Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 9

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Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

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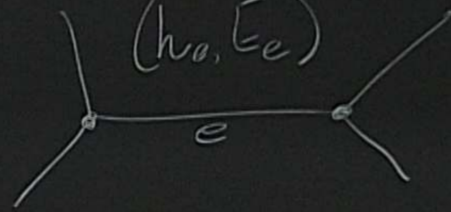
URL: http://pirsa.org/19030076

$(A(x), E(x)) \longrightarrow$



$h_{mn} \rightarrow \hat{h}_{mn} \quad \hat{h}_{mn}|\psi\rangle$

$$[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}} \Longrightarrow$$

Check $[\hat{E}^j, \hat{E}^k]$ ✓

Single Edge:

(h_{mn}, E^j)

$$\{h_{mn}, h_{m'n'}\} = 0$$

$$\{h_{mn}, E^j\} = (hT^j)_{mn}$$

$$\{E^j, E^k\} = \varepsilon_l{}^{jk} E^l$$

Inner Product:

$$\langle \psi_1 | \psi_2 \rangle = \int_{SU(2)} dg_{Haar}\, \overline{\psi_1(g)}\, \psi_2(g)$$

$$\hat{h}_{mn}|\Psi\rangle = |h_{mn}\Psi\rangle$$

$$\widehat{\{f,g\}} \Longrightarrow \hat{E}^j|\Psi\rangle = i\hbar\,|L^j\Psi\rangle$$

$$[\hat{E}^j,\hat{E}^k] \quad \checkmark$$

oduct:

$$\overline{\Psi_1(g)}\,\Psi_2(g)$$

O.N. Basis:

Peter-Weyl $\longrightarrow$ $f(g) = \sum_j d_j f^j_{mn} D^j_{mn}(g)$

$$\mathbb{1}_{SU(2)} = \sum_{j\in \mathbb{N}/2}\sum_{m=-j}^{j}\sum_{n=-j}^{j} |jmn\rangle\langle jmn| \qquad |jmn\rangle = \sqrt{d_j}\,|D^j_{mn}\rangle$$

$\ldots$th $|L^{j}\psi\rangle$

o~~v~~er complete basis.

Basis:

$$\Rightarrow \quad f(g) = \sum_j d_j\, f^j_{mn}\, D^j_{mn}(g)$$

$$|jmn\rangle = \sqrt{d_j}\, |D^j_{mn}\rangle$$

$$\sum_{m=-j}^{j} \sum_{n=-j}^{j} |jmn\rangle\langle jmn|$$

TODAY: Kinematical Hilbert space.

↳ subsystem on edge ⟶ a graph.

↓

length op.

$$\hat{E}^j = i\hbar L^j = i\hbar \frac{d}{dt}\Big|_{t=0} R_{e^{tT_j}}$$

$$(R_h f)(g) = f(gh)$$

$\Psi\rangle = |h_{mn}\Psi\rangle$

$\Rightarrow \quad E^i|\Psi\rangle = i\hbar\,|L^i\Psi\rangle$

o~~v~~er complete basis.

$\nearrow$ Wigner matrices

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$|h_{mn}\psi\rangle$

$E^i|\psi\rangle = i\hbar\,|L^i\psi\rangle$

$\{|jmn\rangle\}$.

o~~v~~er complete ba

O.N. Basis:

→ Wigner matrices

Peter-Weyl → $f(g) = \sum_j d_j\, f^j_{mn}\, D^j_{mn}(g)$

$|jmn\rangle = \sqrt{d_j}\,|D^j_{mn}\rangle$

$\mathbb{1}_{SU(2)} = \sum_{j} \sum_{m}^{j} \sum_{n}^{j} |jmn\rangle\langle jmn|$

space.

edge $\longrightarrow$ a graph.

$$(R_h f)(g) = f(gh)$$

$$g) = D^j_{mn}(gh) = \sum_p \underbrace{D^j_{mp}(g)}_{\langle g|jmp\rangle} D^j_{pn}(g) \Rightarrow R_h|jmn\rangle = \sum_p D^j_{pn}(g)\,|jmp\rangle .$$

$mn\rangle$.

$\{|jmp\rangle,\ p \in \{-j \ldots j\}\}$: subrepresentation space carries a rep D^j

$$_h D^j_{mn}(g) = D^j_{mn}(h^{-1} g)$$

• Left multiplication $\quad L_h D^j_{mn}(g) = D^j_{mn}(h^{-1}g) = \sum_q D^j_{mq}(h^{-1})\, \ldots$

$$\mathcal{H}_e = L^2(SU(2), dg) = \bigoplus_j V_j^* \otimes V_j$$

$$\bullet \xrightarrow[\;|j\, m\, n\rangle\;]{\;(D^j_{mn}(g))\;} \bullet \;\equiv\; \bullet \xrightarrow{\;j\;} \bullet$$

• <u>Left multiplication</u> $L_h D^j_{mn}(g) = D^j_{mn}(h^{-1}g) = \sum_q D^j_{mq}(h^{-1})$

$$\mathcal{H}_e = L^2(SU(2), dg) = \bigoplus_j V_j^* \otimes V_j$$

$$\underset{|j\,m\,n\rangle}{\xrightarrow{\left(D^j_{mn}(g)\right)}} \equiv \xrightarrow{j}$$

$$= \sum_q D^j_{mq}(h^{-1})\, D^j_{qn}(g) \Rightarrow L_h |j\, m\, n\rangle = \sum_q D^j_{mq}(h^{-1})\, |j\, q\, n\rangle$$

$\{|j\, q\, n\rangle\ q=-j \ldots j\}$: subrepresentation space
carries a rep. $(D^j)^*$: dual
contragredient rep.

$$(D^j)^*_{mn}(g) = D^j_{nm}(g^{-1}) = \overline{D^j_{mn}(g)}$$

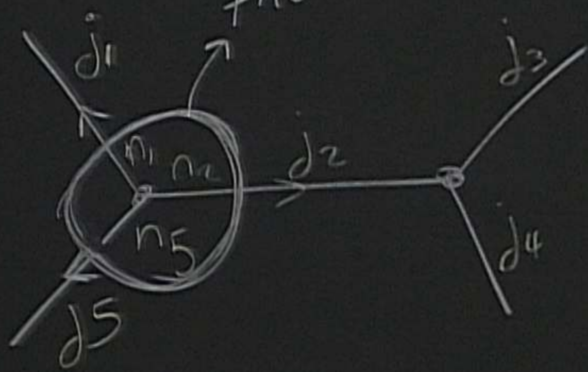
$$= \bigoplus_j V_j^* \otimes V_j$$

$$\equiv \overset{n}{\circ} \xrightarrow{\ j\ } \overset{m}{\circ}$$

(angular momentum associated to both ends of the edge).

• Left multiplication $L_h D^j_{mn}(g) = D^j_{mn}$

this is going to be constrained by the Gauss constraint



$\mathcal{H}_e = L^2(SU(2$

• Left multiplication $\quad L_h D^j_{mn}(g) = D^j_{mn}(h^{-1}g) = \sum_q D^j_{mq}(h^{-1})\, D^j_{qn}(g) \Rightarrow L_h|jmn\rangle =$

$\{|jqn\rangle \; q=-j \dots j\}$ subrep…

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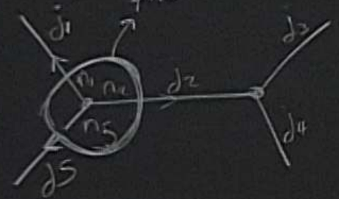


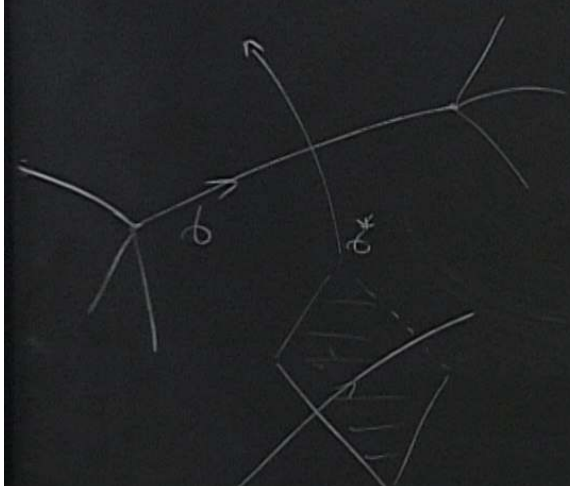
$$\underline{\mathcal{H}_e} = L^2(SU(2), dg) = \bigoplus_j V_j^* \otimes V_j$$

$$\underset{|jmn\rangle}{\xrightarrow{(D^j_{mn}(g))}} \equiv n \xrightarrow{j} m$$

(angular momentum associated to both ends of the …

Length op.

Length op :



ep :

length of a curve γ^* on a 2D surface. $\int_{\gamma^*} \sqrt{\dot{\gamma}^{*a} \dot{\gamma}^{*b} q_{ab}}\, ds$

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$\hookrightarrow q_{ab} = e^i_a e^j_b \delta_{ab}$

op :

length of a curve γ^* on a 2D surface.

$$\int_{\gamma^*} \sqrt{\dot{\gamma}^{*a}\dot{\gamma}^{*b} q_{ab}}\, ds$$

↳ $q_{ab} = e^i_a e^j_b \delta_{ab}$.

$$L \rightarrow\ \sim \text{“} \sum_{n=0}^{N} \sqrt{\sum_k E^k_{e_n} E^k_{e_n}} \text{”}$$

↳ Riemmann approx.

$$\hat{l}^2_{e^*} := \sum_k \hat{E}^k_e \hat{E}^k_e$$

...aa.

$$\int_{\gamma^*} \sqrt{\dot{\gamma}^{*a}\dot{\gamma}^{*b} q_{ab}}\, ds$$

$\hookrightarrow q_{ab} = e^i_a e^j_b \delta_{ab}$.

$\hookrightarrow \sim$ " $\sum_{n=0}^{N} \sqrt{\sum_k E^k_{e_n} E^k_{e_n}}$ "

$\hookrightarrow$ Riemann approx

"heuristic justification"

gth of a curve γ^* on a 2D surface. $\int_{\gamma^*} \sqrt{\dot\gamma^{*a}\dot\gamma^{*b} q_{ab}}\, ds$

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"heuristic justificati

$$\hat{l}^2_{e^*} := \sum_k \hat{E}^k_e \hat{E}^k_e$$

$$E^k |j\,m\,n\rangle = i\hbar \sum_p |j\,m\,p\rangle \frac{d}{dt}\Big|_{t=0} D^j_{pn}(e^{tT^k}) = i\hbar \sum_p |j\,m\,p\rangle D^j_{pn}(T^k)$$

" Right multiplication"

multiplication

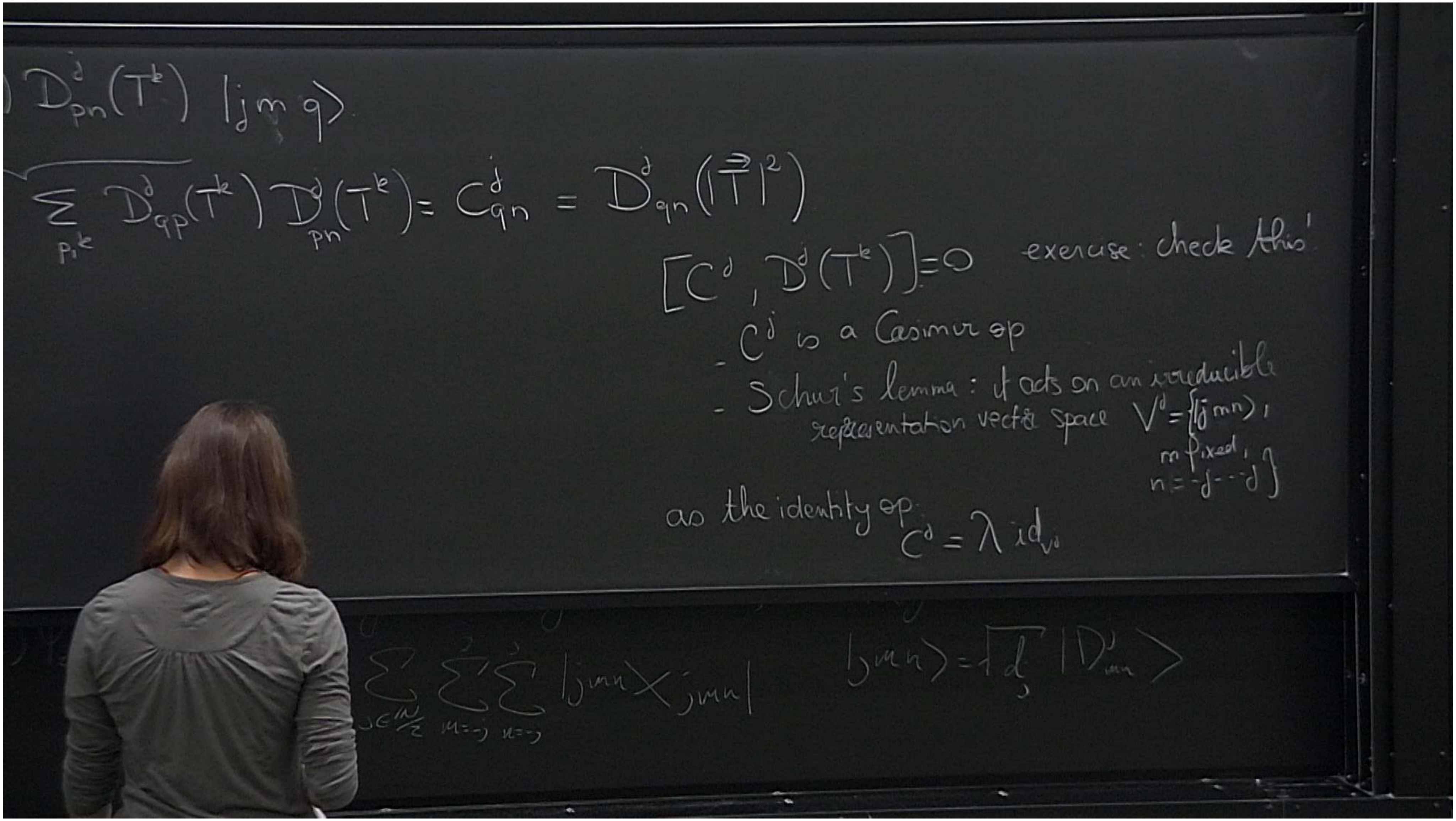
$$\hat{\ell}^2 |jmn\rangle = -\hbar^2 \sum_{p,q,k} D^j_{qp}(T^k)\, D^j_{pn}(T^k)\, |jmq\rangle$$

$$\hookrightarrow \sum_{p,k} D^j_{qp}(T^k)\, D^j_{pn}(T^k) = C^j_{qn} = D^j_{qn}(|\vec{T}|^2)$$

$) \mathcal{D}^{j}_{pn}(T^{k}) \; |j\, m\, q\rangle$

$$\sum_{p,k} \mathcal{D}^{j}_{qp}(T^{k})\, \mathcal{D}^{j}_{pn}(T^{k}) = C^{j}_{qn} = \mathcal{D}^{j}_{qn}(|\vec{T}|^{2})$$

$$[C^{j}, \mathcal{D}^{j}(T^{k})] = 0$$ exercise: check this!


exercise: check this!
C^j is a Casimir op
Schur's lemma: it acts on an irreducible
representation vector space
as the identity op
m fixed,

- C^j is a Casimir op
- Schur's lemma: it acts on representation vector space as the identity op $C^j = \lambda\, id_V$

C^j: su(2) Casimir (modulo a cst)

$\lambda = -j(j+1)$

$$l_{e^*}^2 |j m n\rangle = -\hbar^2 \sum_q |j m q\rangle C^j_{qn} = -\lambda \hbar^2 \sum_q |j m q\rangle \delta_{qn} = +\hbar^2 j(j+1) |j m n\rangle$$

$l_{e^*}^2$: • eigenvectors $\{|j m n\rangle\}$

• spectrum $\left\{ l_P^2\, j(j+1),\ j \in \mathbb{N}/2 \right\}$ is discrete!

$l_P^{3D} = \hbar K$

- C^j is a Casimir op
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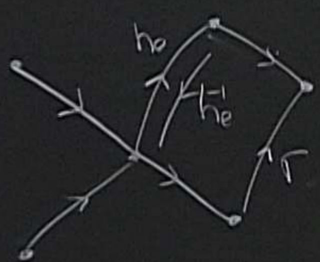
$l_P^{3D} = \hbar \kappa$ ↳ (G, c)

$|j\, m\, n\rangle$

$\mathcal{H}_e = L^2(SU(2), dg)$

# of edges of Γ

$$\mathcal{H}_\Gamma = L^2\left(SU(2)^L\right)$$



$f_\Gamma(h_e)$

$(D^j_{mn}(g))$ $\bullet\!\longrightarrow\!\bullet$ $|j\, m\, n\rangle$ $\equiv$ $n\ \bullet\!\xrightarrow{j}\!\bullet\ m$ associated to both ends

$\mathcal{H}_e = L^2(SU(2), dg)$

\# of edges of Γ

$$\mathcal{H}_\Gamma = L^2\left(SU(2)^E\right) = \bigotimes_e \mathcal{H}_e = \bigotimes_e L^2(SU(2))$$

Inner product

$$\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle = \int_{SU(2)^E} \overline{\Psi_{\Gamma,1}(g_{e_1} \dots g_{e_E})}\, \Psi_{\Gamma,2}(g_{e_1} \dots g_{e_E})\, dg_{e_1} \dots dg_{e_E}$$

Orthonormal basis $|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \dots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$

$$|j\,m\,n\rangle$$

associated to both ends

$$\mathcal{H}_e = L^2(SU(2), dg)$$

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Orthonormal basis

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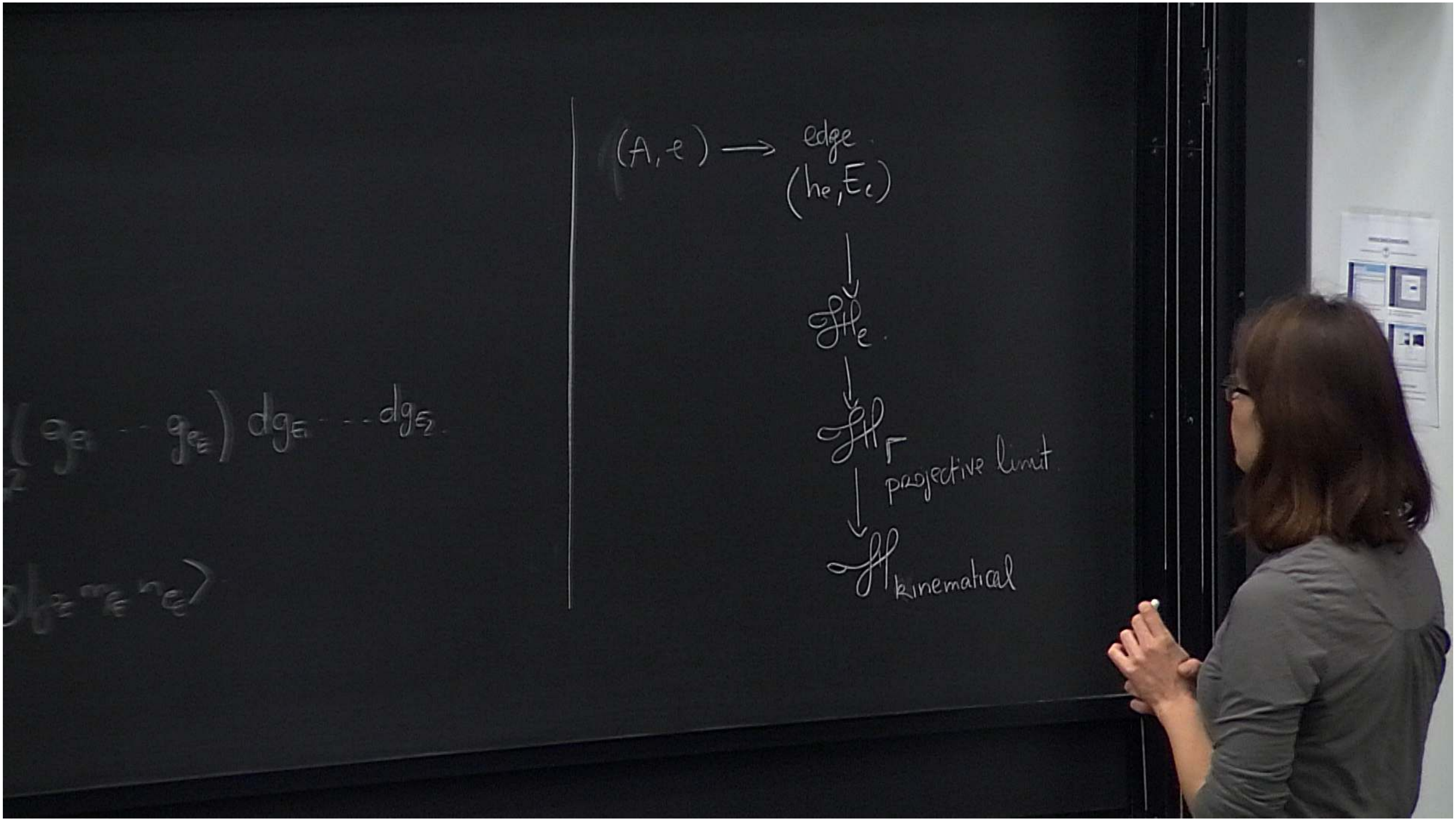
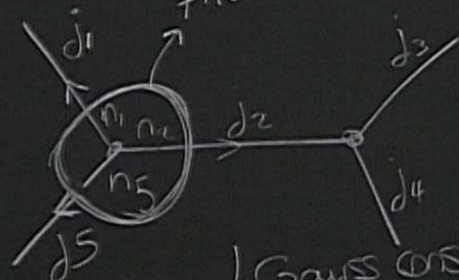
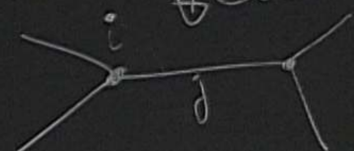

(A, e) ⟶ edge
(h_e, E_e)
H_e
H_Γ
projective limit
H_kinematical
dg_E1 ... dg_E2

• Left multiplication $\quad L_h D^j_{mn}(g) = D^j_{mn}(h^{-1}$

this is going to be constrained by the Gauss constraint



$\downarrow$ Gauss constraints

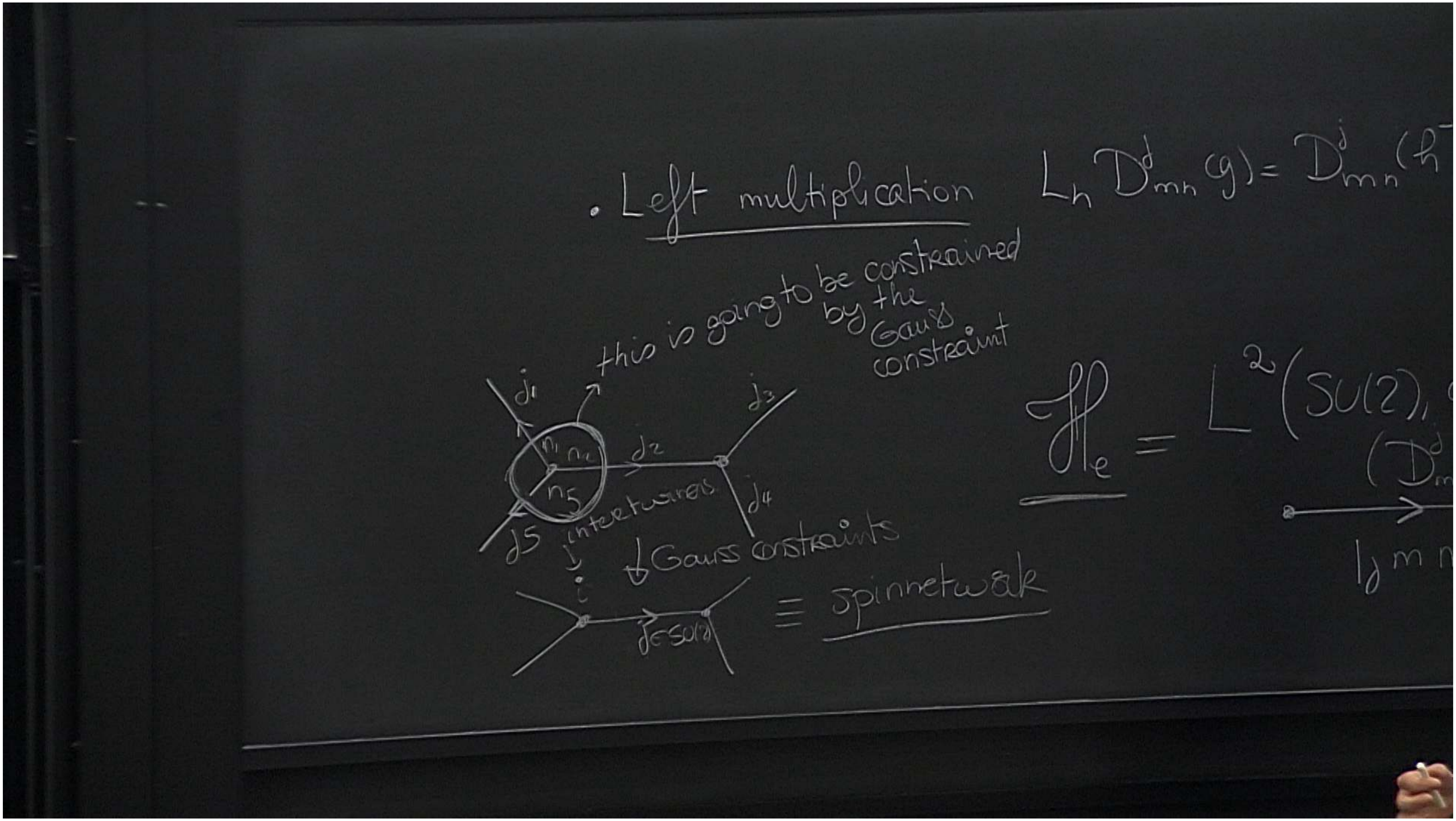


$\equiv$ spinnetwork

$$\mathcal{H}_e = L^2(SU(2),$$

$(D^j_{m}$

$l, j\, m\, n$


• Left multiplication
this is going to be constrained by the Gauss constraint
intertwiners
Gauss constraints
≡ spinnetwork
j ∈ SU(2)
j1
j2
j3
j4
j5
n1
n2
n5
l, j m n

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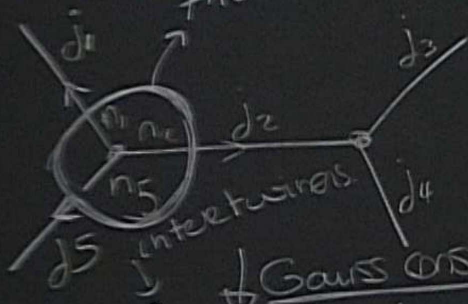
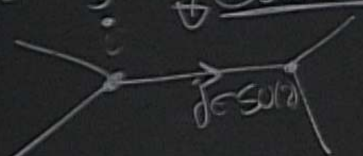


$$\mathcal{H}_e = L^2(SU(2), dg) = \bigoplus_j V_j^* \otimes V_j$$

$(D^j_{mn}(g))$

$|j\, m\, n\rangle$

↓ Gauss constraint ①



$\equiv$ $\boxed{\text{Spinnetwork}}$

↳ ② Flatness contraint $\longrightarrow$ $\boxed{\text{Physical Hilbert space}}$