Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 5

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Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

Date: March 08, 2019 - 10:15 AM

URL: http://pirsa.org/19030072

ACTION! $S = \frac{1}{k}\int R\sqrt{-g}\, d^{4}x$ (4 | 3), $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

First order formalism: $\overline{g_{\mu\nu}(x) = e^{i}_{\mu}(x) e^{j}_{\nu}(x) \delta_{ij}}$ (g), 6 DoF as for the

we have an add. t

↳ e^{i}_{μ}

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

counting DoF. 6 for both.

$\Rightarrow$ In 3D $\Rightarrow R_{\mu\nu} = R_{\mu\nu\alpha\beta} = 0$

↳ Gravity is a Topological Theory in 3D (No spacetime dynamic for vacuum solutions).

, 6 DoF as for the fundamental object

we have an additional SO(3) symmetry

↳ $e'^{i}_{\mu}(x) = R^{i}_{j}(x)\, e^{j}_{\mu}(x) \Rightarrow$ Action Remains Invariant.

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

counting DoF. 6 for both.

$\Rightarrow \Sigma_N$ 3D $\Rightarrow R_{\mu\nu} = R_{\mu\nu\alpha\beta} = 0$

↳ Gravity is a Topological Theory in 3D (No spacetime dynamic for vacuum solutions).

i.e. 6 DoF as for the fundamental object

we have an additional $SO(3)$ symmetry

↳ $e'^i_\mu(x) = R^i_j(x)\, e^j_\mu(x) \Rightarrow$ Action Remains Invariant.

Spin connections: Tensor in internal spin indices.

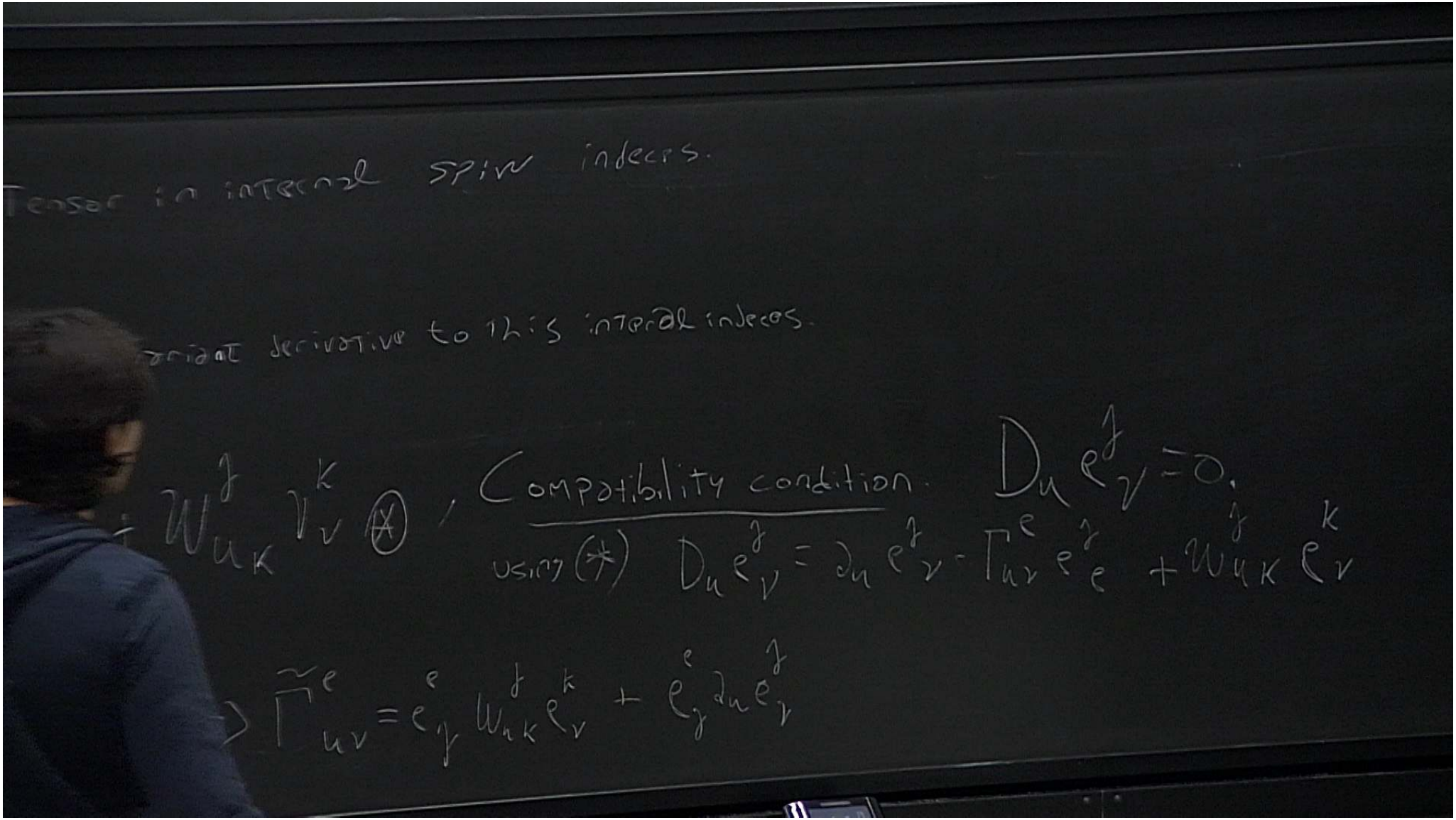
$W^{i}_{\mu k}$ - "expand" the notion of covariant derivative to this internal indices.

$$D_\mu v^\gamma_\nu = \partial_\mu v^\gamma_\nu - \Gamma^{\rho}_{\mu\nu} v^\gamma_\rho + W^\gamma_{\mu k} v^k_\nu \quad (*)$$

Compatibility condit

using $(*)$ $D_\mu e^\gamma_\nu = \partial_\mu$

$$\Rightarrow \tilde{\Gamma}^{\rho}_{\mu\nu} = e^\rho_\gamma W^\gamma_{\mu k} e^k_\nu + e^\rho_\gamma \partial_\mu e^\gamma_\nu$$


Tensor in internal spin indices.
variant derivative to this internal indices.
$= W^{j}_{\mu k} V^{k}_{\nu}$ (*) / Compatibility condition. $D_\mu e^{j}_{\nu} = 0$.
using (*) $D_\mu e^{j}_{\nu} = \partial_\mu e^{j}_{\nu} - \Gamma^{\rho}_{\mu\nu} e^{j}_{\rho} + W^{j}_{\mu k} e^{k}_{\nu}$
$\Rightarrow \tilde{\Gamma}^{\rho}_{\mu\nu} = e^{\rho}_{j} W^{j}_{\mu k} e^{k}_{\nu} + e^{\rho}_{j} \partial_\mu e^{j}_{\nu}$

Tensor in internal spin indices.

...ion of covariant derivative to this internal indices.

$$\partial_\mu V_\epsilon^\gamma + W_{\mu K}^{\ \ \gamma} V_\nu^K \quad (*)$$

Compatibility condition. $D_\mu e^\gamma_\nu = 0.$

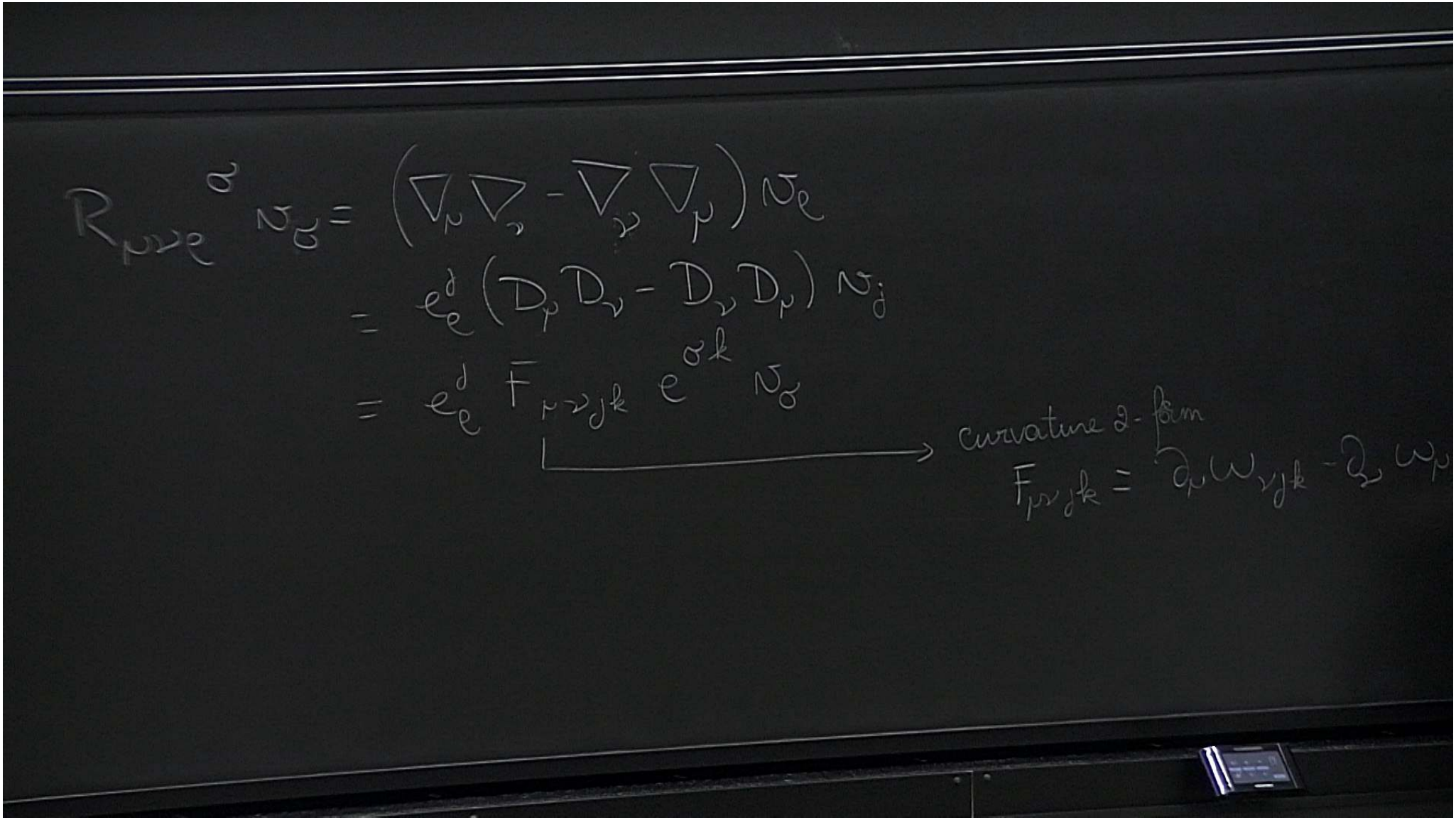
using $(*)$ $D_\mu e^\gamma_\nu = \partial_\mu e^\gamma_\nu - \Gamma^\epsilon_{\mu\nu} e^\gamma_\epsilon + W_{\mu K}^{\ \ \gamma} e^K_\nu$

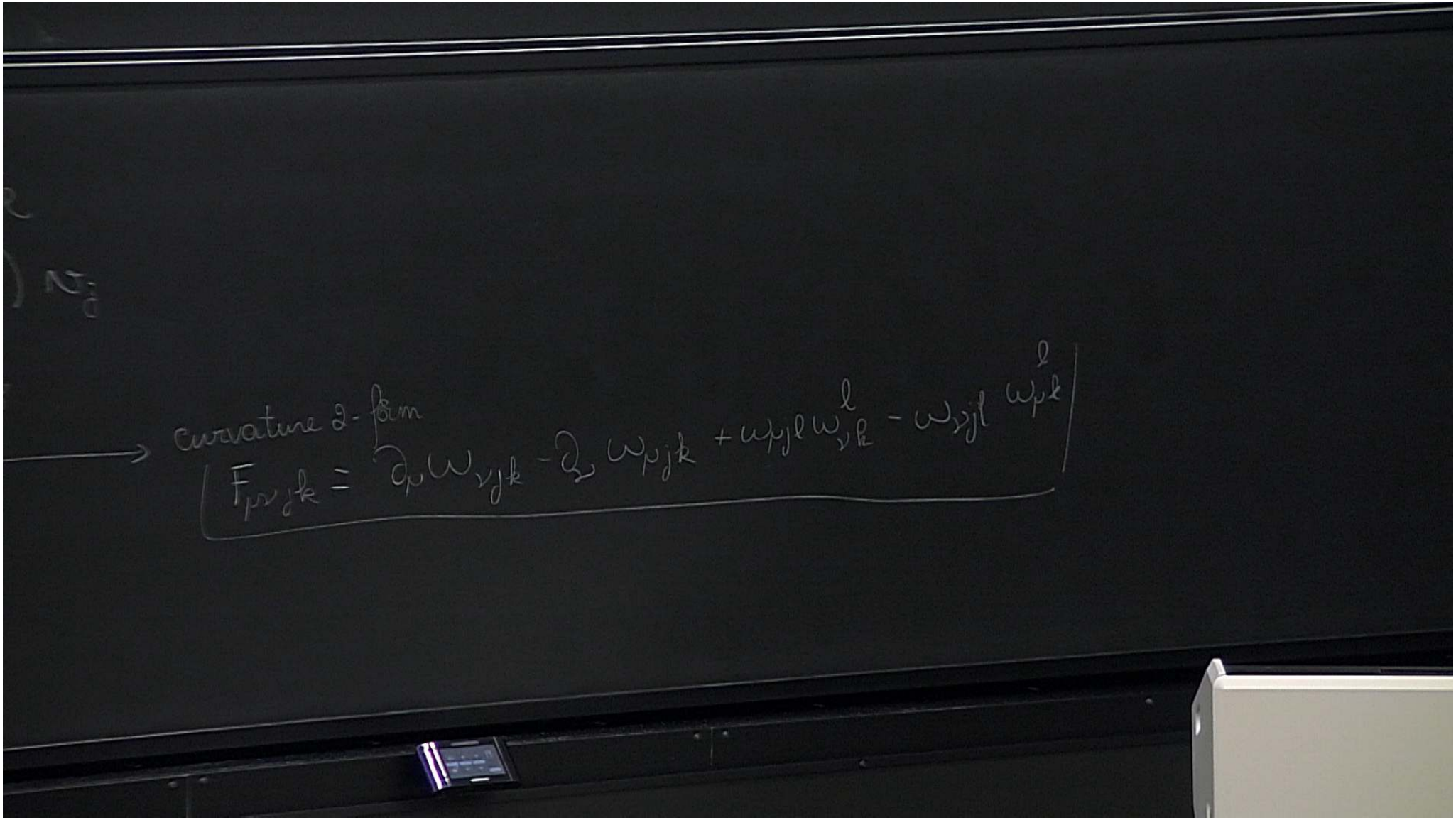
$$\Rightarrow \tilde{\Gamma}^\epsilon_{\mu\nu} = e^\epsilon_\gamma W_{\mu K}^{\ \ \gamma} e^K_\nu + e^\epsilon_\gamma \partial_\mu e^\gamma_\nu$$

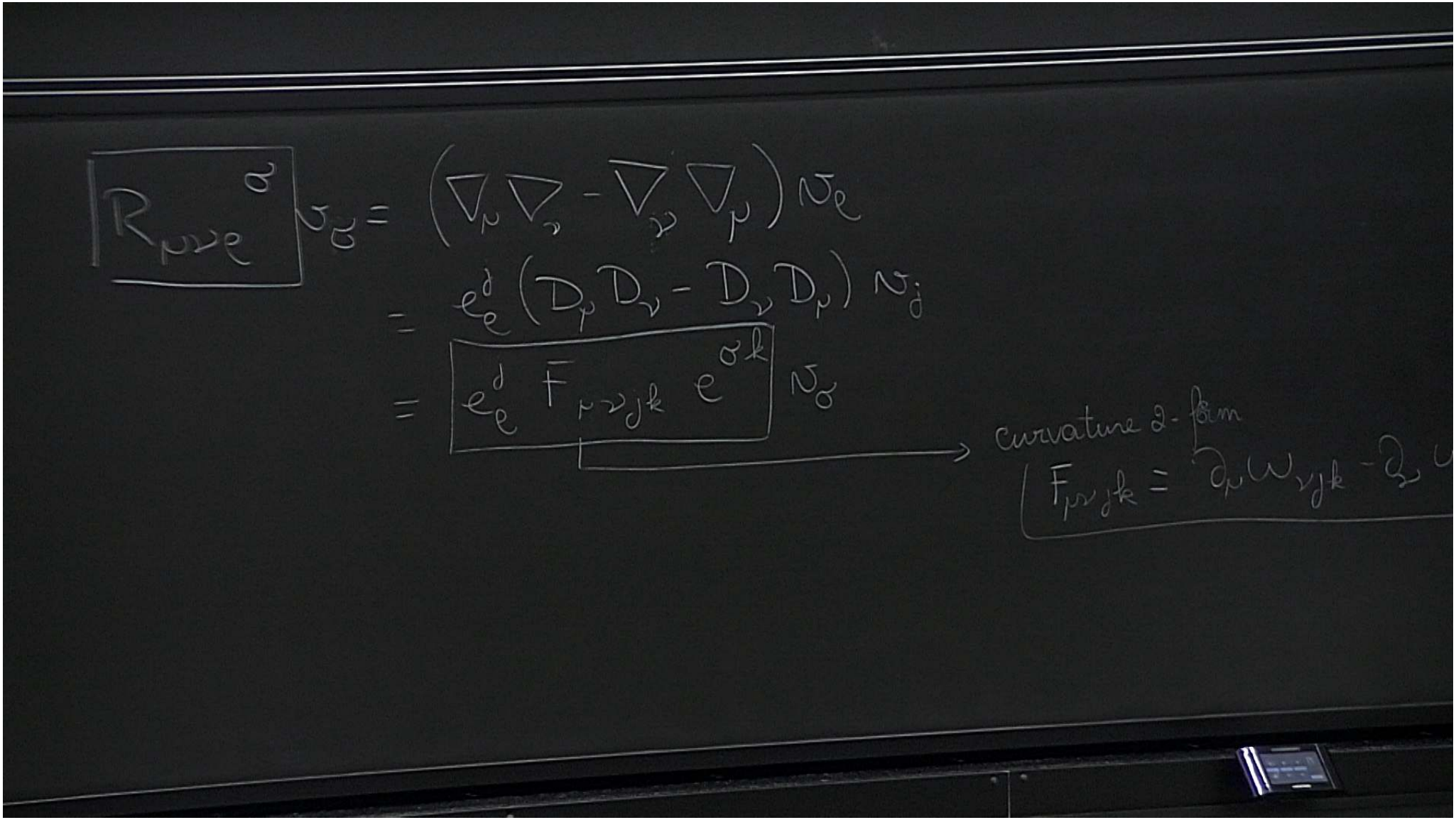
TODAY: 3D first order action for gravity

Canonical analysis

Symmetries / Constraints


$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) v_\rho$
$= e^a_\rho \left(D_\mu D_\nu - D_\nu D_\mu\right) v_a$
$= e^a_\rho \, \bar{F}_{\mu\nu a}{}^{k} \, e^{\sigma k} \, v_\sigma$
curvature 2-form
$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_\mu$


Curvature 2-form
$F_{\mu\nu j}{}^{k} = \partial_\mu \omega_{\nu j}{}^{k} - \partial_\nu \omega_{\mu j}{}^{k} + \omega_{\mu j}{}^{l}\omega_{\nu l}{}^{k} - \omega_{\nu j}{}^{l}\omega_{\mu l}{}^{k}$


$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_\rho$
$= e^d_\rho (D_\mu D_\nu - D_\nu D_\mu) v_d$
$= e^d_\rho \bar{F}_{\mu\nu d k} e^{\sigma k} v_\sigma$
curvature 2-form
$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk}$

$$R_{\mu\nu\rho}{}^{\sigma}\, v_\sigma = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) v_\rho$$

$$= e^{\sigma}_{\rho}\left(D_\mu D_\nu - D_\nu D_\mu\right) v_\sigma$$

$$= \boxed{e^{\sigma}_{\rho}\, \bar{F}_{\mu\nu jk}\, e^{jk}}\, v_\sigma$$

curvature 2-form

$$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu$$

3D : enforce the antisymmetry of $\omega_{\mu jk}$ and of $\bar{F}_{\mu\nu jk}$

$$\omega_{\mu jk} = \varepsilon_{jlk}\, \omega_\mu^{\ l}$$

$$\bar{F}_{\mu\nu jk} = \varepsilon_{jlk}\, F_{\mu\nu}$$

curvature 2-form

$$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu jl}\omega^{l}_{\nu k} - \omega_{\nu jl}\omega^{l}_{\mu k}$$

ε_{ijk} totally antisymmetric tensor

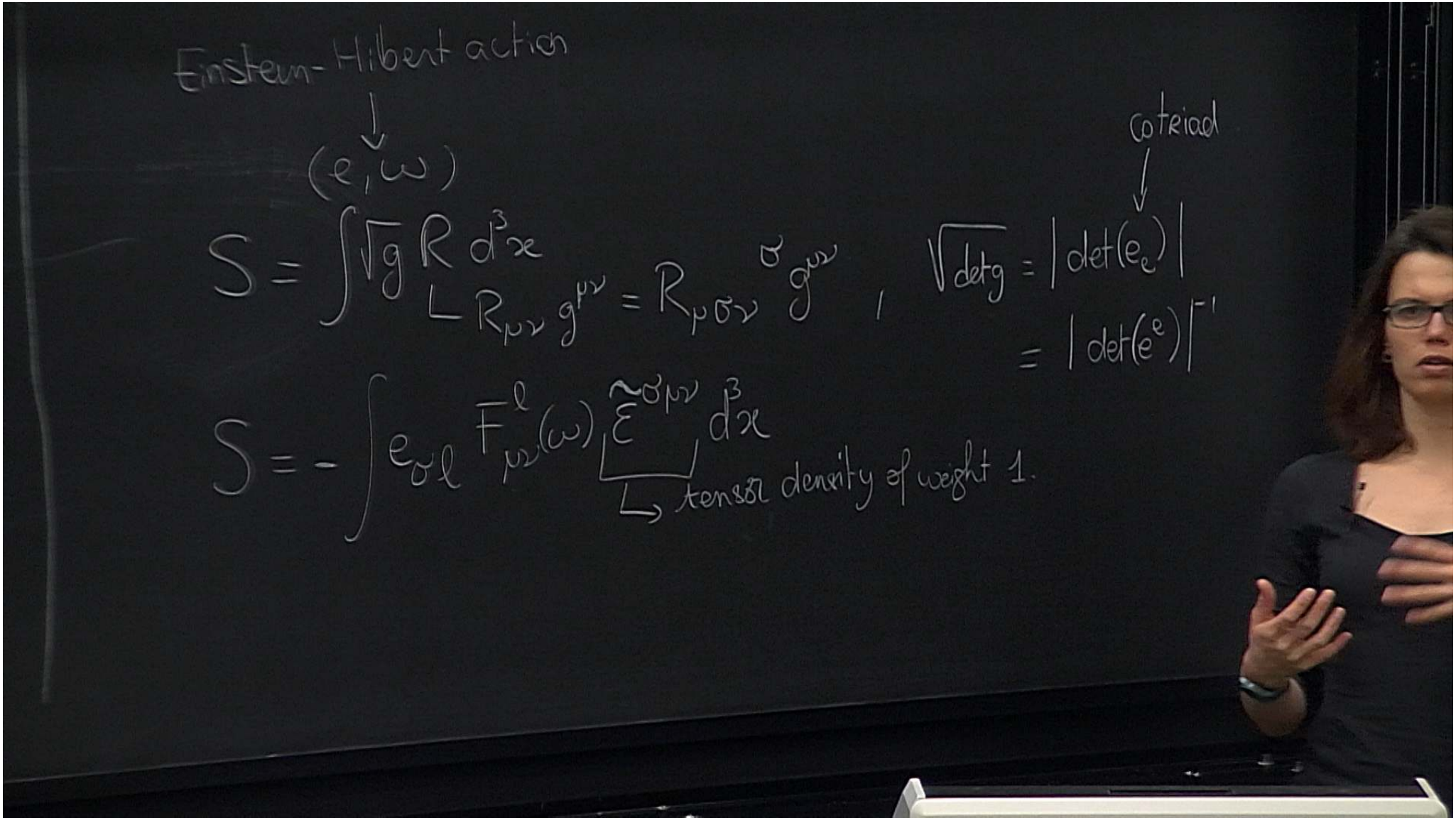
$\varepsilon_{123} = 1$

ε_{ijk} constant structure of $su(2)$, $so(3)$.

$$D_\mu \phi_j = \partial_\mu \phi_j + \varepsilon_{jkl}\, \omega_\mu^k\, \phi^l$$

curvature term $\longleftarrow$ $$F^l_{\mu\nu} = \partial_\mu \omega^l_\nu - \partial_\nu \omega^l_\mu + \varepsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}$$

torsion term $\longleftarrow$ $$T^l_{\mu\nu} = \partial_\mu e^l_\nu - \partial_\nu e^l_\mu + \varepsilon^{ljk}\, \omega_{\mu j}\, e_{\nu k} - \varepsilon^{ljk}\, \omega_{\nu j}\, e_{\mu k}$$


Einstein-Hilbert action
(e, ω)
$S = \int \sqrt{g} R \, d^3x$
$R_{\mu\nu} g^{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma} g^{\mu\nu}$
$\sqrt{\det g} = |\det(e^a_\mu)|$
$= |\det(e^a)|^{-1}$
cotriad
$S = -\int e_{\sigma l} F^{l}_{\mu\nu}(\omega) \tilde{\varepsilon}^{\sigma\mu\nu} d^3x$
tensor density of weight 1.

Einstein-Hilbert action

(e, ω)

$$S = \int \sqrt{g}\, R\, d^3x$$

$$R_{\mu\nu} g^{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma} g^{\mu\nu}$$

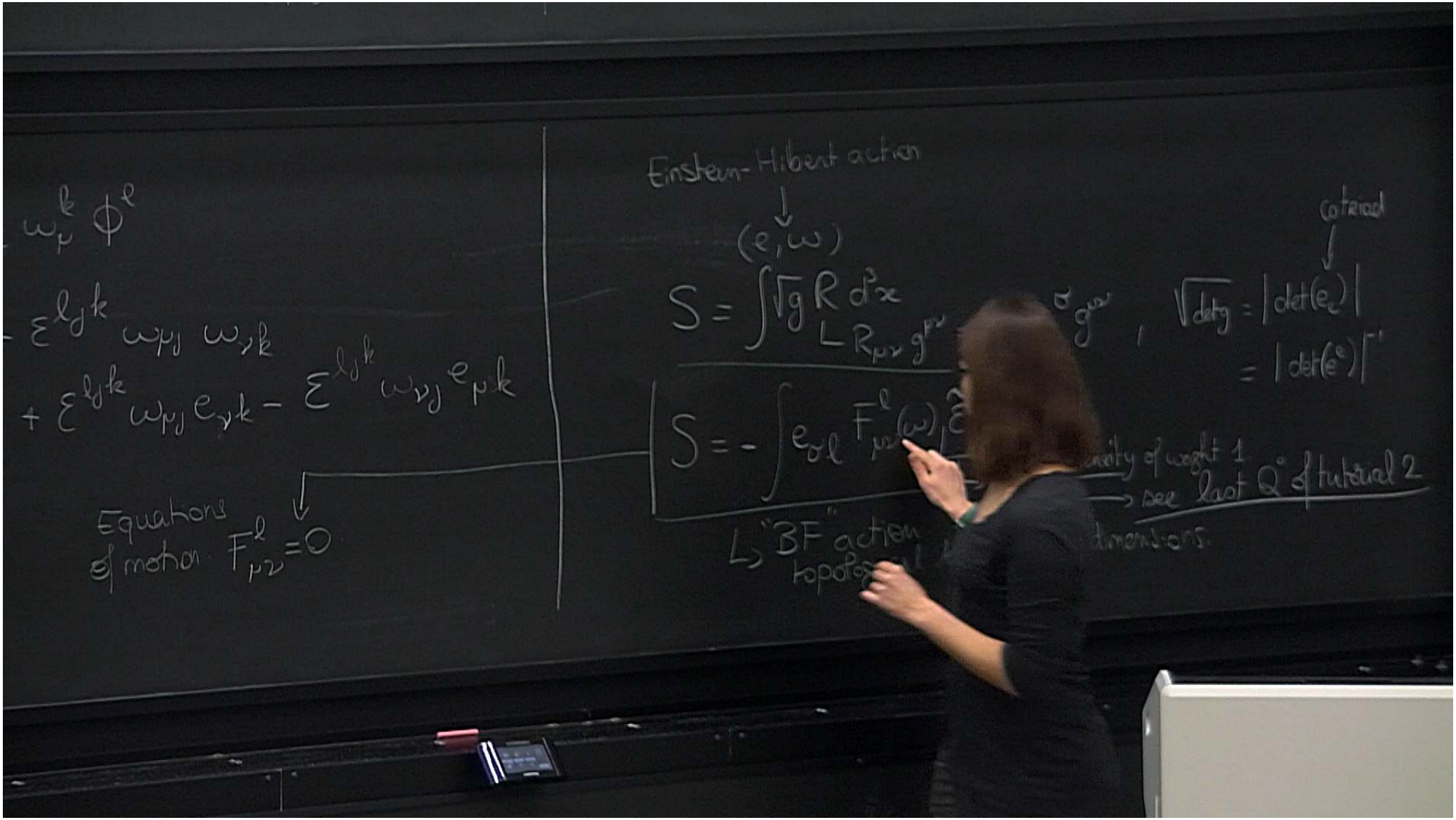
cotriad

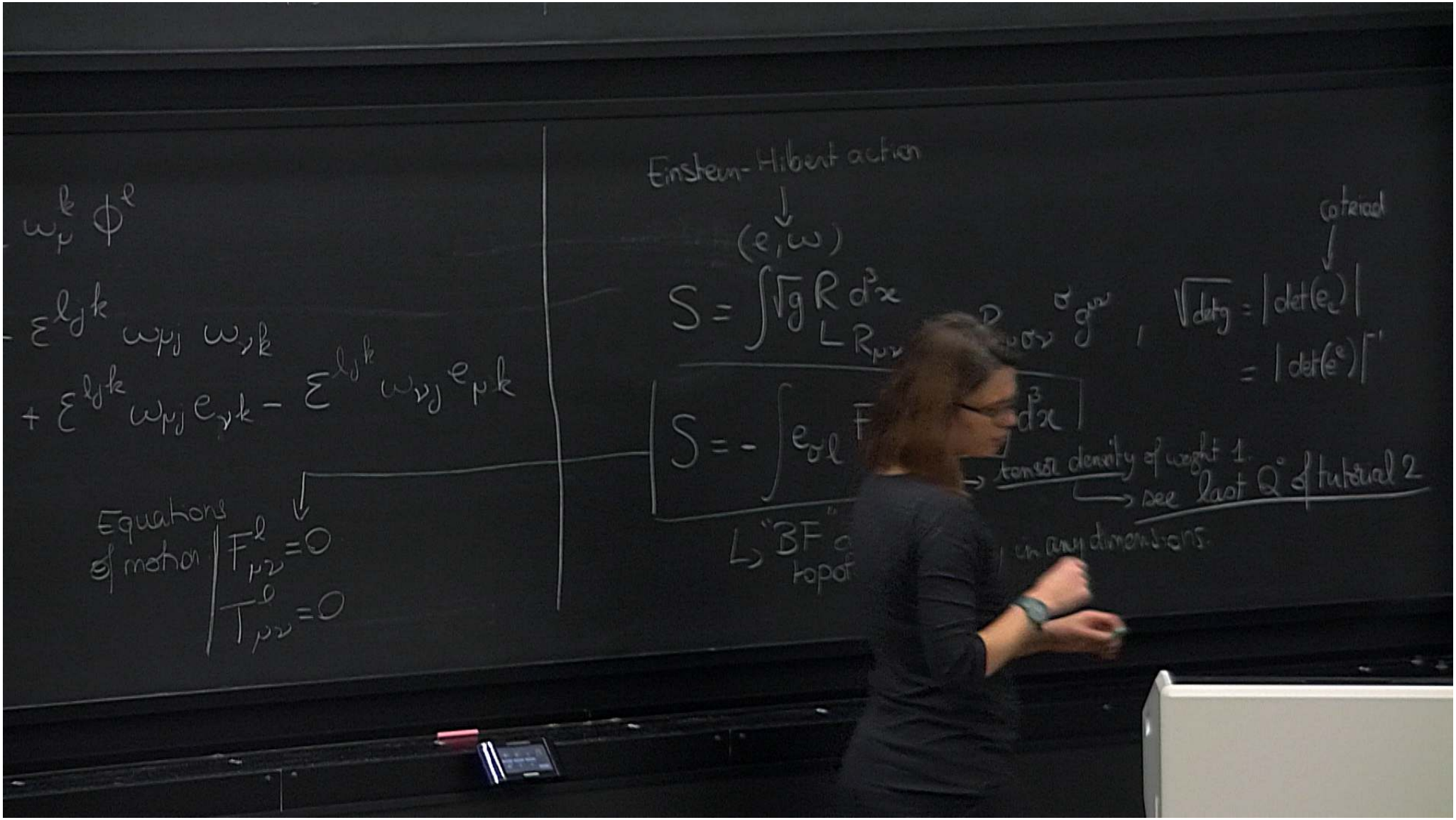
$$\sqrt{\det g} = |\det(e^a_\mu)| = |\det(e^a)|^{-1}$$

$$S = -\int e_{al}\, F^{l}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\mu\nu}\, d^3x$$

$\tilde{\varepsilon}^{\mu\nu}$: tensor density of weight 1. → see last Q° of tutorial 2

↳ "BF" action: topological theory in any dimensions.


Einstein-Hilbert action
(e, ω)
$S = \int \sqrt{g} R \, d^3x$
$L R_{\mu\nu} g^{\mu\nu}$
$S = -\int e_{al} F^{l}_{\mu\nu}(\omega)$
"BF" action
topolo
Equations of motion $F^{l}_{\mu\nu} = 0$
cotriad
$\sqrt{\det g} = |\det(e^a_\mu)|$
$= |\det(e^a_\mu)|^{-1}$
density of weight 1
→ see last Q° of tutorial 2
dimensions.


Einstein-Hilbert action
(e, ω)
cotriad
tensor density of weight 1
see last Q° of tutorial 2
Equations of motion
in any dimensions

Symmetries : * rotation in the internal index
* diffeomorphism
* translation $\longrightarrow N^j$

$$\delta_N^T e_N^j = D_N N^j$$

$$\delta_N^T \omega_N^j = 0$$

in the internal index

...dium

...tion $\longrightarrow N^i$

$$\delta_N^T e_\nu^i = D_\nu N^i$$

$$\delta_N^T \omega_\nu^i = 0$$

ex: check that action is invariant.

Symmetries

* rotation in the internal index
* diffeomorphism
* translation $\longrightarrow N^i$

$$\delta^T_N e^i_\mu = D_\mu N^i$$

$$\delta^T_N \omega^i_\mu = 0$$

ex: check that action is invariant

Canonical analysis

Hamiltonian analysis:

$$\mathcal{M} = \mathbb{R} \times \Sigma \; ; \; \mu \longrightarrow (0, \underbrace{a}_{\text{spatial coordinates}})$$

$$S = \frac{1}{2} \int e_{ab} \, \tilde{F}^{ab}_{\mu\nu} \, \tilde{E}^{\sigma\mu\nu} \, d^3x$$

we have an additional $SO(3)$ symmetry

$$\hookrightarrow e'^i_a(x) = R^i_j(x) \, e^j_a(x) \Longrightarrow$$

Symmetries : * rotation in the internal index
* diffeomorphism
* translation $\longrightarrow N^a$ $\quad \delta^T_N e^a_\mu = D_\mu N^a$ $\quad \delta^T_N \omega^a_\mu = 0$

ex: check that action is invariant.

Canonical analysis

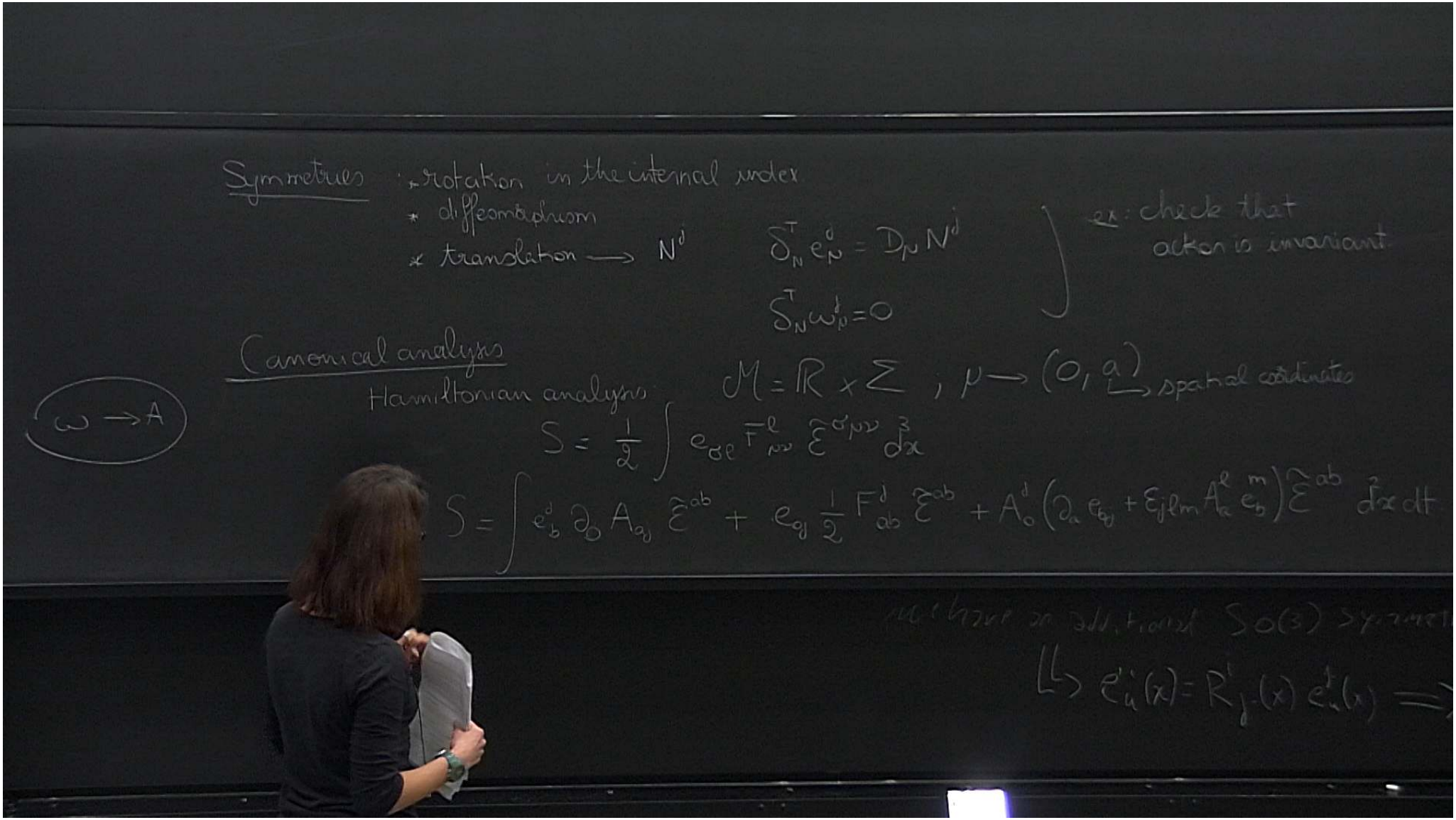
Hamiltonian analysis: $\mathcal{M} = \mathbb{R} \times \Sigma$; $\mu \longrightarrow (0, a)$ ↳ spatial coordinates

$$S = \frac{1}{2} \int e_{ae} \, \bar{F}^{el}_{\mu\nu} \, \tilde{\varepsilon}^{a\mu\nu} \, d^3x$$

$$S = \int e^a_b \, \partial$$

we have an additional $SO(3)$ symmetry

↳ $e'^i_\mu(x) = R^i_j(x) \, e^j_\mu(x) \Longrightarrow$


Symmetries : * rotation in the internal index
* diffeomorphism
* translation ⟶ N^j
$\delta^T_N e^a_\mu = D_\mu N^j$
$\delta^T_N \omega^j_\mu = 0$
ex: check that action is invariant
Canonical analysis
Hamiltonian analysis
$\mathcal{M} = \mathbb{R} \times \Sigma$, $\mu \longrightarrow (0, \underset{\hookrightarrow}{a})$ spatial coordinates
$S = \frac{1}{2} \int e_{ae} \tilde{F}^l_{\mu\nu} \tilde{\mathcal{E}}^{a\mu\nu} d^3x$
$\omega \to A$
$S = \int e^d_b \partial_0 A_{aj} \tilde{\mathcal{E}}^{ab} + e_{aj} \frac{1}{2} F^j_{ab} \tilde{\mathcal{E}}^{ab} + A^j_0 \left(\partial_a e_{bj} + \epsilon_{jlm} A^l_a e^m_b \right) \tilde{\mathcal{E}}^{ab} \, d^3x \, dt$
We have an additional SO(3) symmetry
$\hookrightarrow e'_\mu(x) = R^j_i(x) e^i_\mu(x) \Longrightarrow$

$\delta_N^T \omega_\mu^I = 0$

Canonical analysis

Hamiltonian analysis: $\mathcal{M} = \mathbb{R} \times \Sigma$; $\mu \rightarrow (0, a)$ → spatial coordinates

$$S = \frac{1}{2} \int e_{IJ} F^{IJ}_{\mu\nu} \tilde{\epsilon}^{\mu\nu} d^3x$$

$$S = \int e^I_b \partial_0 A_a^J \tilde{\epsilon}^{ab} + e_a^J \frac{1}{2} F^J_{ab} \tilde{\epsilon}^{ab} + A_0^J \left(\partial_a e_b^J + \epsilon_{jlm} A_a^l e_b^m\right) \tilde{\epsilon}^{ab} \, d^3x \, dt$$

Canonical pair: $\{A_a^I(x), E_k^b(y)\} = \delta_k^I \delta_a^b \delta(x,y)$

↳ $E^a_J = \frac{\delta S}{\delta(\partial_0 A_a^J)} = \tilde{\epsilon}^{ab} e^J_b$ densitized ad

* translation $\longrightarrow$ $N^{\cdots}$ $\quad \delta_N e^I_\mu = D_\mu N^{\cdots}$ $\quad$ action is invariant

$$\delta^T_N \omega^{IJ}_\mu = 0$$

cal analysis

Hamiltonian analysis: $\mathcal{M} = \mathbb{R} \times \Sigma$, $\mu \longrightarrow (0, a)$ $\hookrightarrow$ spatial coordinates

$$S = \frac{1}{2} \int e_{IJ} F^{IJ}_{\mu\nu} \tilde{\epsilon}^{\mu\nu\rho} d^3x$$

$$S = \int e^I_b \partial_0 A^{Ib}_a \tilde{\epsilon}^{ab} + e_{0I} \left[\frac{1}{2} F^I_{ab} \tilde{\epsilon}^{ab} \right] + A^I_0 \left[\left(\partial_a e_{bI} + \epsilon_{jlm} A^l_a e^m_b \right) \tilde{\epsilon}^{ab} \right] d^3x\, dt$$

Lagrange multipliers

Canonical pair: $\{ A^i_a(x), E^b_k(y) \} = \delta^i_k \delta^b_a \delta(x,y)$

$\hookrightarrow$ $E^a_j = \dfrac{\delta S}{\delta(\partial_0 A^j_a)} = \tilde{\epsilon}^{ab} e^j_b$ = densitized triad

Constraints: $\mathcal{F}^I = \frac{1}{2} F^I_{ab} \tilde{\epsilon}^{ab}$ $\quad \vec{\mathcal{F}}^I = \tilde{\epsilon}_{ab} \left(\partial_a A^I_b + \frac{1}{2} \epsilon^I{}_{kl} A^k_a A^l_b \right)$: Flatness constraint

$\mathcal{G}^I = \frac{1}{2} T^I_{ab} \tilde{\epsilon}^{ab}$ $\quad \mathcal{G}^I = \partial_a E^a_I + \epsilon_{Jme} A^l_a E^{am}$. Gauss constraint

Constraints: $\mathcal{G}^j = \frac{1}{2} T^j_{ab} \hat{\mathcal{E}}^{ab}$

$\mathcal{F}^j = \mathcal{E}_{ab} (\partial_a A_b + \frac{1}{2} C^{\cdots}_{kl} A_a$

$\mathcal{G}^j = \partial_a E^a_j + \mathcal{E}_{jme} A^l_a E^a$

Hamiltonian: $H = -\int d^3x \left(N^j \mathcal{F}_j + \Lambda^j \mathcal{G}_j \right)$

↳ totally constrained system.

$\mathcal{G}^j = \partial_a E^a_j + \varepsilon_{jme} A^l_a E^{am}$ · Gauss constraint

$\Lambda^j \mathcal{G}_j$)

system.

see · ROMANO's article