Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 9

Speakers: MaÃ¯tÃ© Dupuis

Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

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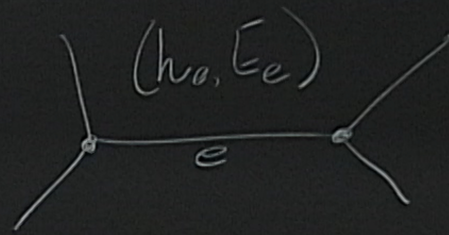
URL: http://pirsa.org/19030076

$(A(x), E(x)) \longrightarrow$



$h_{mn} \longrightarrow \hat{h}_{mn} \qquad \widehat{h_{mn}|\Psi\rangle}$

$[\hat{f}, \hat{g}] = i\hbar \{f, g\} \Longrightarrow$

Check $[\hat{E}^j, \hat{E}^k]$ ✓

Single Edge:

(h_{mn}, E^j)

$$\{h_{mn}, h_{m'n'}\} = 0$$

$$\{h_{mn}, E^j\} = (hT^j)_{mn}$$

$$\{E^j, E^k\} = \varepsilon_l{}^{jk} E^l$$

Inner Product:

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{SU(2)} dg_{Haar} \, \overline{\Psi_1(g)} \, \Psi_2(g)$$

$$\hat{h}_{mn}|\Psi\rangle = |h_{mn}\Psi\rangle$$

$$\widehat{\{f,g\}} \Longrightarrow \hat{E}^j|\Psi\rangle = i\hbar|L^j\Psi\rangle$$

$$[\hat{E}^j, \hat{E}^k] \checkmark$$

<u>O.N. Basis:</u>

oduct:

$$\overline{\Psi_1}(g)\Psi_2(g)$$

Peter-Weyl $\longrightarrow$ $f(g) = \sum_j d_j f^j_{mn} D^j_{mn}(g)$

$$\mathbb{1}_{SU(2)} = \sum_{j\in \mathbb{N}/2}\sum_{m=-j}^{j}\sum_{n=-j}^{j} |jmn\rangle\langle jmn|$$

$$|jmn\rangle = \sqrt{d_j}\,|D^j_{mn}\rangle$$

$\ldots h\,|L^{j}\psi\rangle$

~~o~~ver complete basis.

Basis :

$\Rightarrow \quad f(g) = \sum_j d_j f^{j}_{mn} D^{j}_{mn}(g)$

$\sum_{m=-j}^{j} \sum_{n=-j}^{j} |jmn\rangle\langle jmn|$

$|jmn\rangle = \sqrt{d_j}\,|D^{j}_{mn}\rangle$

TODAY: Kinematical Hilbert space.

↳ subsystem on edge ⟶ a graph.

↓

length op.

$$\hat{E}^a = i\hbar L^a = i\hbar \frac{d}{dt}\Big|_{t=0} R_{e^{t\tau_a}}$$

$$(R_h f)(g) = f(gh).$$

$|\psi\rangle = |h_{mn}\psi\rangle$

$\Rightarrow \quad E^i|\psi\rangle = i\hbar|L^i\psi\rangle$

over complete basis.

O.N. Basis:

$\rightarrow$ Wigner matrices

Peter-Weyl $\rightarrow \quad f(g) = \sum d_j f^j_{mn} D^j_{mn}(g)$

$\mathbb{1}_{SU(2)} = \sum_{j\in\frac{\mathbb{N}}{2}} \sum_{m=-j}^{j} \sum_{n=-j}^{j} |jmn\rangle\langle jmn| \qquad |jmn\rangle = \sqrt{d_j}\,|D^j_{mn}\rangle$

$|h_{mn}\psi\rangle$

$\hat{E}^i|\psi\rangle = i\hbar|L^i\psi\rangle$

$\{|jmn\rangle\}$. over complete ba

O.N. Basis:

Wigner matrices

Peter-Weyl $\longrightarrow$ $f(g) = \sum_j d_j f^j_{mn} D^j_{mn}(g)$

$|jmn\rangle = \sqrt{d_j}\,|D^j_{mn}\rangle$

$1_{SU(2)} = \sum_j \sum_m^j \sum_n^j |jmn\rangle\langle jmn|$

space.

edge $\longrightarrow$ a graph.

$$(R_h f)(g) = f(gh)$$

$$g) = D^j_{mn}(gh) = \sum_p \underbrace{D^j_{mp}(g)}_{\langle g | jmp \rangle} D^j_{pn}(g) \Rightarrow R_h |jmn\rangle = \sum_p D^j_{pn}(g)\, |jmp\rangle .$$

$mn\rangle$.

$\{ |jmp\rangle, p \in \{-j \dots j\} \}$: subrepresentation space carries a rep D^j

$${}_h D^j_{mn}(g) = D^j_{mn}(h^{-1} g)$$

- <u>Left multiplication</u> $L_h D^j_{mn}(g) = D^j_{mn}(h^{-1}g) = \sum_q D^j_{mq}(h^{-1})\,D$

$$\mathcal{H}_e = L^2(SU(2), dg) = \bigoplus_j V_j^* \otimes V_j$$

$$\underset{|j\,m\,n\rangle}{\overset{(D^j_{mn}(g))}{\bullet\!\longrightarrow\!\bullet}} \equiv \bullet\!\xrightarrow{\;j\;}\!\bullet$$

- **Left multiplication** $L_h D^j_{mn}(g) = D^j_{mn}(h^{-1}g) = \sum_q D^j_{mq}(h^{-1})\, D$

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$$\underset{|jmn\rangle}{\bullet \xrightarrow{\left(D^j_{mn}(g)\right)} \bullet} \;\equiv\; \bullet \xrightarrow{j} \bullet$$

$$= \sum_q D^j_{mq}(h^{-1})\, D^j_{qn}(g) \;\Rightarrow\; L_h |j\, m\, n\rangle = \sum_q D^j_{mq}(h^{-1})\, |j\, q\, n\rangle$$

$\{|j\, q\, n\rangle \;\; q=-j \dots j\}$: subrepresentation space
carries a rep. $(D^j)^*$: dual
contragredient rep.

$$(D^j)^*_{mn}(g) = D^j_{nm}(g^{-1}) = \overline{D^j_{mn}(g)}$$

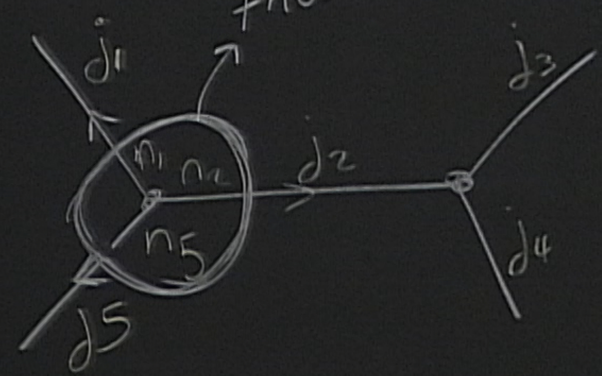
$$= \bigoplus_j V_j^* \otimes V_j$$

$$\equiv \; n \xrightarrow{\;j\;} m$$

(angular momentum associated to both ends of the edge).

• Left multiplication $\quad L_h D^j_{mn}(g) = D^j_{mn \ldots}$

this is going to be constrained by the Gauss constraint



$$\mathcal{H}_e = L^2(SU(2 \ldots$$

$|j \ldots$

• Left multiplication $L_h D^j_{mn}(g) = D^j_{mn}(h^{-1} g) = \sum_q D^j_{mq}(h^{-1})\, D^j_{qn}(g) \Rightarrow L_h |j\, m\, n\rangle =$

$\{ |j\, q\, n\rangle \;\; q = -j \dots j \}$ subrep

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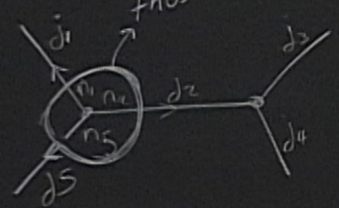

$$\underline{\mathcal{H}_e} = L^2(SU(2), dg) = \bigoplus_j V_j^* \otimes V_j$$

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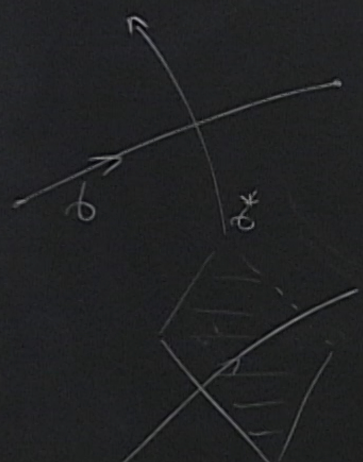
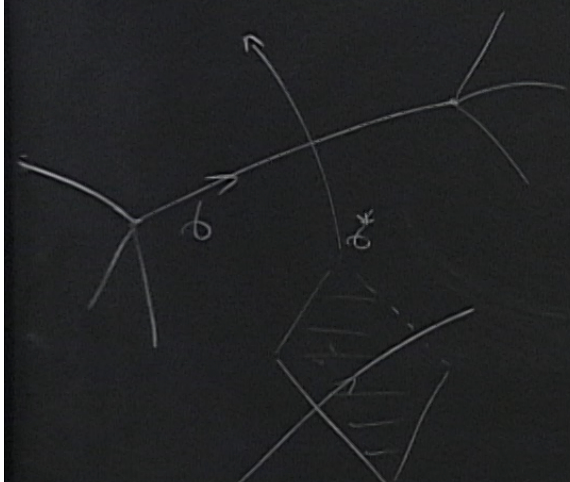
(angular momentum associated to both ends of the

Length op.



Length op :



ep :

length of a curve γ^* on a 2D surface. $\int_{\gamma^*} \sqrt{\dot{\gamma}^{*a}\dot{\gamma}^{*b} q_{ab}}\, ds$

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$\hookrightarrow q_{ab} = e^i_a e^j_b \delta_{ab}$.

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$\hookrightarrow \sim$ " $\sum_{n=0}^{N} \sqrt{\sum_k E^k_{e_n} E^k_{e_n}}$ "

$\hookrightarrow$ Riemann approx.

$$\hat{l}^2_{e^*} := \sum_k \hat{E}^k_e \hat{E}^k_e$$

aa.

$$\int_{\gamma^*} \sqrt{\dot{\gamma}^{*a}\,\dot{\gamma}^{*b}\, q_{ab}}\; ds$$

↳ $q_{ab} = e^i_a\, e^j_b\, \delta_{ab}$

↳ $\sim$ " $\displaystyle\sum_{n=0}^{N} \sqrt{\sum_k E^k_{e_n} E^k_{e_n}}$ "

↳ Riemann approx

"heuristic justification"

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$\hookrightarrow$ Riemmann approx

"heuristic justification"

$$\hat{l}^2_{e^*} := \sum_k \hat{E}^k_e \hat{E}^k_e$$

$$E^k |j\,m\,n\rangle = i\hbar \sum_p |j\,m\,p\rangle \frac{d}{dt}\Big|_{t=0} D^j_{pn}(e^{tT^k}) = i\hbar \sum_p |j\,m\,p\rangle D^j_{pn}(T^k)$$

" Right multiplication "

t multiplication"

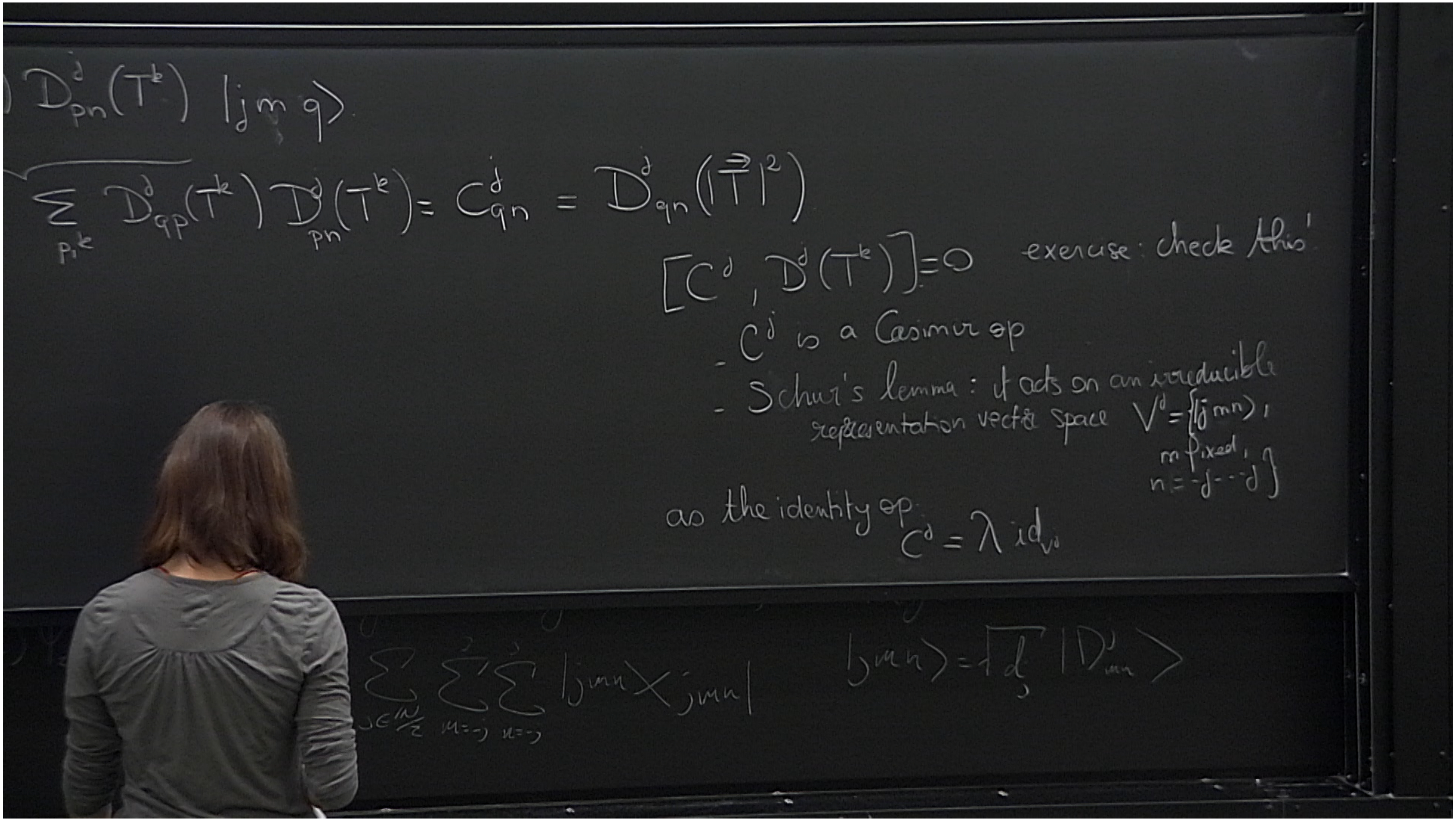
$$\hat{\ell}^2 |j m n\rangle = -\hbar^2 \sum_{p,q,k} D^j_{qp}(T^k)\, D^j_{pn}(T^k)\, |j m q\rangle$$

$$\hookrightarrow \sum_{p,k} D^j_{qp}(T^k)\, D^j_{pn}(T^k) = C^j_{qn} = D^j_{qn}\left(|\vec{T}|^2\right)$$

$) D^j_{pn}(T^k) \; |j \, m \, q\rangle$

$$\sum_{p,k} D^j_{qp}(T^k)\, D^j_{pn}(T^k) = C^j_{qn} = D^j_{qn}(|\vec{T}|^2)$$

$[C^j, D^j(T^k)] = 0$ exercise: check this!


$D^j_{pn}(T^k)\ |j\,m\,q\rangle$
$\sum_{p,k} D^j_{qp}(T^k)\, D^j_{pn}(T^k) = C^j_{qn} = D^j_{qn}(|\vec{T}|^2)$
$[C^j, D^j(T^k)] = 0$
exercise: check this!
- C^j is a Casimir op
- Schur's lemma: it acts on an irreducible representation vector space $V^j = \{|j\,m\,n\rangle,\ m$ fixed, $n = -j \dots j\}$
as the identity op. $C^j = \lambda\, \mathrm{id}_{V^j}$
$\sum_{j \in \mathbb{N}/2} \sum_{m=-j}^{j} \sum_{n=-j}^{j} |j\,m\,n\rangle\langle j\,m\,n|$
$|j\,m\,n\rangle = \sqrt{d_j}\, |D^j_{mn}\rangle$

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as the identity op. $C^j = \lambda\, id_{V}$

C^j: $su(2)$ Casimir (modulo a cst)

$\lambda = -j(j+1)$

$$l_{e^*}^2 |j m n\rangle = -\hbar^2 \sum_q |j m q\rangle C^j_{qn} = -\lambda \hbar^2 \sum_q |j m q\rangle \delta_{qn} = +\hbar^2 j(j+1) |j m n\rangle$$

$l_{e^*}^2$: • eigenvectors $\{|j m n\rangle\}$

• spectrum $\left\{ l_P^2\, j(j+1),\ j \in \mathbb{N}/2 \right\}$ is discrete!

$l_P^{3D} = \hbar \kappa$

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C^j: su(2) Casimir (modulo a cst)

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$$l_{e^*}^2 |jmn\rangle = -\hbar^2 \sum_q |jmq\rangle C^j_{qn} = -\lambda \hbar^2 \sum_q |jmq\rangle \delta_{qn} = +\hbar^2 j(j+1) |jmn\rangle$$

$l_{e^*}^2$: • eigenvectors $\{|jmn\rangle\}$

• spectrum $\{ l_P^2 j(j+1), j \in \mathbb{N}/2 \}$ is discrete!

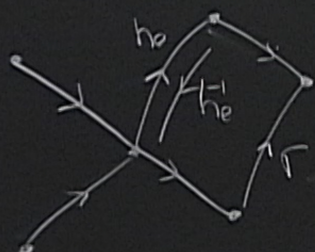
$l_P^{3D} = \hbar \kappa$ ↳ (G, c)

$|j\, m\, n\rangle$

$\mathcal{H}_e = L^2(SU(2), dg)$

\# of edges of Γ

$$\mathcal{H}_\Gamma = L^2\left(SU(2)^L\right)$$



$f_\Gamma(h_e)$

$|j\, m\, n\rangle$

associated to both ends

$\mathcal{H}_e = L^2(SU(2), dg)$

\# of edges of Γ

$$\underline{\mathcal{H}_\Gamma} = L^2\left(SU(2)^E\right) = \bigotimes_e \mathcal{H}_e = \bigotimes_e L^2(SU(2))$$

Inner product

$$\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle = \int_{SU(2)^E} \overline{\Psi_{\Gamma,1}(g_{e_1} \cdots g_{e_E})}\, \Psi_{\Gamma,2}(g_{e_1} \cdots g_{e_E})\, dg_{e_1} \cdots dg_{e_E}$$

Orthonormal basis

$$|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \cdots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$$

$(D_{mn}(g))$ associated to both ends

$|j\,m\,n\rangle \equiv$ n → m

$\mathcal{H}_e = L^2(SU(2), dg)$

\# of edges of Γ

$$\mathcal{H}_\Gamma = L^2(SU(2)^E) = \bigotimes_e \mathcal{H}_e = \bigotimes_e L^2(SU(2))$$

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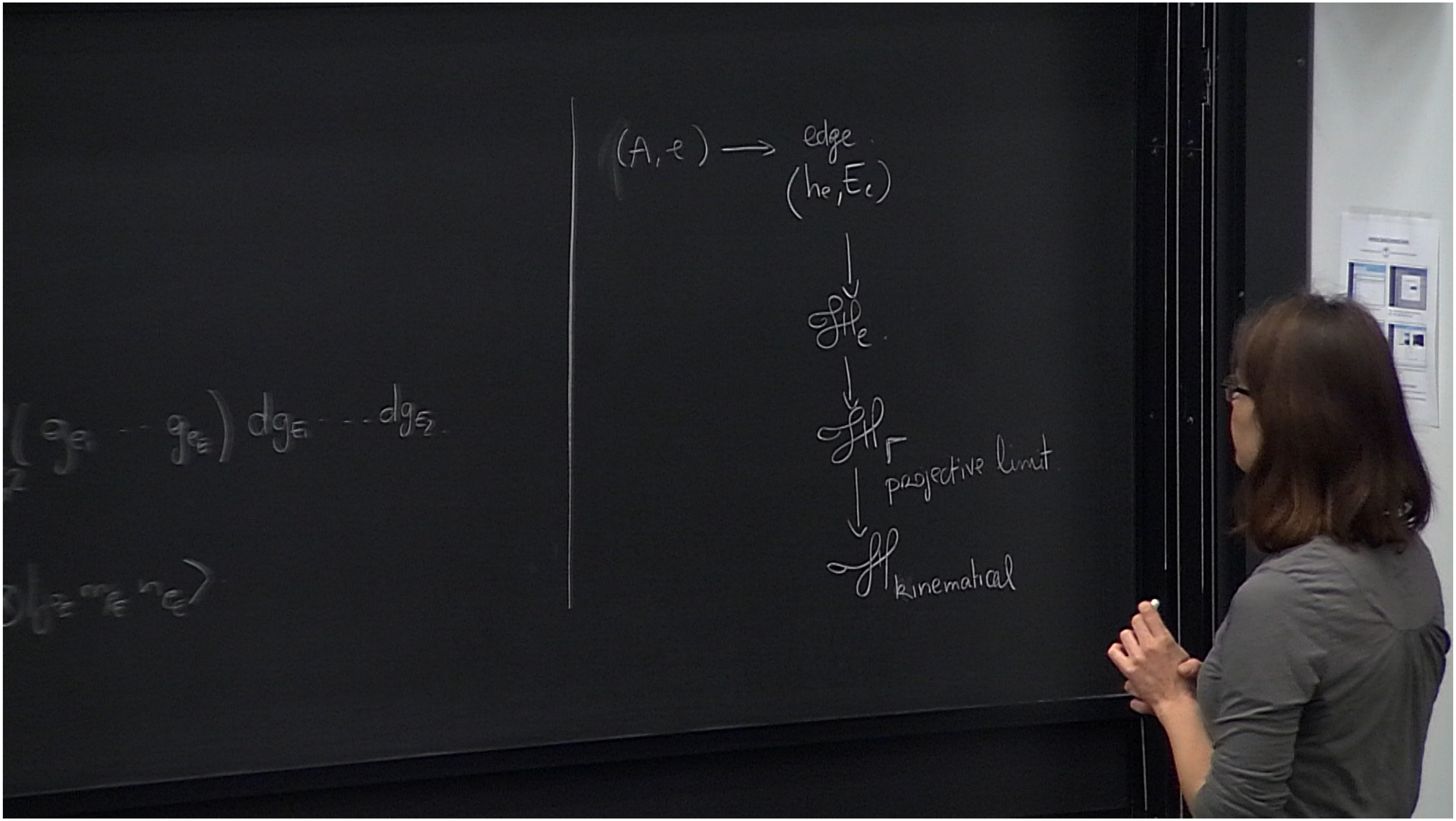
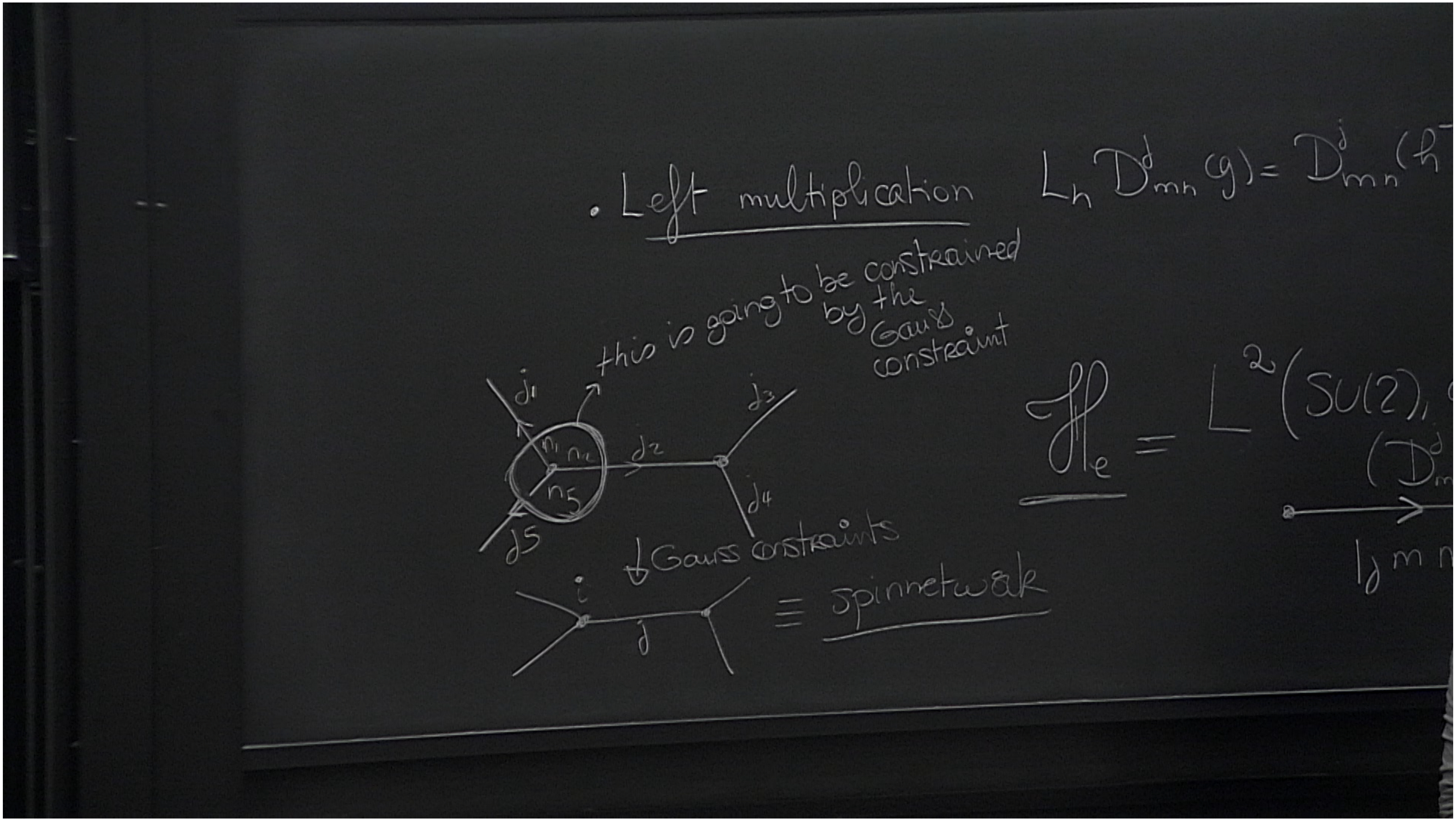
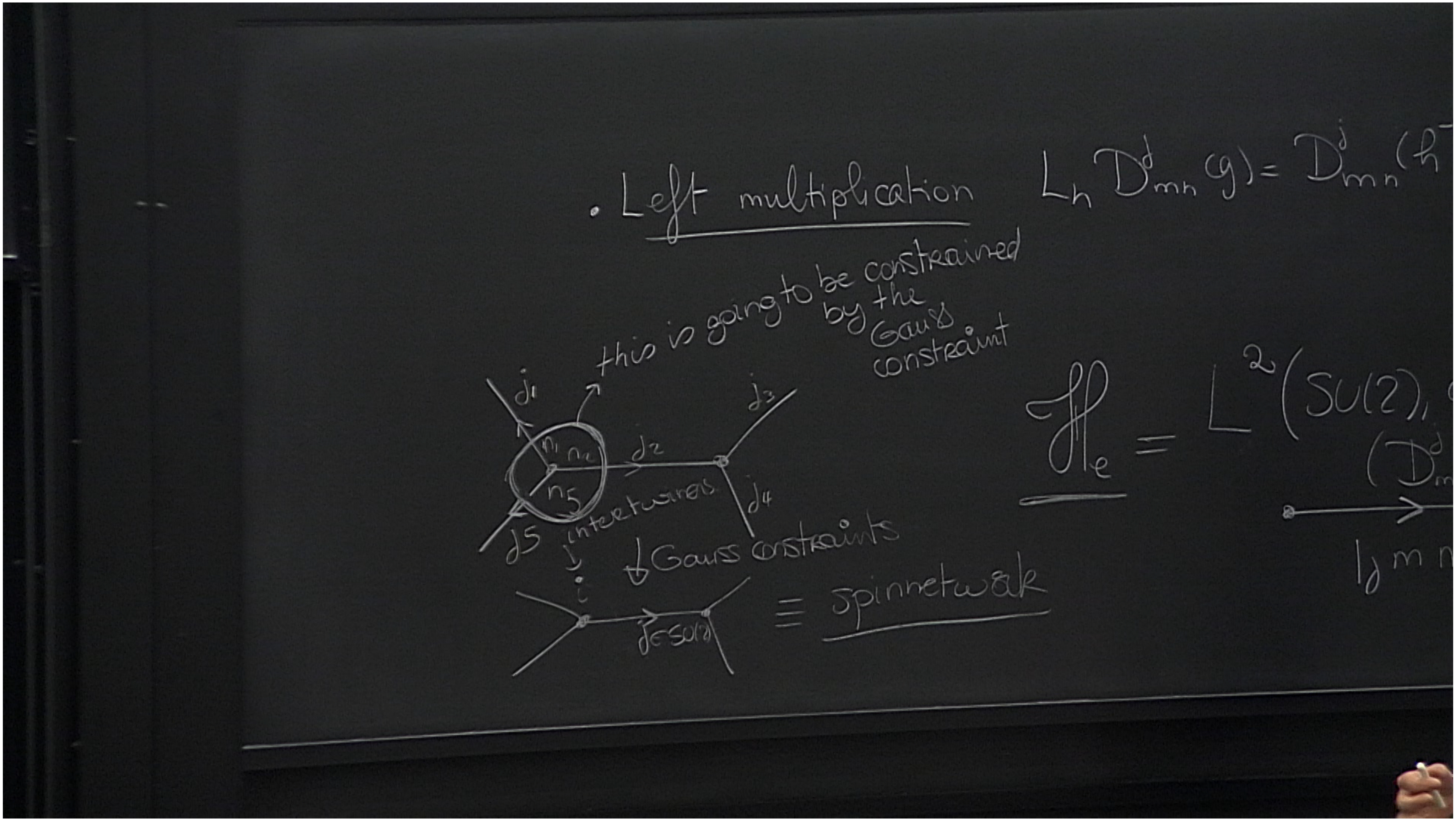
Orthonormal basis $|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \cdots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$


(A, e) → edge
(h_e, E_e)
H_e
H_Γ
projective limit
H_kinematical


• Left multiplication
L_h D^j_mn (g) = D^j_mn (h
this is going to be constrained by the Gauss constraint
j1
n1
n2
j2
n5
j5
j3
j4
Gauss constraints
i
j
≡ spinnetwork
H_e = L^2(SU(2),
(D^j_m
l, j m n



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j_1
j_2
j_3
j_4
j_5
n_1
n_2
n_5
intertwiners
i
Gauss constraints
$j \in SU(2)$
≡ spinnetwork
$\mathcal{H}_e = L^2(SU(2),$
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$l, m\ n$

• **Left multiplication** $L_h D^j_{mn}(g) = D^j_{mn}(h^{-1} g) = \sum_q D^j_{mq}(h^{-1}\ldots$

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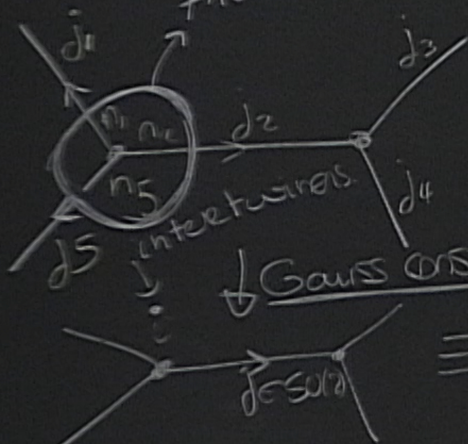

$$\mathcal{H}_e = L^2(SU(2), dg) = \bigoplus_j V_j^* \otimes V_j$$

$(D^j_{mn}(g))$ $\equiv$ $n \xrightarrow{j} m$

$|j\, m\, n\rangle$

↓ Gauss constraint ① $\equiv$ $\boxed{\text{spinnetwork}}$

↳ ② Flatness contraint $\longrightarrow$ $\boxed{\text{Physical Hilbert space}}$