Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 7

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Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

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LAST TIME:

$$H = -\int d^2x \; N^j \mathcal{F}_j + \Lambda^j G_j$$

INTERPRET CONSTRAINTS.

$$\left(\mathcal{F}[N] = \int d^2x \, N^j \mathcal{F}_j \, , \quad G[\Lambda] = \int d^2x \, \Lambda^j G_j \right)$$

$$\{ A^j_a(x), E^b_k(y) \} = \delta^j_k \, \delta^b_a \, \delta(x-y)$$

$$\delta_\Lambda E = \{ E \, , G[\Lambda] \} = \epsilon_{jnk} E^{am} \Lambda^k(x)$$

$$\delta_\Lambda A = -(\partial \Lambda + \epsilon \Lambda A$$

$$\delta_N E^a_j = \tilde{\varepsilon}^{ca} D_c N_j(x)$$

$$\delta_N A = 0$$

Λ: ROTATIONS

N: TRANSLATIONS

$$\delta_\Lambda A = -(\partial\Lambda + \epsilon \Lambda A$$

$$\delta_N E_j^a = \tilde{\varepsilon}^{ca} D_c N_j(x)$$

$$\delta_N A = 0$$

Λ: ROTATIONS

N: TRANSLATIONS

DIFFEO: $\left(\delta^R_{A(v)} + \delta^T_{e(v)}\right) A = \delta^D_v e^j_\mu + T^j_{\mu\nu} v^\nu$

$$\left(\delta^R_{A(v)} + \delta^T_{e(v)}\right) A = \delta^D_v A^j_\mu + F^j_{\mu\nu} v^\nu$$

CONSTRAINT ALGEBRA.

$$\{G[\Lambda], G[\Lambda']\} = G([\Lambda, \Lambda'])$$

$$\{\mathcal{F}[N], G[\Lambda]\} = \mathcal{F}([N, \Lambda'])$$

$$\{\mathcal{F}[N], \mathcal{F}[N']\} = 0$$

Configuration variables - holonomies

- $A^i_a(x) \longrightarrow h_e[A] = P \exp - \int_e A \quad \in SU(2)$

- Transformation under a finite rotation in terms of a SU(2) gp elt.

T^j are su(2) generators.

$g \in SU(2)$

→ $R_k^{\ l}(g)\, T_l = g^{-1} \cdot T_k \cdot g$

→ Lie-algebra valued objects: $E^a = E^a_j T^j$, $A_a = A^j_a T_j$, $F_{ab} = F^j_{ab} T_j$

$E^a_j = \tilde{\varepsilon}^{ab} e_{bj}$

• Transformation under a finite rotation in terms of a SU(2) gp elt.

→ $R_k^{\ \ell}(g)\, T_\ell =$

→ Lie-algebra

internal vector

$$\underbrace{v^k R_k^{\ \ell}(g)}_{(R^{-1}(g).v)^\ell} T_\ell = g^{-1}\cdot\left(v^k T_k\right)\cdot g \Rightarrow \left(R(g).v\right)^j T_j = g\cdot\left(v^k T_k\right)\cdot g^{-1}$$

T^j are su(2) generators

$\rightarrow R^{\ell}_{\ k}(g)\, T_\ell = g^{-1} \cdot T_k \cdot g \qquad g \in SU(2)$

s of a SU(2) gp elt.

$\rightarrow$ Lie-algebra valued objects: $E^a = E^a_j T^j$, $A_a = A^j_a T_j$, $F_{ab} = F^j_{ab} T_j$

$E^a_j = \tilde{\epsilon}^{ab} e_{bj}$

$\Rightarrow (R(g) \cdot v)^j T_j = g \cdot \underline{(v^k T_k)} \cdot g^{-1}$

$\Rightarrow E^a \rightarrow g\, E^a g^{-1}$

$F_{ab} \rightarrow g\, F_{ab}\, g^{-1}$

$\longrightarrow h_e[A] = P\exp -\int_e A \in SU(2)$

e

...rmation under a finite rotation in terms of a SU(2) gp elt. $\rightarrow R_k^{\ell}(g)\, T_\ell = g^{-1}\cdot T_k \cdot g$

...internal vector

$$\underbrace{v^k R_k^{\ell}(g)}_{(R^{-1}(g)\cdot v)^{\ell}} T_\ell = g^{-1}\cdot(v^k T_k)\cdot g \Rightarrow (R(g).v)^j T_j = g\cdot\underline{(v^k T_k)}\cdot g^{-1}$$

$\rightarrow$ Lie-algebra valued objects: E^a

$\Rightarrow E^a \rightarrow g E^a g^{-1}$

$F_{ab} \rightarrow g F_{ab} g^{-1}$

connection: $A_a \longrightarrow g A_a g^{-1} + g\,\partial_a g^{-1}$

Ex: check this last transformation rule.

$V^j T_j$

connection: $A_a \longrightarrow g A_a g^{-1} + g \partial_a g^{-1}$

Ex: check

$V = V^j T_j$

$s \in [0,1]$

tangent vector

$\dot{\gamma}(s) = \frac{d}{ds}\gamma(s)$

parallel transport

$V(s) = h(s)\, V(0)\, h(s)^{-1}$

↳ holonomy from $\gamma(0)$ to $\gamma(s)$

Ex: check this last transformation rule.

under gauge transf. $\quad V(0) \to g(0)\, V(0)\, g(0)^{-1} = V'(0)$

$$V(s) \to g(s)\, V(s)\, g(s)^{-1} = V'(s)$$

gauge transfo for a holonomy?

$$V'(s) = h'(s)\, V'(0)\, h'^{-1}(s)$$

$$= h'(s)\, g(0)\, V(0)\, g(0)^{-1}\, h'^{-1}(s) = g(s)\, h(s)\, V(0)\, h^{-1}(s)\, g(s)^{-1}$$

$$\left[h(s) \to h'(s) = g(s)\, h(s)\, g(0)^{-1} = g(\gamma(s))\, h(s)\, g(\gamma(0))^{-1} \right]$$

Definition of the holonomy

V parallel transported along γ: $0 = \dot{\gamma}^a D_a V(s)$

$D_a V = \partial_a V + [A_a, V]$

$\dot{\gamma}_a \partial_a = \frac{d}{ds}$

$= \left(\frac{d}{ds} h(s)\right) V(0)\, h^{-1}(s) - h(s)\, V(0)\, h^{-1}(s) \frac{d h(s)}{ds} h^{-1}(s)$

N · TRANSLATIONS

$\{\mathcal{F}[N], \mathcal{F}[N']\} = 0$

Definition of the holonomy

V parallel transported along γ: $\quad 0 = \dot\gamma^a D_a V(s) \qquad D_a V = \partial_a V + [A_a, V]$

$\dot\gamma_a \partial_a = \frac{d}{ds}$

$\longrightarrow \frac{d}{ds}(h h^{-1}) = 0$

$= \left(\frac{d}{ds} h(s)\right) V(0)\, h^{-1}(s) - h(s) V(0)\, h^{-1}(s) \frac{d h(s)}{ds} h^{-1}(s) + \dot\gamma^a A_a h(s) V(0) h(s)^{-1} - h(s) V(0) h(s)^{-1} \dot\gamma^a A_a$

$= 0$

Defining differential eq for the holonomy

$$\boxed{\frac{d}{ds} h(s) = -\dot\gamma^a A_a(\gamma(s))\, h(s)}$$

$h(0) = \mathbb{1}$

$$D_a V = \partial_a V + [A_a, V]$$

$$\longrightarrow \frac{d}{ds}(h h^{-1}) = 0.$$

$$V(0)\, h^{-1}(s) - h(s)\, V(0)\, h^{-1}(s) \frac{d h(s)}{ds} h^{-1}(s) + \dot\gamma^a A_a\, h(s) V(0) h(s)^{-1} - h(s) V(0) h(s)^{-1} \dot\gamma^a A_a$$

$$\boxed{-\dot\gamma^a A_a(\gamma(s))\, h(s)}$$

$h(0) = \mathbb{1} \Rightarrow$ well-known solution:

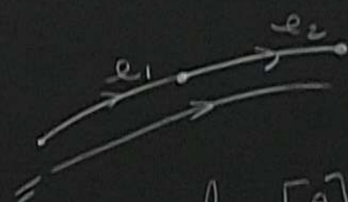
$$h_\gamma(s) = P \exp\left(-\int_\gamma A\right)$$

$$A(s) = \dot\gamma(s)\, A_a(\gamma(s))$$

Important properties of holonomy

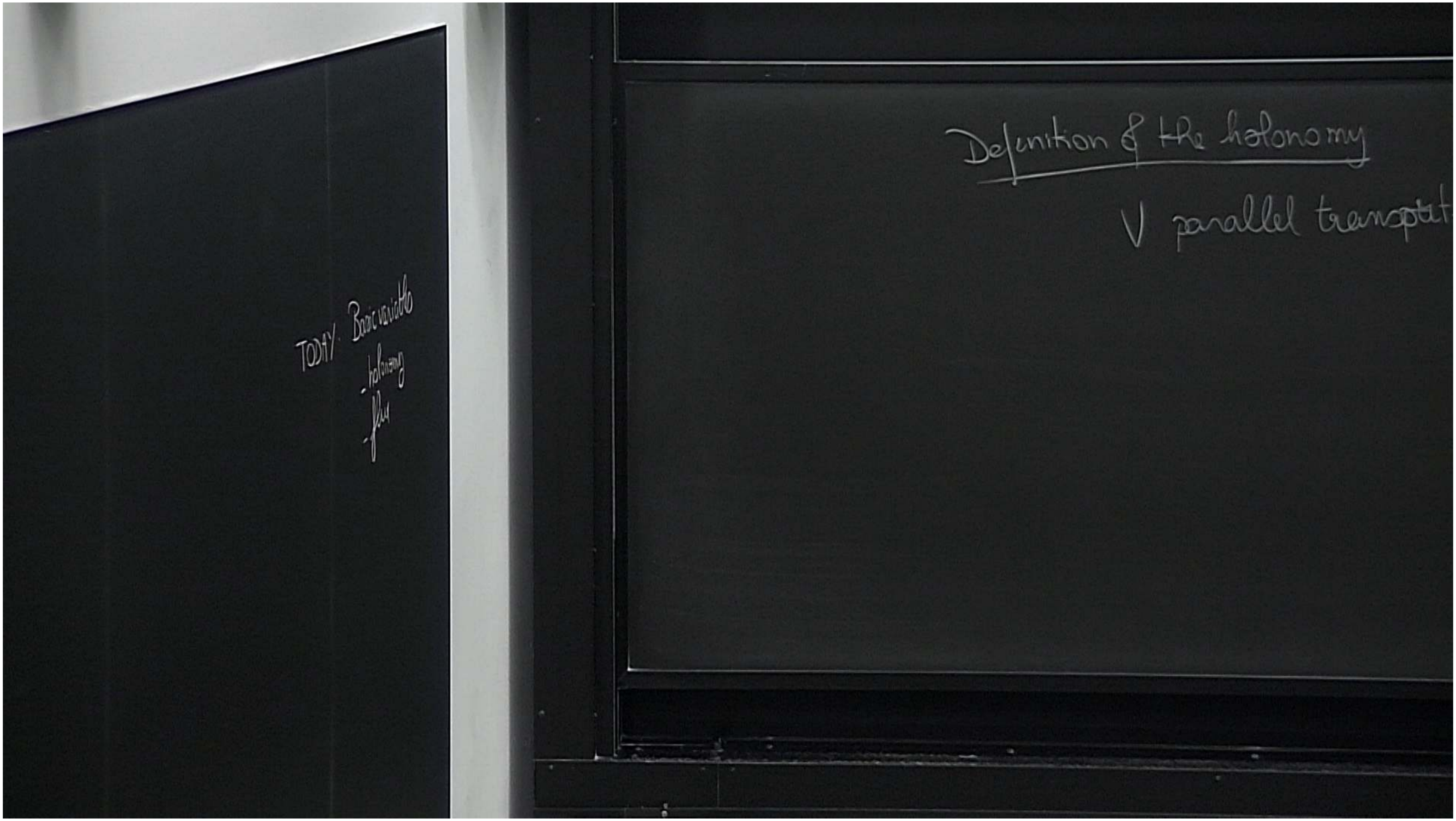
i) definition $h_\gamma[A]$ independent of parametrization of path γ.

ii) γ is a point $h_\gamma = \mathbb{1}$.

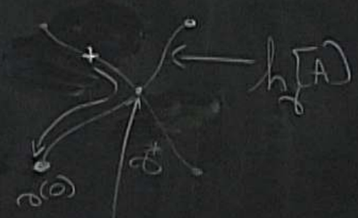


$$h_e[A] = h_{e_1}[A]\, h_{e_2}[A].$$

$$h_{e^{-1}}[A] = h_e^{-1}[A]$$


TODAY: Basic variables
- holonomy
- flux
Definition of the holonomy
∇ parallel transport

Conjugated variables – fluxes



flux sitting at $\gamma(0)$

$$E_\gamma = \int_{\gamma^*} h_{\gamma\gamma^*}(s)\, e_a(\gamma^*(s)) \left(h_{\gamma\gamma^*}(s)\right)^{-1}$$

↳ // transport from $\gamma^*(s)$ to $\gamma(0)$

- $s = s' = \frac{1}{2}$
- $\det(\dot\gamma^a, \dot\gamma^{*b}) > 0$

$\gamma(0)$

Rmk $E_\gamma \longrightarrow g(\gamma(0))\, E_\gamma\, g(\gamma(0))^{-1}$

$h_{\gamma^{-1}} = h_\gamma^{-1}$, $E_{\gamma^{-1}} = -\, h_\gamma E_\gamma h_\gamma^{-1}$

to extract component

$tr\,(T_j T_k) = -\frac{1}{2}\delta_{jk}$

$T_j = -\frac{i}{2}\sigma_j$

$E_{\gamma j} = -2\, T_R(E_\gamma T^j)$

$\{(E_\gamma)_j , h_\gamma\} = h_\gamma T_j$

$\{(h_\gamma)_{MN} , (h_\gamma)_{M'N'}\} = 0$

$\{E^i_\gamma , E^j_\gamma\} = \epsilon^{ij}{}_k E^k_\gamma$

$T^* SU(2)$