Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 8

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Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

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Why are we doing iT?

- 3D Gravity Action

$$S = \int d^2x\, dt \left(e_b^j \partial_0 A_{aj} \tilde{\epsilon}^{ab} + e_{aj} \frac{1}{2} F_{ab}^j \tilde{\epsilon}^{ab} + A_a^j \left(\partial_a e_{bj} + \epsilon_{jlm} A_a^l e_b^m \right) \right)$$

- Conjugated "momentum"

$$E_j^a = \frac{\delta S}{\delta(\partial_0 A_{aj})} = \tilde{\epsilon}^{ab} e_{bj}$$

- $\{A_a^j(x), E_k^b(y)\} = \delta_k^j \delta_a^b \delta(x,y)$

$$\int d^3x \sqrt{-g}$$

Holonomies

$$\frac{SU(2)}{\mathbb{Z}_2} \approx SO(3)$$

$$E^a = E^{aj} T_j \to g E^a g^{-1}$$

$$F_{ab} = F_{ab}^j T_j \to g F_{ab} g^{-1}$$

$$A_a = A_a^j T_j \to g A_a g^{-1} + g \partial_a g^{-1}$$

Parallel Transport

$V(s) = h(s) V(0) h(s)^{-1}$

$s \in [0,1]$

$h(s) \to g(\gamma(s)) h(s) g(\gamma(0))^{-1}$

$V(\gamma(0))$, $\gamma(s)$

$\cdot \dot\gamma^a D_a V = 0 \to \frac{d}{ds} h(s) = -\dot\gamma^a A_a(\gamma(s)) h(s)$

$\cdot h(0) = \mathbb{1}$ $\to h_\gamma(s) = P e^{-\int_\gamma A} = \sum_{n=0}^{\infty} (-1)^n \int_0^s ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 A(s_n) A(s_{n-1}) \cdots A(s_2) A(s_1)$

$\gamma(s)$ Fluxes γ

$E_\gamma = \int_{\gamma^*} h_{\gamma\gamma^*} e_a \dot\gamma^{*a} h^{-1} ds$ $E_\gamma \to g E_\gamma g^{-1}$

$h_{\gamma\gamma^*} = h_\gamma(\tfrac{1}{2})^{-1} h_{\gamma^*}(\tfrac{1}{2}; s)$

$E_\gamma = E_\gamma^j T_j$

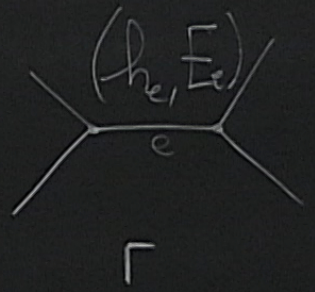
$\{h_\gamma m_n, h_{\gamma^* m^*}\} = 0$

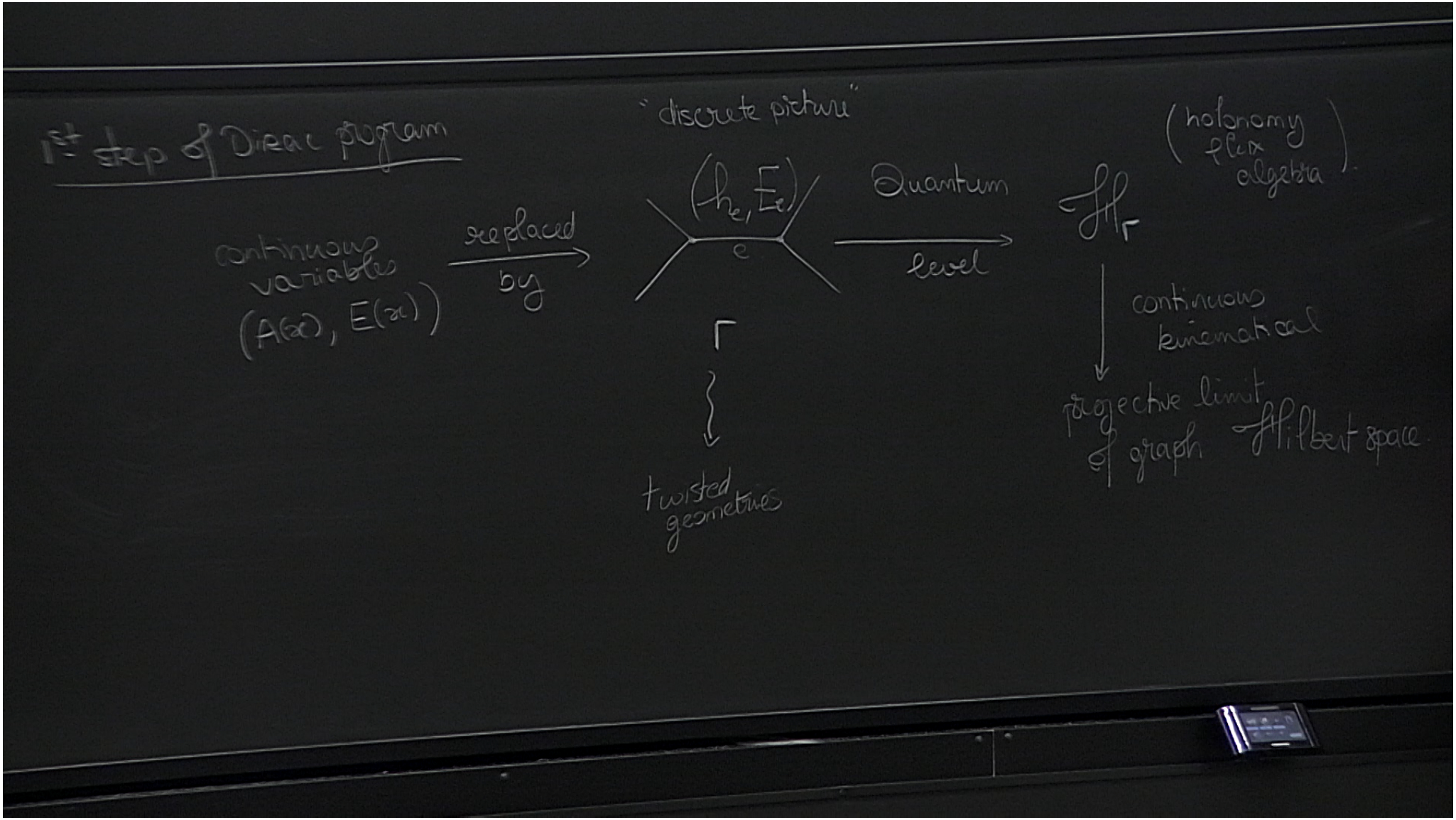
$\{E_\gamma^i, h_{\gamma^* m^*}\} = (h_\gamma T^i)_{m^*}$

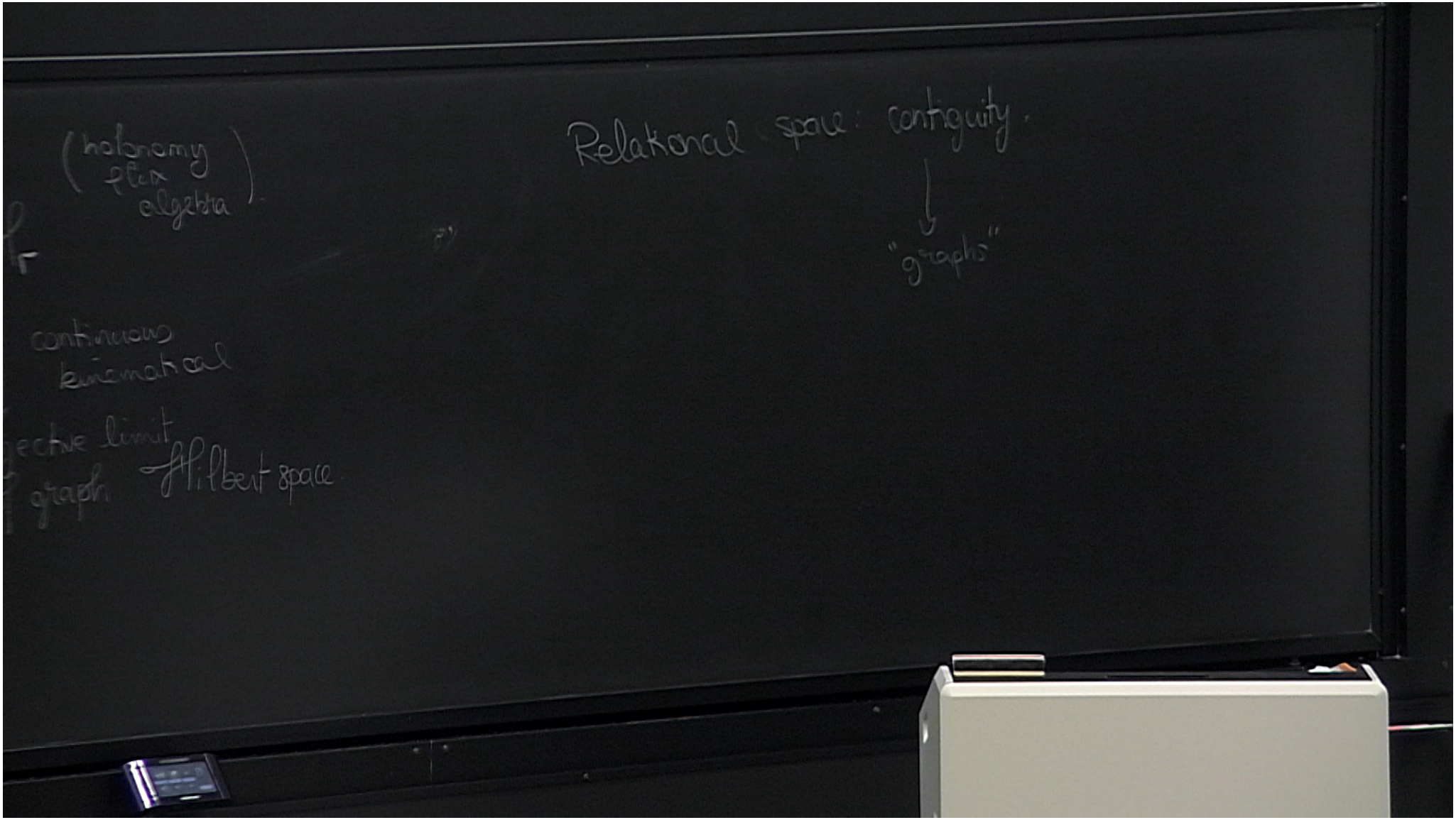
$\{E_\gamma^i, E_{\gamma^*}^j\} = C^{ij}{}_k E_{\gamma\gamma^*}^k$

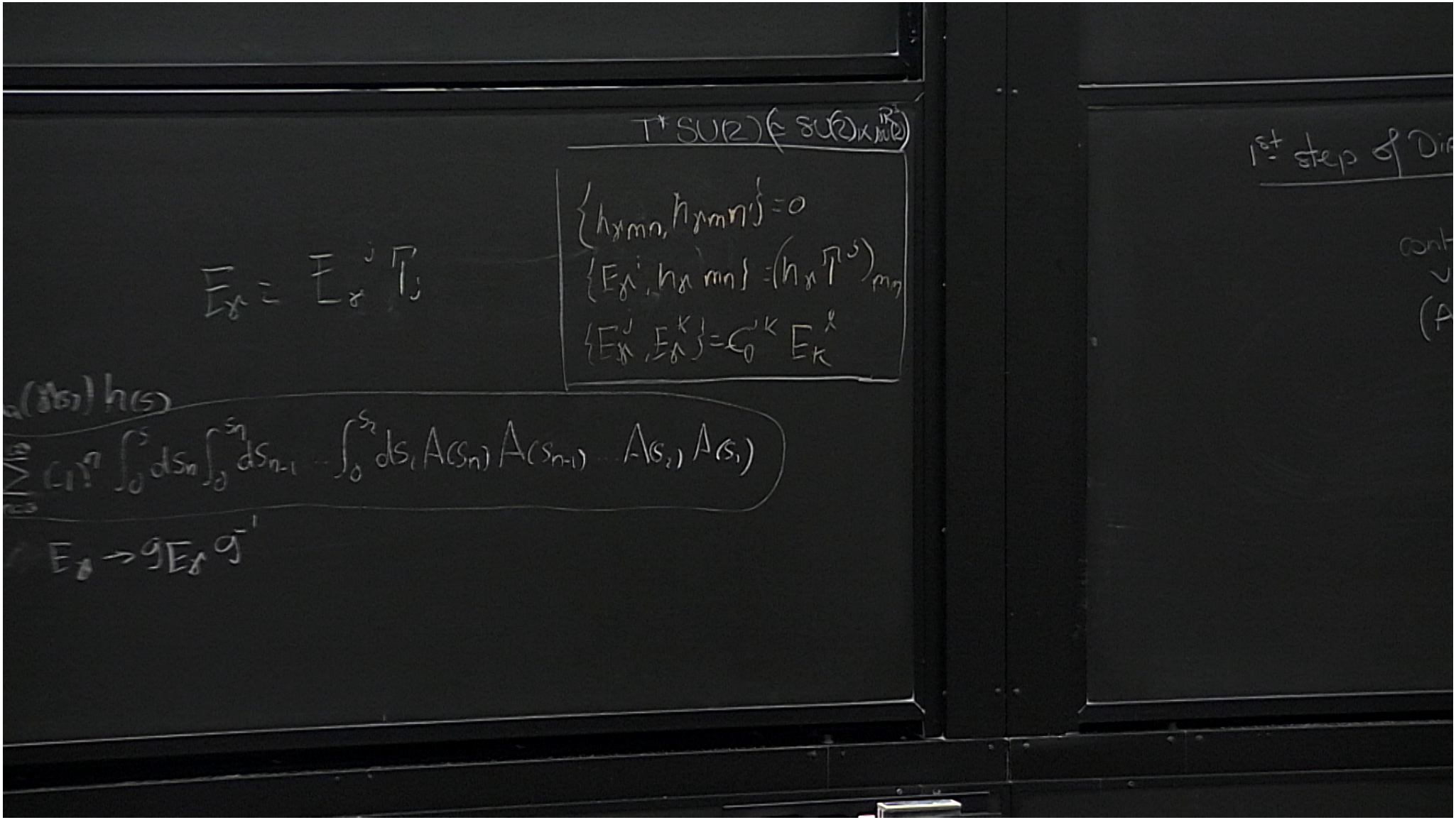
1st step of Dirac program

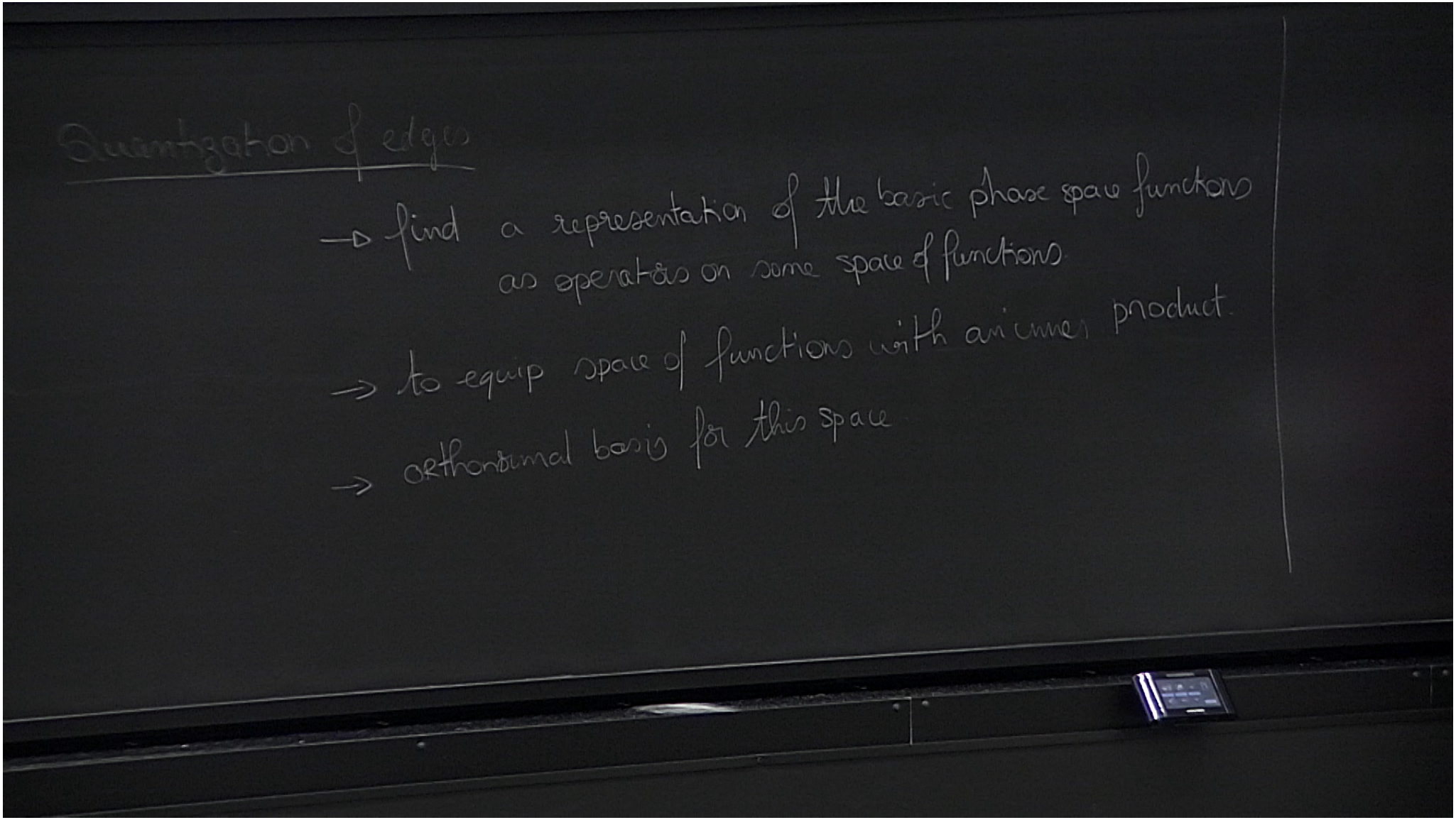
continuous variables $(A(x), E(x))$ → replaced by




1st step of Dirac program
continuous variables (A(x), E(x))
replaced by
"discrete picture"
(h_e, E_e)
e
Γ
twisted geometries
Quantum level
H_Γ
(holonomy flux algebra)
continuous kinematical
projective limit of graph
Hilbert space


(holonomy
flux
algebra)
Relational space: contiguity
"graphs"
continuous
kinematical
ective limit
graph
Hilbert space


$T^*SU(2) \subset SU(2) \times \mathbb{R}^3$
$E_\delta = E_\delta^j T_j$
$\{h_{\gamma mn}, h_{\gamma' m'n'}\} = 0$
$\{E_\delta^j, h_{\gamma mn}\} = (h_\gamma T^j)_{mn}$
$\{E_\delta^j, E_{\delta'}^k\} = C_0^{jk} E_k^{\delta}$
$\sum (-1)^n \int_0^s ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 A(s_n) A(s_{n-1}) \cdots A(s_2) A(s_1)$
$E_\delta \to g E_\delta g^{-1}$
1st step of


Quantization of edges
→ find a representation of the basic phase space functions as operators on some space of functions
→ to equip space of functions with an inner product
→ orthonormal basis for this space

$h_e \in SU(2)$

Quantize function on the gp $SU(2)$ $f: G \to \mathbb{C}$

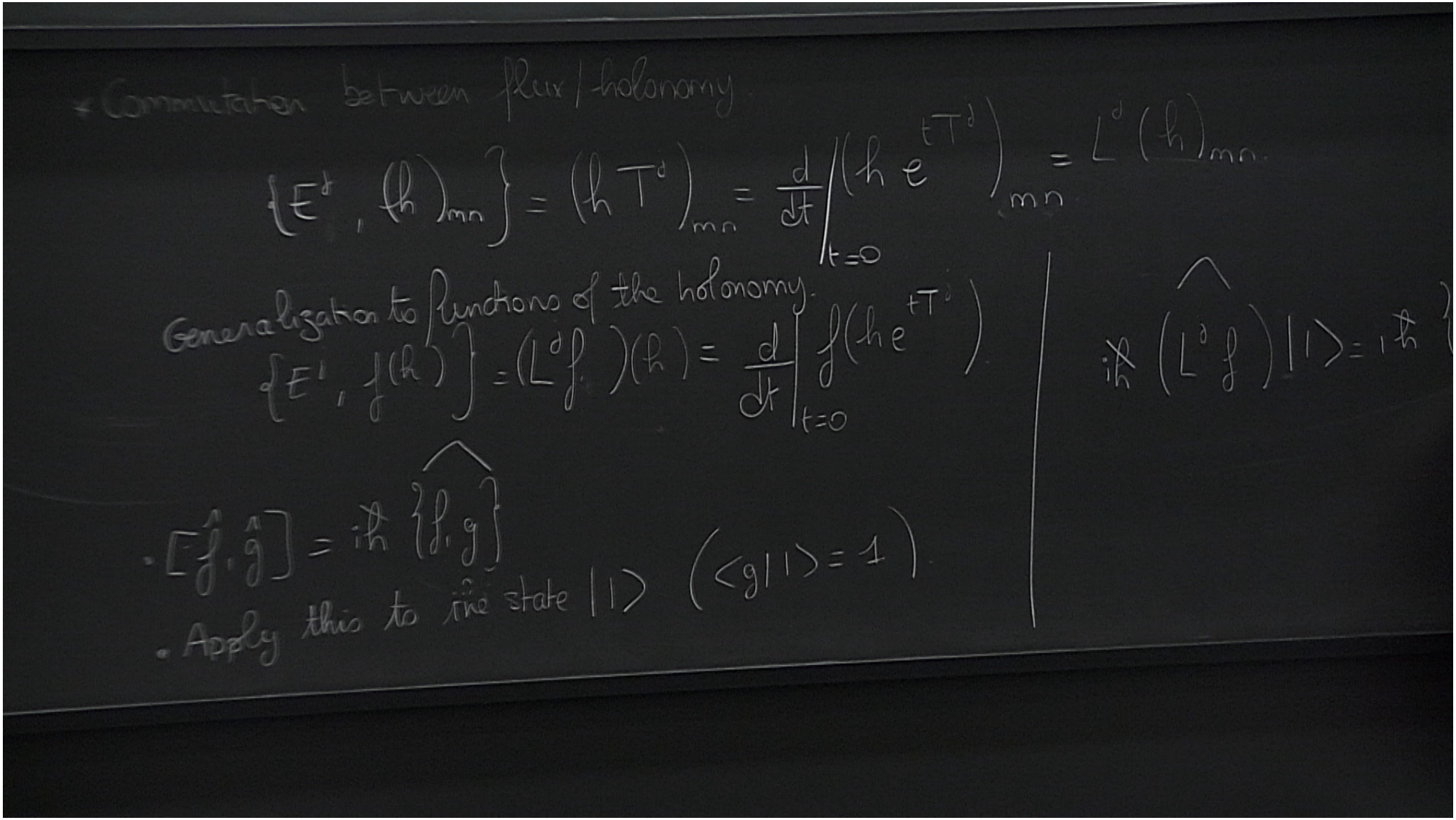
↳ quantized as multiplication op.

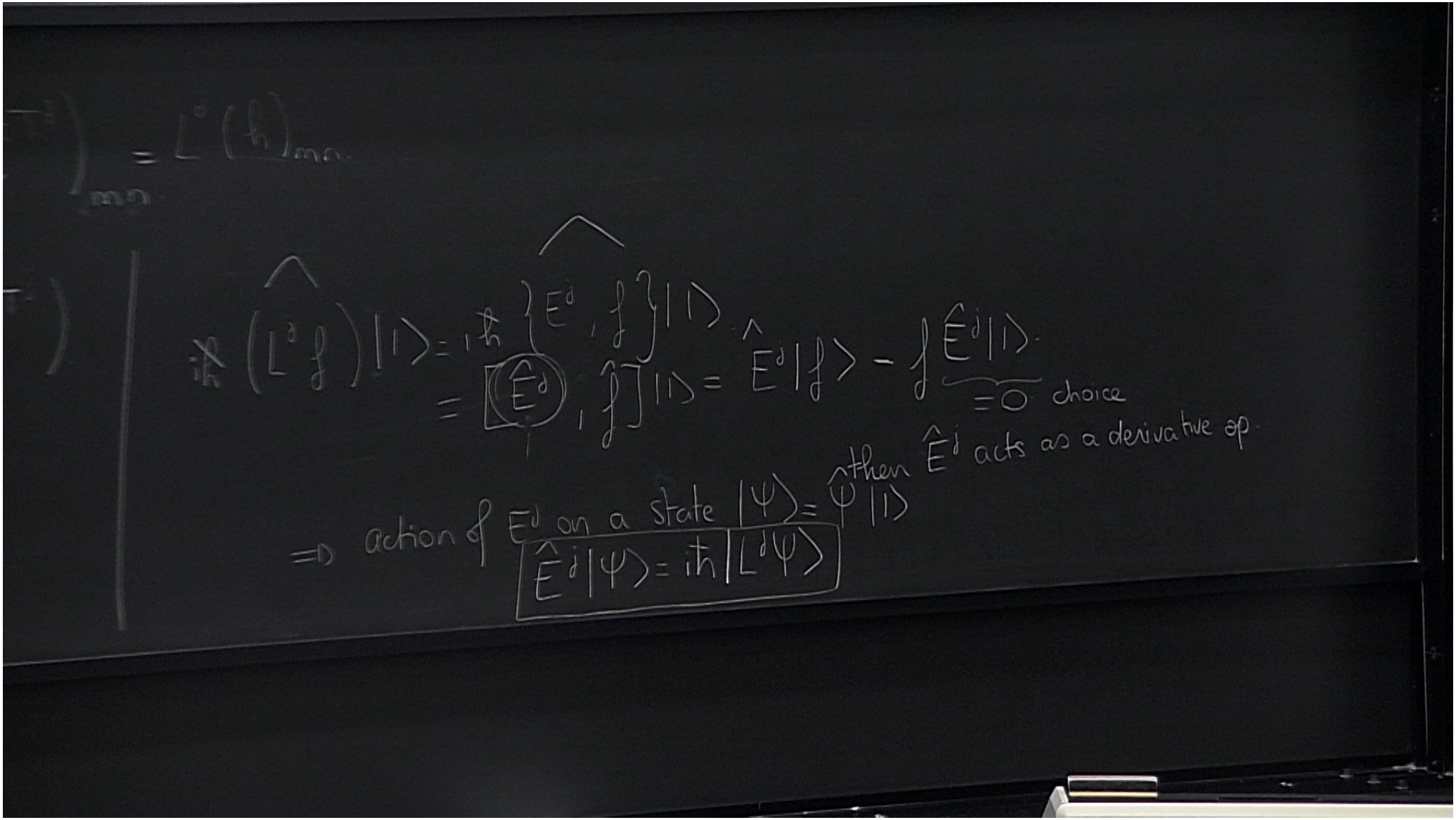
$\psi_e : SU(2) \to \mathbb{C}$

$g_e \mapsto \psi_e(g_e)$

$|\psi_e\rangle$ holonomy rep $\langle g_e | \psi_e \rangle = \psi(g_e)$

- $\hat{f}\, \psi(g_e) = f(g_e)\, \psi(g_e)$
- $\hat{f} |\psi_e\rangle = f |\psi_e\rangle$
- Multiplication op: commute OK!


Commutation between flux / holonomy
$\{E^a, (h)_{mn}\} = (h T^a)_{mn} = \frac{d}{dt}\Big|_{t=0} (h e^{tT^a})_{mn} = L^a (h)_{mn}$
Generalization to functions of the holonomy
$\{E^a, f(h)\} = (L^a f)(h) = \frac{d}{dt}\Big|_{t=0} f(h e^{tT^a})$
$\widehat{(L^a f)}|1\rangle = i\hbar$
$[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}}$
Apply this to the state $|1\rangle$ $(\langle g|1\rangle = 1)$


$= L^a(h)_{mn}$
$i\hbar\,\widehat{(L^a f)}|1\rangle = i\hbar\,\{\hat{E}^a, f\}|1\rangle$
$= [\hat{E}^a, f]|1\rangle = \hat{E}^a|f\rangle - f\hat{E}^a|1\rangle$
$=0$ choice
$\Rightarrow$ action of $\hat{E}^a$ on a state $|\psi\rangle = \hat{\psi}|1\rangle$ then $\hat{E}^a$ acts as a derivative op.
$\hat{E}^a|\psi\rangle = i\hbar|L^a\psi\rangle$

Consistency of the final commutation relation between the fluxes.

$$[\hat{E}^i, \hat{E}^k]\,|\psi\rangle = \left(\hat{E}^i \hat{E}^k - \hat{E}^k \hat{E}^i\right)|\psi\rangle = (i\hbar)^2 \left|\left(L^i \triangleright L^k - L^k \triangleright L^i\right)\psi\right\rangle$$

$$\stackrel{?}{=} i\hbar \{E^i, E^k\}\,|\psi\rangle$$

$$[\hat{E}^i, \hat{E}^k] = i\hbar\, \varepsilon^{ik}{}_l\, \hat{E}^l = i\hbar \{E^i, E^k\}$$

evaluation on matrix elements.

$$\left(L^i \triangleright L^k - L^k \triangleright L^i\right)(h)_{mn} = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \left(\left(h\, e^{sT^i} e^{tT^k}\right)_{mn} - \left(h\, e^{tT^k} e^{sT^i}\right)_{mn}\right)$$

$$= \left(h_{re}\,[T^i, T^k]\right)_{mn} = \varepsilon^{ikl} \left(h_{re}\, T^l\right)_{mn} = \varepsilon^{ikl} \frac{d}{dt}\Big|_{t=0} \left(h\, e^{tT^l}\right)_{mn} = \varepsilon^{ikl}\, L^l (h)_{mn}$$

Consistency of the final commutation relation between the fluxe

$h \in SU(2)$

h_{mn}

$$[\hat{E}^j, \hat{E}^k]|\psi\rangle = (\hat{E}^j \hat{E}^k - \hat{E}^k \hat{E}^j)|\psi\rangle = (i\hbar)^2$$

$\stackrel{?}{=} i\hbar$

evaluation on matrix elements.

$$(L^j \triangleright L^k - L^k \triangleright L^j)(h)_{mn} = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} ((h$$

$$= (h_e [T^j, T^k]$$

$= (\quad)_{mn}$

→ Haar measure $\int_{SU(2)} dg = 1$

$$\int dg = \int d(hg) = \int d(gh) = \int dg^{-1}$$

$$\langle \psi_1 | \psi_2 \rangle = \int \overline{\psi_1(g)}\, \psi_2(g)\, dg$$

↳ Hilbert space for one edge $\mathcal{H}_e = L^2(SU(2), dg)$ → ON basis?

An orthonormal basis - Peter-Weyl theorem

Given a function $f \in L^2(SU(2))$

$$f(g) = \sum_j d_j \hat{f}^j_{mn} D^j_{mn}(g)$$

$$\hat{f}^j_{mn} = d_j \int_{SU(2)} dg \, D^j_{mn}(g^{-1}) f(g)$$

$$d_j = 2j+1$$

$\langle g | jmn \rangle = \sqrt{d_j}\, D^j_{mn}(g)$

$\{|j\,mn\rangle\}$

- $\int_{SU(2)} D^j_{mn}(g)\, D^{j'}_{m'n'}(g)\, dg = \frac{1}{d_j}\, \delta^{jj'}\, \delta_{mm'}\, \delta_{nn'}$
- $\sum_{j \in \frac{\mathbb{N}}{2}} \sum_{m=-j}^{j} \sum_{n=-j}^{j} |jmn\rangle\langle jmn| = \mathbb{1}_{L^2(SU(2), dg)}$