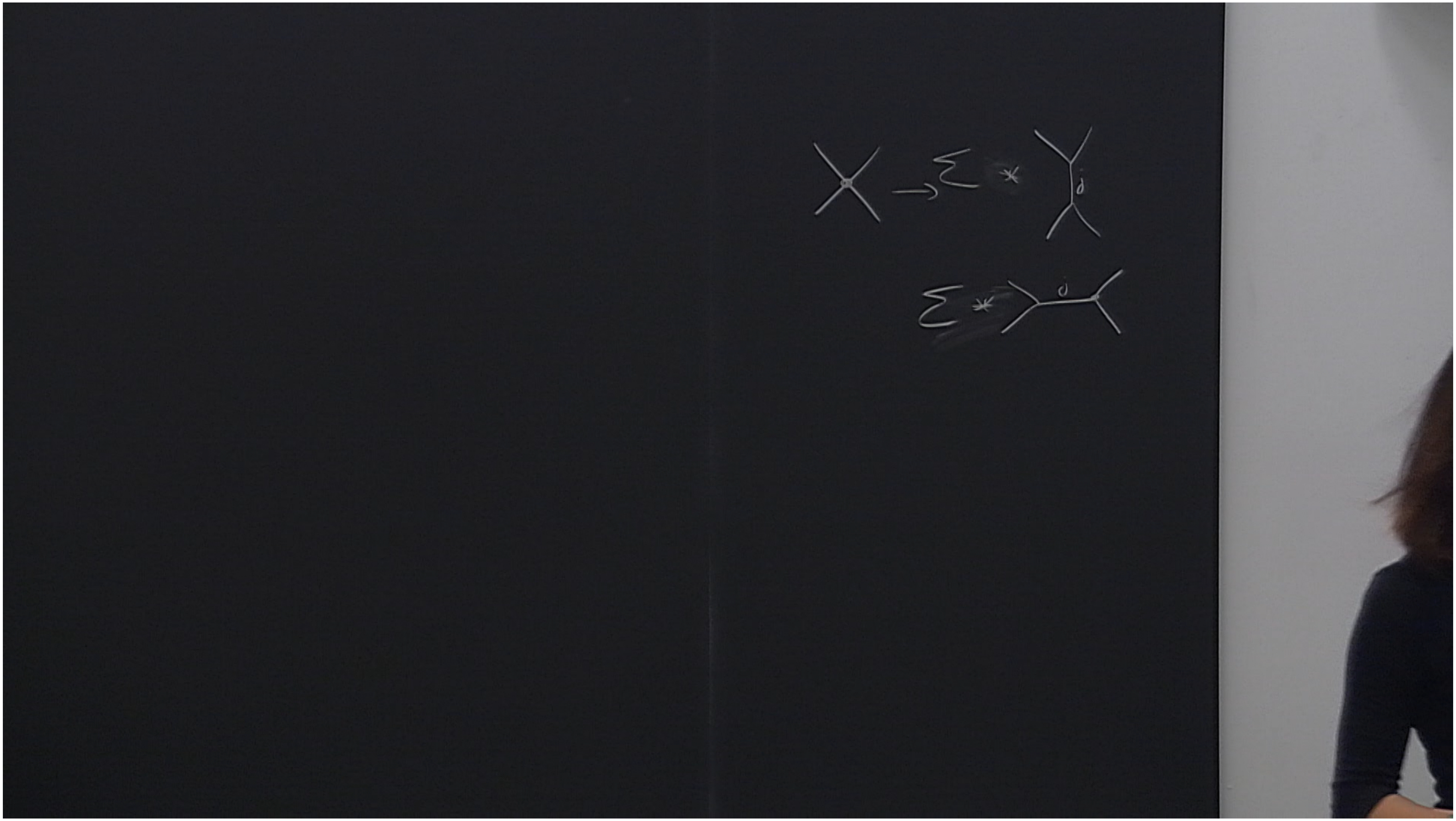
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_1 + 1}}$$

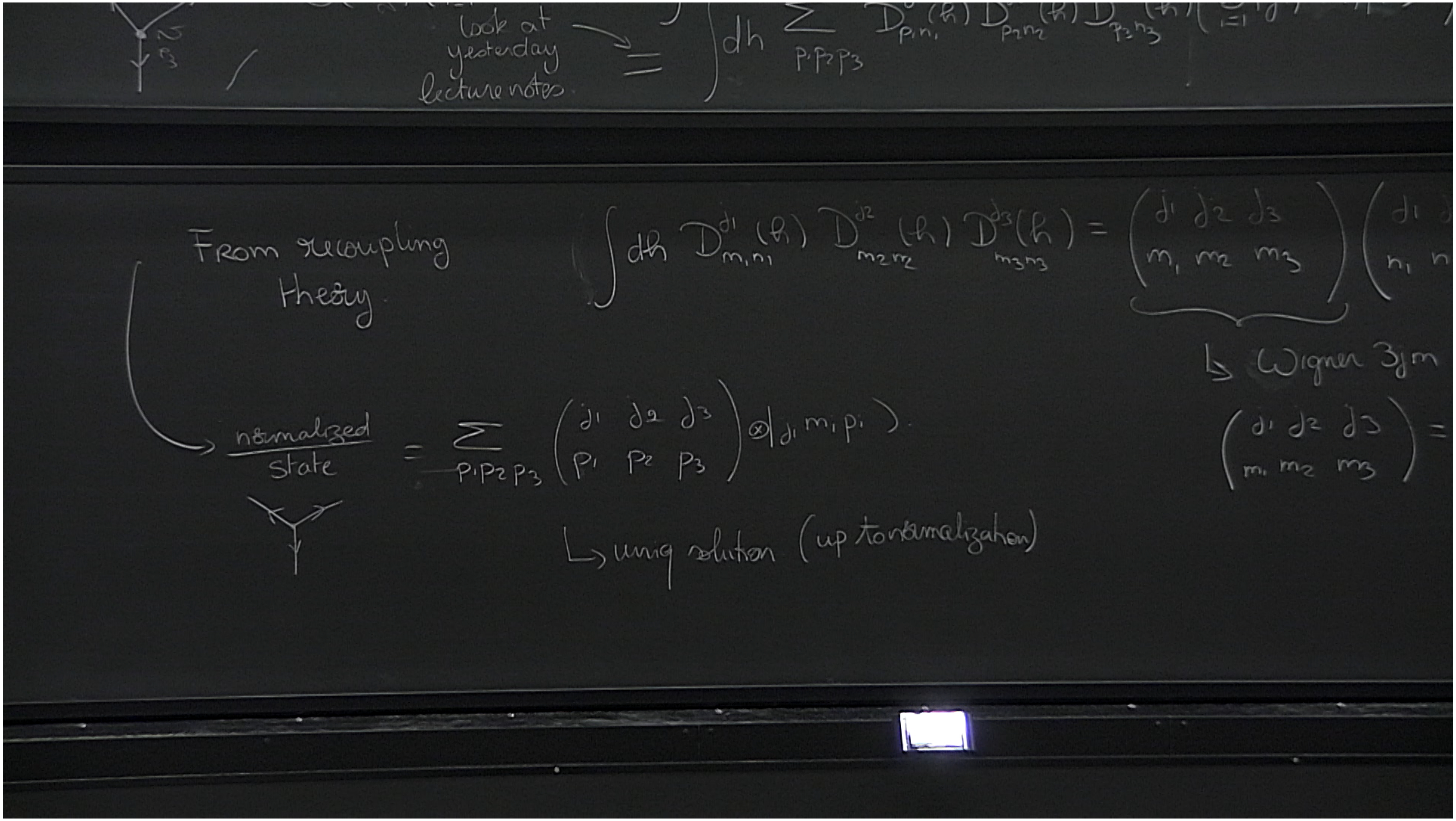
$$= \int dh \sum_{p_1 p_2 p_3} D_{p_1 n_1}(h) \, D_{p_2 n_2}(h) \, D_{p_3 n_3}(h) \left(\bigotimes_{i=1} | j_i, \ldots, p_i \rangle \right) \qquad R_n^e$$

$$\int dh \, D^{j_1}_{m_1 n_1}(h) \, D^{j_2}_{m_2 n_2}(h) \, D^{j_3}_{m_3 n_3}(h) = \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

↳ Wigner 3jm symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_1 + 1}} \underbrace{\langle j_1 m_1 ; j_2 m_2 | j_3, -m_3 \rangle}_{\text{C-G coeff}}$$




look at yesterday lecture notes
$= \int dh \sum_{p_1 p_2 p_3} D_{p_1 n_1}(h) D_{p_2 n_2}(h) D_{p_3 n_3}(h)$
From recoupling theory
$\int dh\, D^{j_1}_{m_1 n_1}(h)\, D^{j_2}_{m_2 n_2}(h)\, D^{j_3}_{m_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & \\ n_1 & n \end{pmatrix}$
↳ Wigner 3jm
normalized state $= \sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \otimes |j_1 m_1 p_1 \rangle$
↳ uniq solution (up to normalization)
$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =$

• … constant in q.)

• intertwiner

$$i^{p_1 p_2 p_3} =: \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$

• spinnetwork

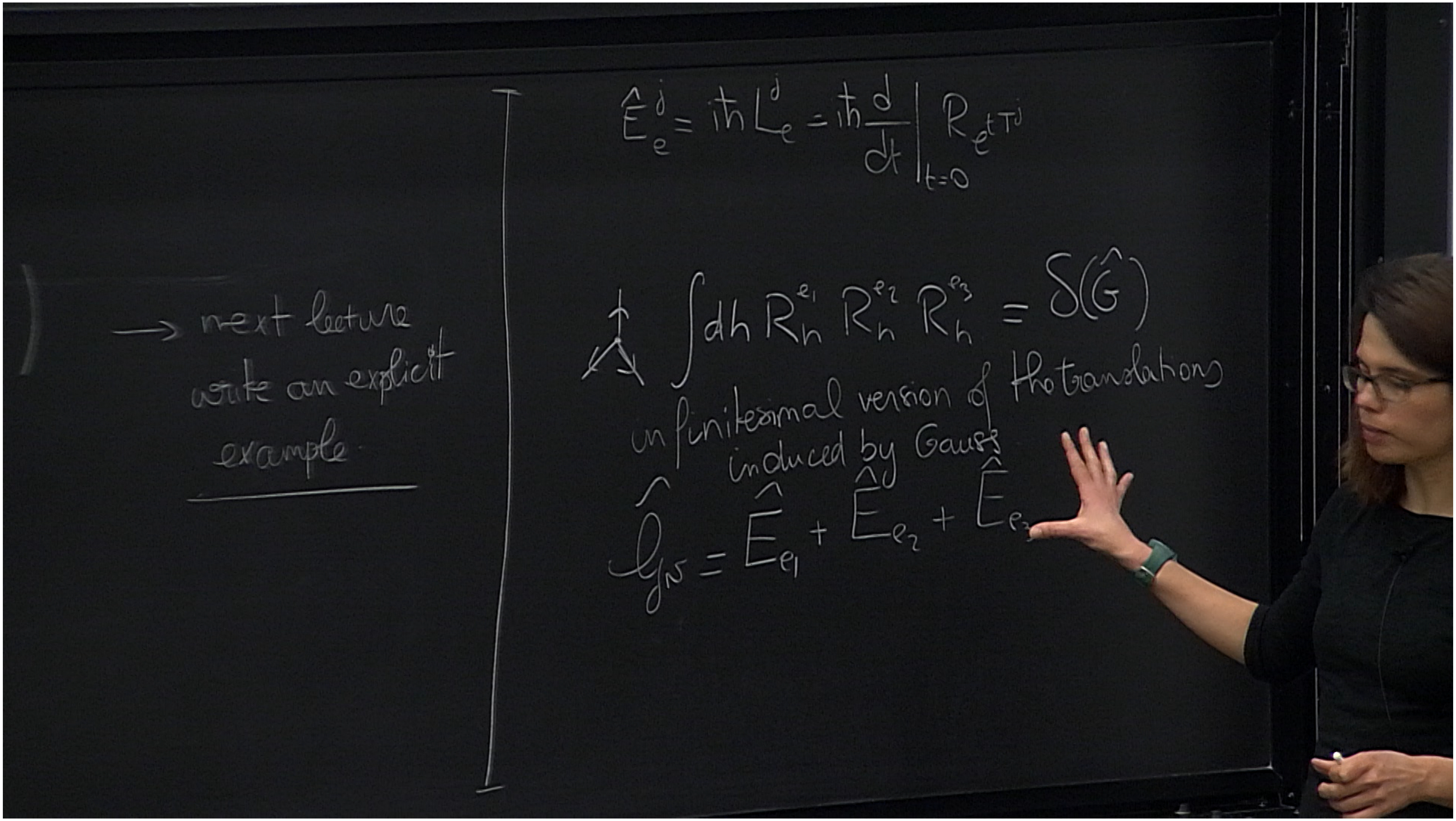
↓

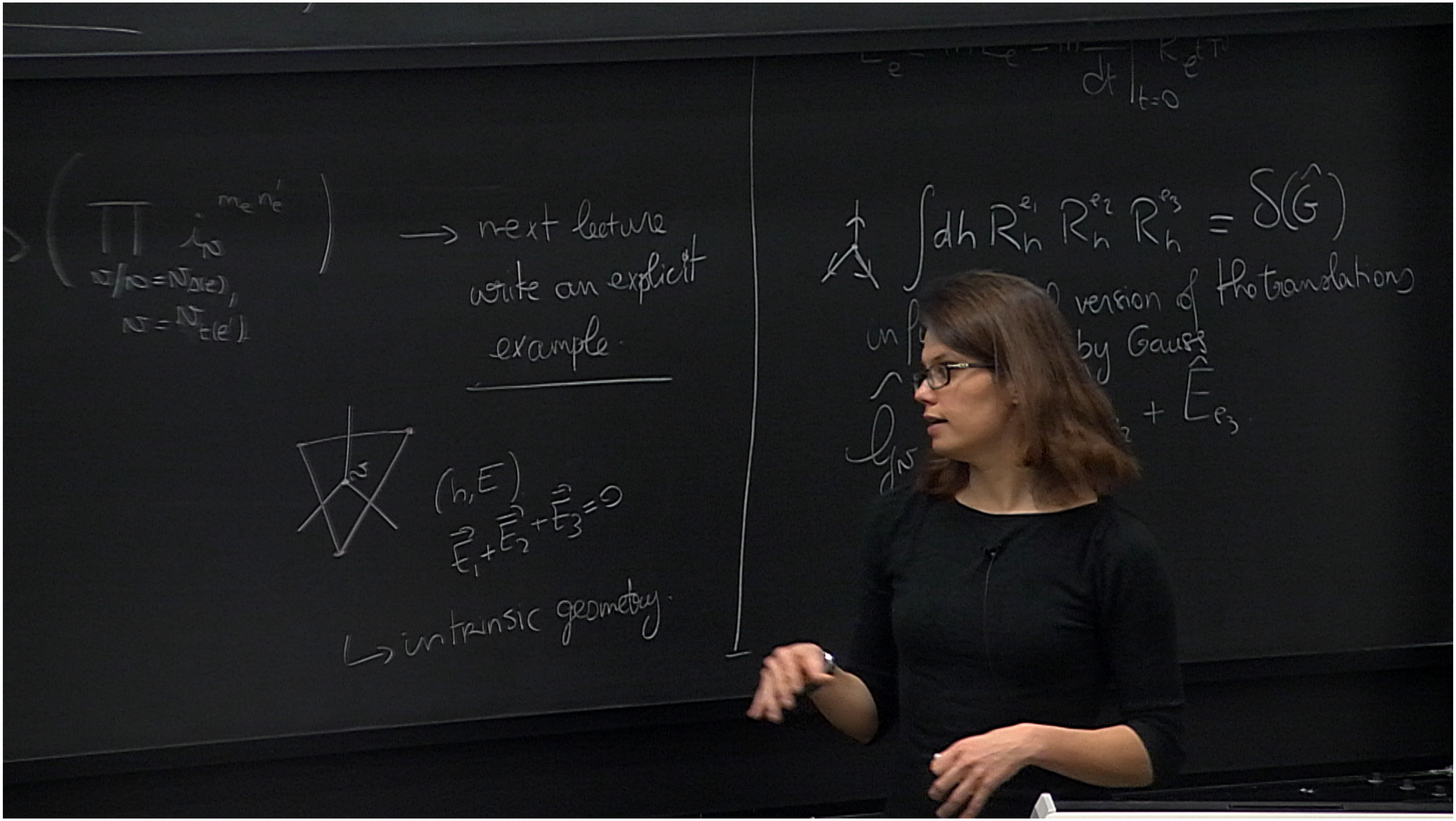
SU(2)-inv function

$$S_\Gamma^{\{j_e, i_v\}}(g_1 \cdots g_E) = \sum \prod_{e=1}^{E} \langle g_e | j_e\, m_e n_{e'} \rangle \left(\prod_{v / v = v_{s(e)},\ v = v_{t(e')}} i_v^{m_e n_{e'}} \right)$$

(Γ ← graph)

→ next lecture write an explicit example


→ next lecture
write an explicit
example
$\hat{E}_e^j = i\hbar L_e^j = i\hbar \frac{d}{dt}\Big|_{t=0} R_{e^{tT^j}}$
$\int dh\, R_h^{e_1} R_h^{e_2} R_h^{e_3} = \delta(\hat{G})$
infinitesimal version of the translations
induced by Gauss
$\hat{G}_v = \hat{E}_{e_1} + \hat{E}_{e_2} + \hat{E}_{e_3}$


→ next lecture
write an explicit
example.
(h, E)
intrinsic geometry.
version of the translations
by Gauss