Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 5

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Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

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ACTION! $S = \frac{1}{k}\int R\sqrt{-g}\, d^{4}x$ (4 dim / 3 dim), $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

First order formalism: $g_{\mu\nu}(x) = \overbrace{e^{i}_{\mu}(x)\, e^{j}_{\nu}(x)}^{g}\, \delta_{ij}$, 6 DoF as for the

we have an add. t

↳ e^{i}_{μ}

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

counting DoF. 6 for both.

$\Rightarrow$ In 3D $\Rightarrow$ $R_{\mu\nu} = R_{\mu\nu\alpha\beta} = 0$

↳ Gravity is a Topological Theory in 3D (No spacetime dynamics for vacuum solutions).

6 DoF as for the fundamental object

We have an additional SO(3) symmetry

↳ $e'^{i}_{\mu}(x) = R^{i}_{j}(x)\, e^{j}_{\mu}(x) \Rightarrow$ Action Remains Invariant.

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

$\Rightarrow \Sigma_N$ 3D $\Rightarrow R_{\mu\nu} = R_{\mu\nu\alpha\beta} = 0$

counting DoF. 6 for both.

↳ Gravity is a Topological Theory in 3D (No spacetime dynamic for vacuum solutions).

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We have an additional $SO(3)$ symmetry

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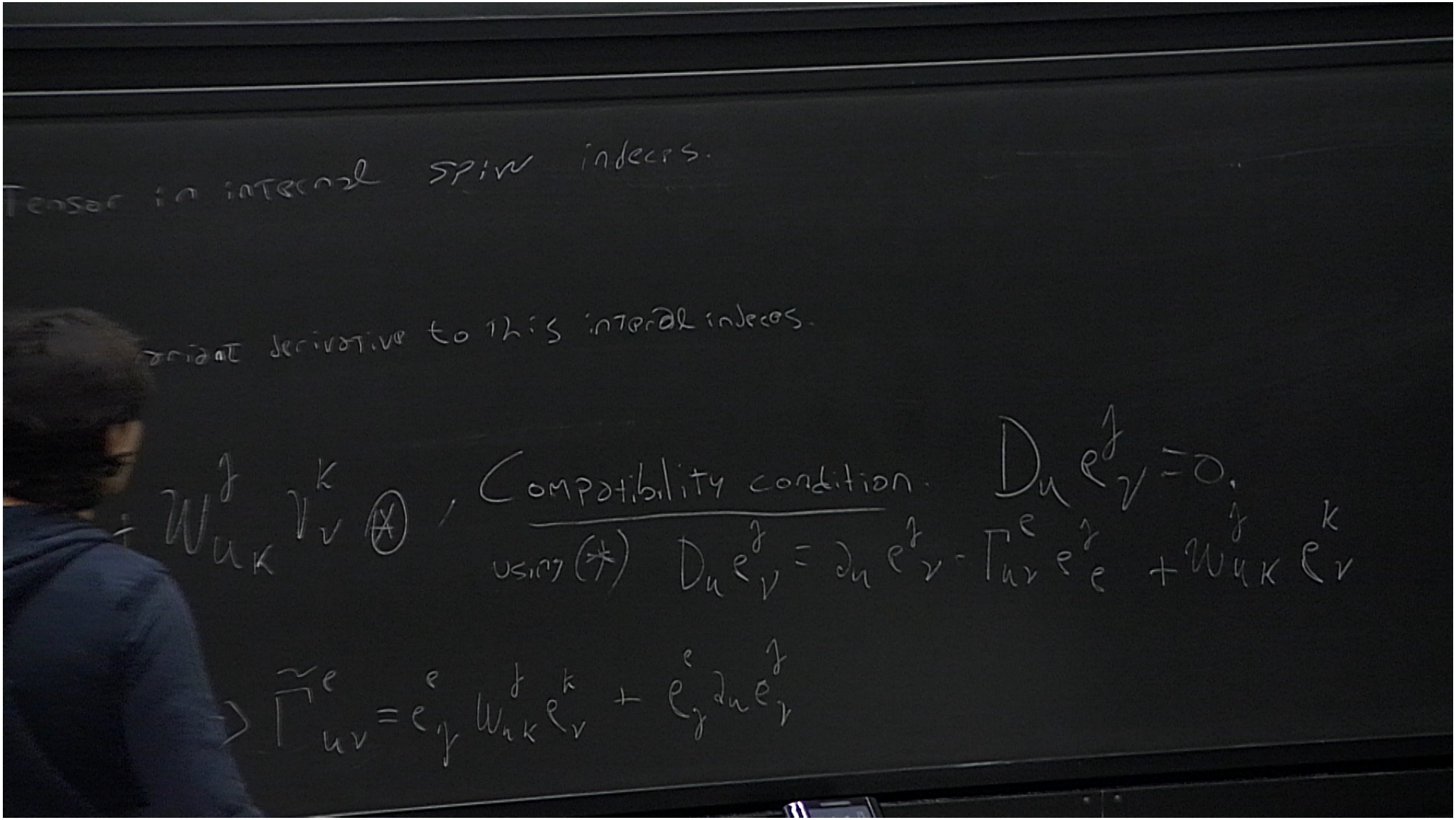
Spin connections: Tensor in internal spin indices.

$W^{i}_{\mu k}$ - "expand" the notion of covariant derivative to this internal indices.

$$D_\mu v^\gamma_\nu = \partial_\mu v^\gamma_\nu - \Gamma^{\rho}_{\mu\nu} v^\gamma_\rho + W^\gamma_{\mu k} v^k_\nu \quad (*)$$ Compatibility condit

using $(*)$ $D_\mu e^\gamma_\nu = \partial_\mu$

$$\Rightarrow \tilde{\Gamma}^{\rho}_{\mu\nu} = e^\rho_\gamma W^\gamma_{\mu k} e^k_\nu + e^\rho_\gamma \partial_\mu e^\gamma_\nu$$


Tensor in internal spin indices.
variant derivative to this internal indices.
$= W^{j}_{\mu k} V^{k}_{\nu}$ (*) / Compatibility condition. $D_\mu e^{j}_{\nu} = 0$.
using (*) $D_\mu e^{j}_{\nu} = \partial_\mu e^{j}_{\nu} - \Gamma^{\rho}_{\mu\nu} e^{j}_{\rho} + W^{j}_{\mu k} e^{k}_{\nu}$
$\Rightarrow \tilde{\Gamma}^{\rho}_{\mu\nu} = e^{\rho}_{j} W^{j}_{\mu k} e^{k}_{\nu} + e^{\rho}_{j} \partial_\mu e^{j}_{\nu}$

Tensor in internal spin indices.

...ion of covariant derivative to this internal indices.

$$\partial_\mu V_c^\gamma + W_{\mu K}^{\gamma}\, V_\nu^K \quad (*)$$

Compatibility condition. $D_\mu e^\gamma_\nu = 0.$

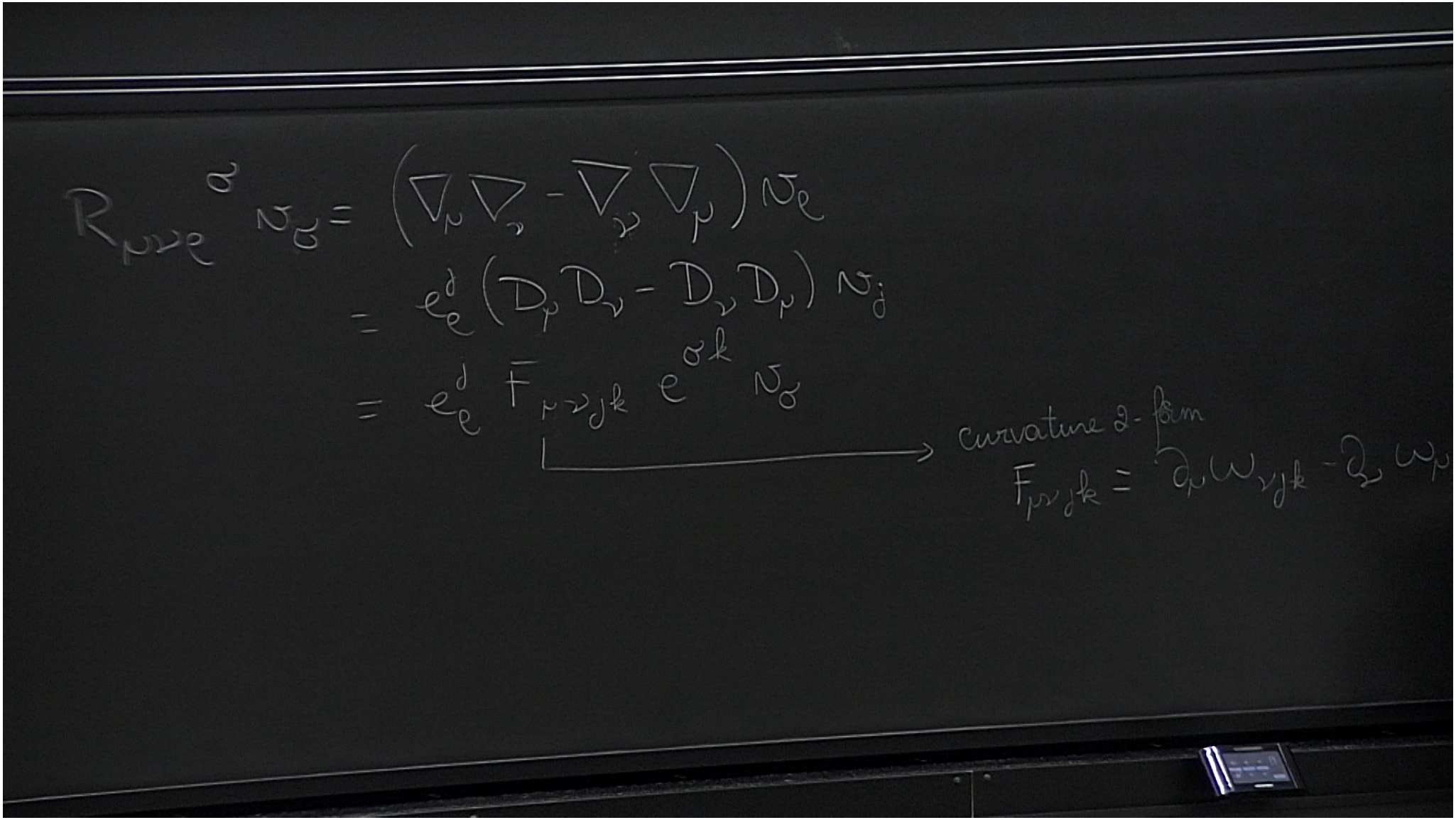
using $(*)$ $D_\mu e^\gamma_\nu = \partial_\mu e^\gamma_\nu - \Gamma^{\rho}_{\mu\nu} e^\gamma_\rho + W^{\gamma}_{\mu K} e^K_\nu$

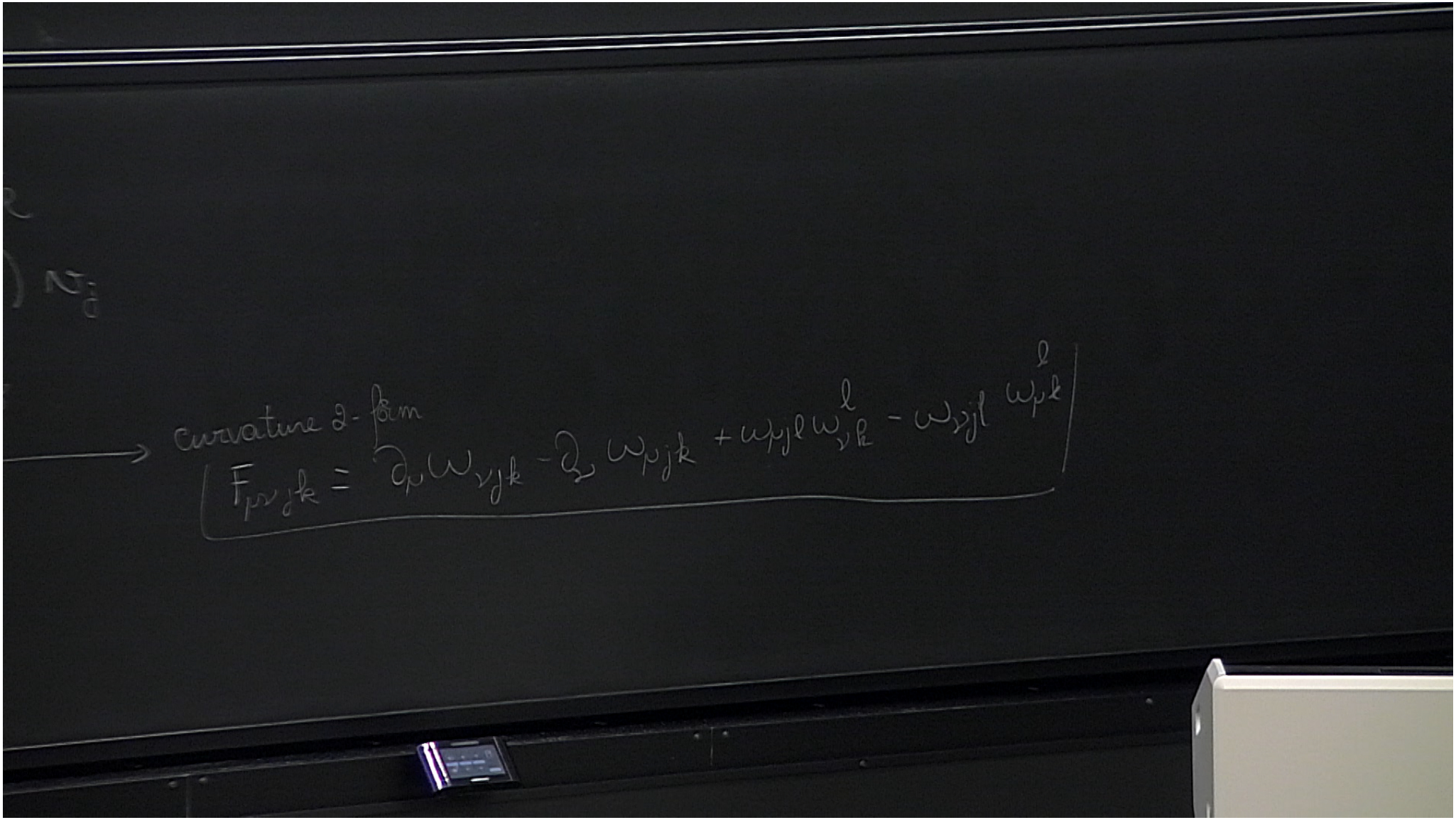
$$\Rightarrow \tilde{\Gamma}^{\rho}_{\mu\nu} = e^\rho_\gamma W^{\gamma}_{\mu K} e^K_\nu + e^\rho_\gamma \partial_\mu e^\gamma_\nu$$

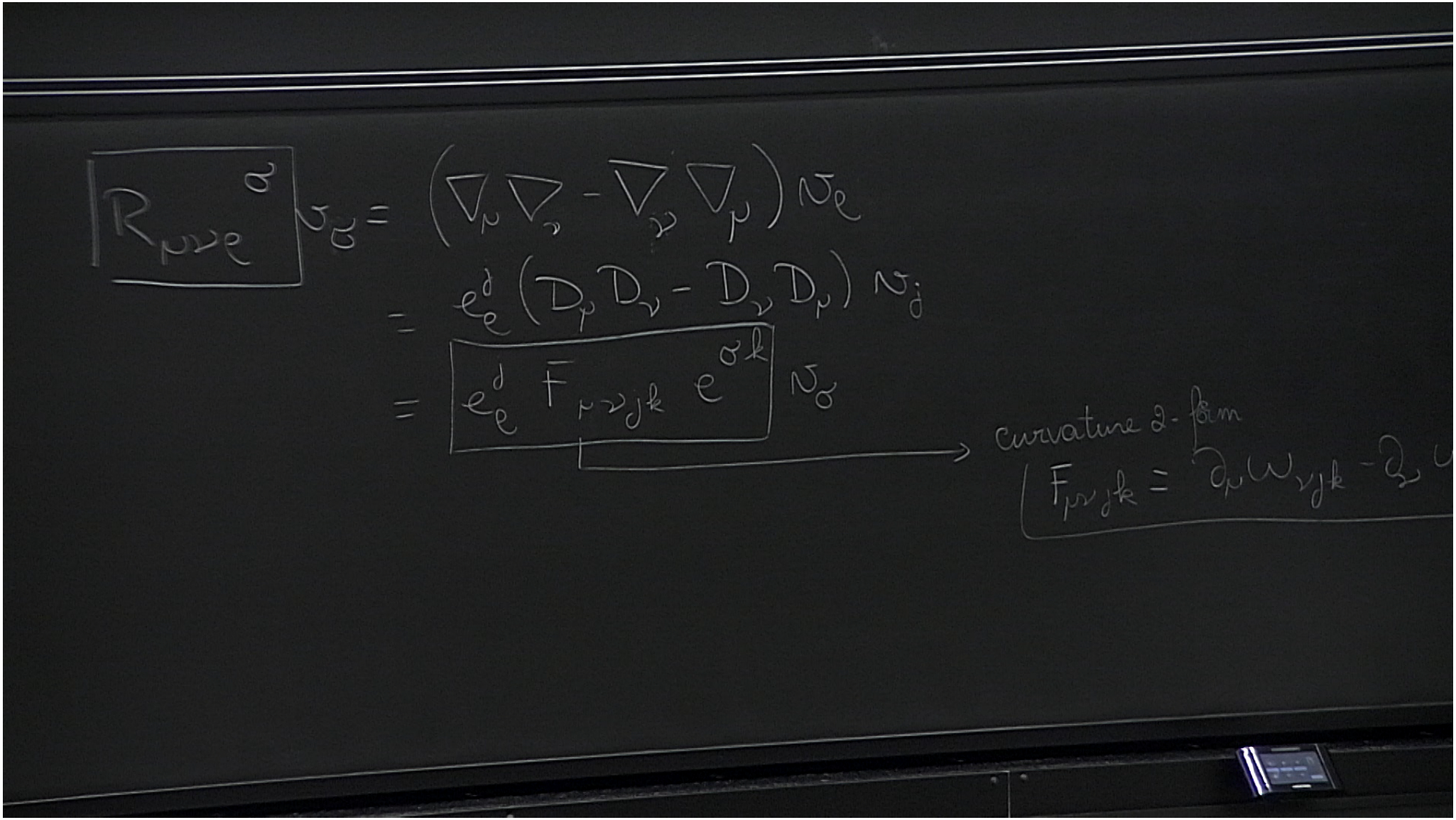
TODAY: 3D first order action for gravity

Canonical analysis

Symmetries / Constraints


$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) v_\rho$
$= e^d_\rho \left(D_\mu D_\nu - D_\nu D_\mu\right) v_d$
$= e^d_\rho \, \bar{F}_{\mu\nu d k} \, e^{\sigma k} \, v_\sigma$
curvature 2-form
$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_\mu$


curvature 2-form
$F_{\mu\nu}{}^{j}{}_{k} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j l}\omega^{l}_{\nu k} - \omega_{\nu j l}\omega^{l}_{\mu k}$


$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_\rho$
$= e^d_\rho (D_\mu D_\nu - D_\nu D_\mu) v_d$
$= e^d_\rho \bar{F}_{\mu\nu d k} e^{\sigma k} v_\sigma$
curvature 2-form
$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu$

$$R_{\mu\nu\rho}{}^{\sigma}\, v_\sigma = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) v_\rho$$

$$= e^{\sigma}_{\rho}\left(D_\mu D_\nu - D_\nu D_\mu\right) v_\sigma$$

$$= \boxed{e^{\sigma}_{\rho}\, \bar{F}_{\mu\nu jk}\, e^{jk}}\, v_\sigma$$

curvature 2-form

$$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu$$

3D: enforce the antisymmetry of $\omega_{\mu jk}$ and of $\bar{F}_{\mu\nu jk}$

$$\omega_{\mu jk} = \varepsilon_{jlk}\, \omega_\mu^{\ l}$$

$$\bar{F}_{\mu\nu jk} = \varepsilon_{jlk}\, F_{\mu\nu}$$

curvature 2-form

$$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu jl}\omega^{l}_{\nu k} - \omega_{\nu jl}\omega^{l}_{\mu k}$$

ε_{ijk} totally antisymmetric tensor

$\varepsilon_{123} = 1$

ε_{ijk} constant structure of $su(2)$, $so(3)$.

$$T_{\mu\nu jk} = e_{jkl} T_{\mu\nu}$$

$$D_\mu \phi_j = \partial_\mu \phi_j + \varepsilon_{jkl} \omega_\mu^k \phi^l$$

curvature term $\longleftarrow$ $$F^l_{\mu\nu} = \partial_\mu \omega^l_\nu - \partial_\nu \omega^l_\mu + \varepsilon^{ljk} \omega_{\mu j} \omega_{\nu k}$$

torsion term $\longleftarrow$ $$T^l_{\mu\nu} = \partial_\mu e^l_\nu - \partial_\nu e^l_\mu + \varepsilon^{ljk} \omega_{\mu j} e_{\nu k} - \varepsilon^{ljk} \omega_{\nu j} e_{\mu k}$$

Einstein-Hilbert action

(e, ω)

$$S = \int \sqrt{g}\, R\, d^3x$$

$R_{\mu\nu}\, g^{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma}\, g^{\mu\nu}$,

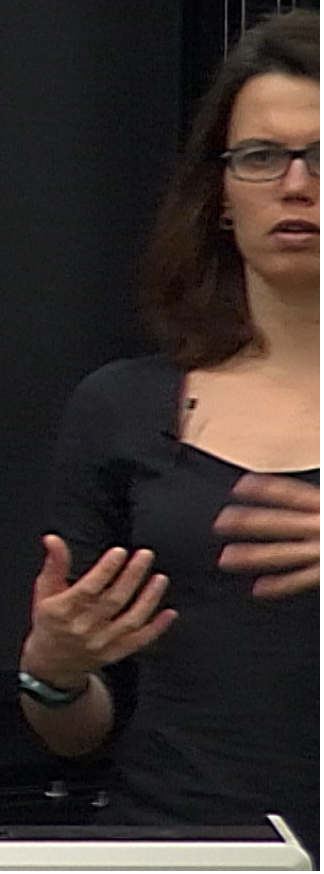
cotriad

$$\sqrt{\det g} = |\det(e^a_i)|$$

$$= |\det(e^i_a)|^{-1}$$

$$S = -\int e_{\sigma l}\, F^{l}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x$$

$\tilde{\varepsilon}^{\sigma\mu\nu}$: tensor density of weight 1.



Einstein-Hilbert action

(e, ω)

$$S = \int \sqrt{g} R \, d^3x$$

$$R_{\mu\nu} g^{\mu\nu} = R_{\mu a \nu}{}^{a} g^{\mu\nu}$$

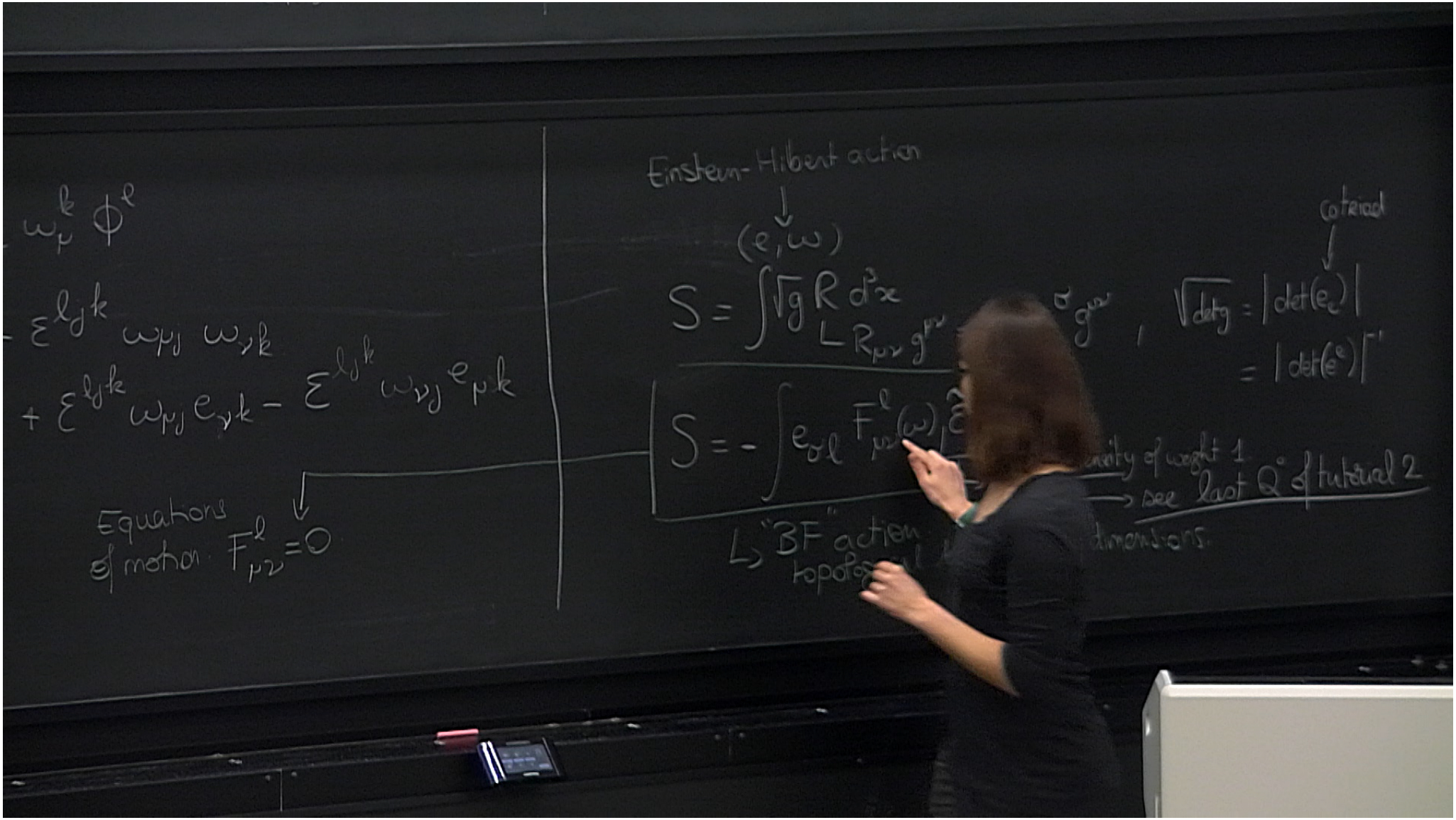
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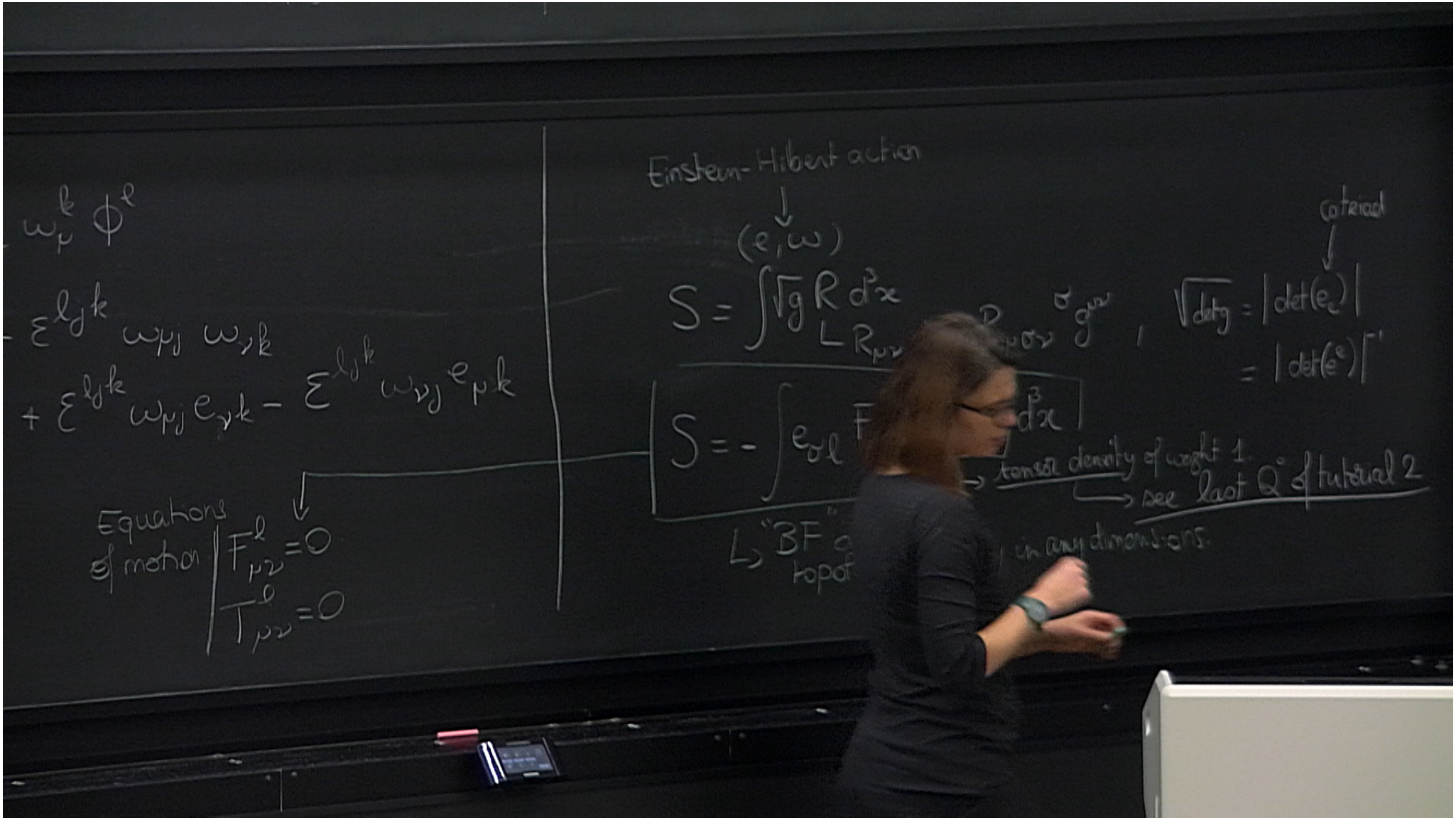
$$\sqrt{\det g} = |\det(e^a_\mu)| = |\det(e^a)|^{-1}$$

$$S = -\int e_{al} F^{l}_{\mu\nu}(\omega) \tilde{\varepsilon}^{a\mu\nu} d^3x$$

$\tilde{\varepsilon}^{a\mu\nu}$: tensor density of weight 1. → see last Q° of tutorial 2

↳ "BF" action: topological theory in any dimensions.


Einstein-Hilbert action
(e, ω)
S = ∫ √g R d³x
L R_μν g^μν
Equations of motion F^k_μν = 0
"BF" action
topolog
cotriad
√detg = |det(e^a_μ)|
= |det(e^a)|^-1
→ see last Q° of tutorial 2
dimensions.


Einstein-Hilbert action
(e, ω)
cotriad
Equations of motion
tensor density of weight 1
see last Q° of tutorial 2
in any dimensions

Symmetries : * rotation in the internal index

* diffeomorphism

* translation $\longrightarrow N^j$ $\qquad \delta^T_N e^j_\nu = D_\nu N^j$

$\delta^T_N \omega^j_\nu = 0$

in the internal index

$\longrightarrow N^j$

$$\delta^T_N e^j_\nu = D_\nu N^j$$

$$\delta^T_N \omega^j_\nu = 0$$

ex: check that action is invariant.

Symmetries

* rotation in the internal index
* diffeomorphism
* translation $\longrightarrow N^d$

$$\delta^T_N e^d_\nu = D_\nu N^d$$

$$\delta^T_N \omega^d_\nu = 0$$

ex: check that action is invariant

Canonical analysis

Hamiltonian analysis:

$$\mathcal{M} = \mathbb{R} \times \Sigma \; ; \; \nu \longrightarrow (0, a) \hookrightarrow \text{spatial coordinates}$$

$$S = \frac{1}{2} \int e_{abc} \, \bar{F}^{ab}_{\mu\nu} \, \tilde{E}^{c\mu\nu} \, d^3x$$

we have an additional $SO(3)$ symmetry

$$\hookrightarrow e^i_a(x) = R^i_j(x) \, e^j_a(x) \Longrightarrow$$

Symmetries

- rotation in the internal index
- * diffeomorphism
- * translation $\longrightarrow N^d$

$$\delta_N^T e^a_\nu = D_\nu N^a$$

$$\delta_N^T \omega^a_\nu = 0$$

ex: check that action is invariant

Canonical analysis

Hamiltonian analysis:

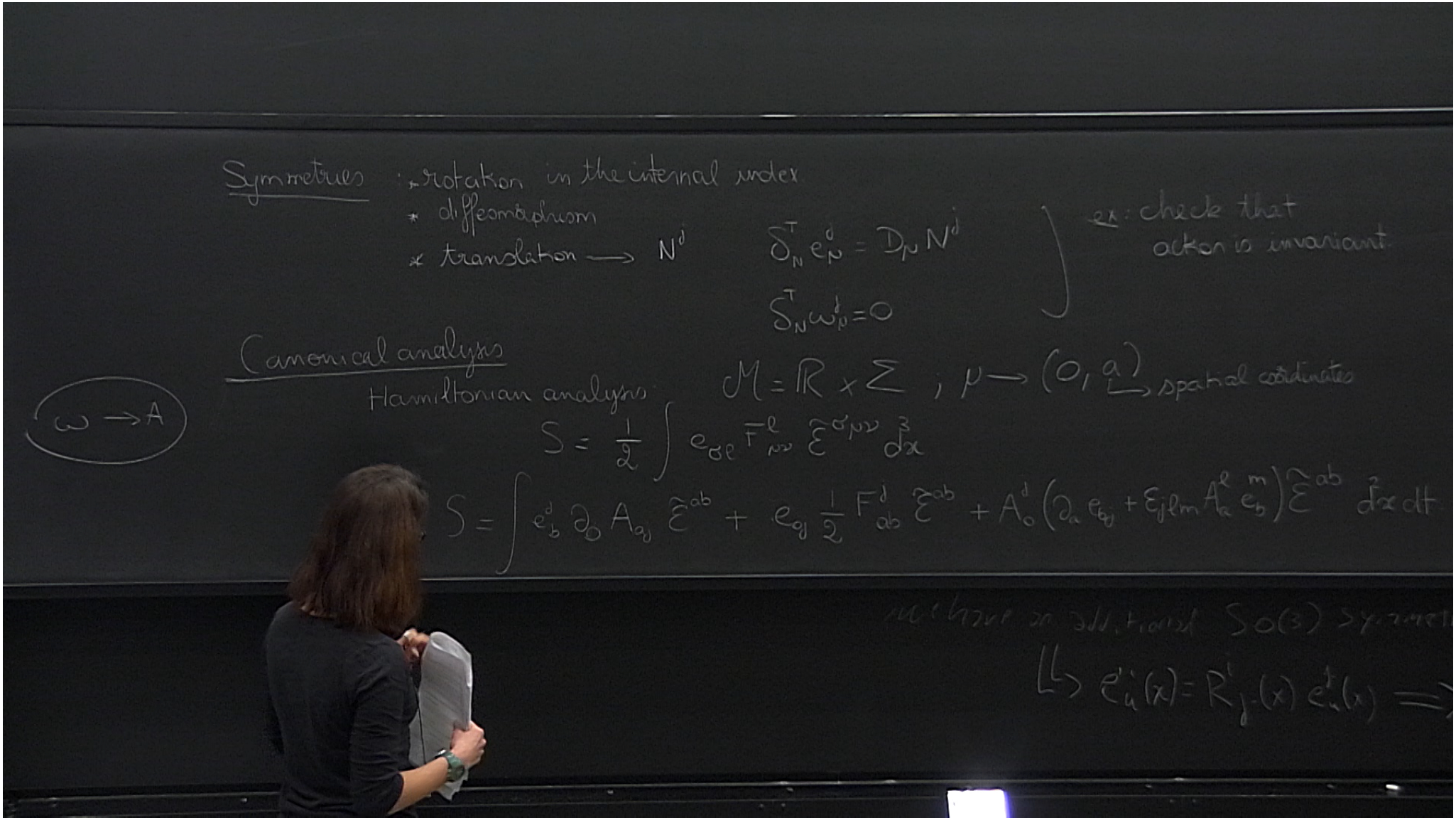
$$\mathcal{M} = \mathbb{R} \times \Sigma \; ; \; \nu \longrightarrow (0, a) \hookrightarrow \text{spatial coordinates}$$

$$S = \frac{1}{2} \int e_{ae} \, \bar{F}^{el}_{\mu\nu} \, \tilde{\mathcal{E}}^{a\mu\nu} \, d^3x$$

$$S = \int e^a_b \, \partial$$

M: have an additional $SO(3)$ symmetry

$$\hookrightarrow e'^i_a(x) = R^i_j(x) \, e^j_a(x) \Longrightarrow$$


Symmetries : * rotation in the internal index
* diffeomorphism
* translation ⟶ N^j
$\delta_N^T e_\mu^a = D_\mu N^j$
$\delta_N^T \omega_\mu^j = 0$
ex : check that action is invariant
Canonical analysis
Hamiltonian analysis
$\mathcal{M} = \mathbb{R} \times \Sigma$; $\mu \longrightarrow (0, a)$ ⟶ spatial coordinates
$S = \frac{1}{2} \int e_{ae} \tilde{F}^{l}_{\mu\nu} \tilde{\mathcal{E}}^{a\mu\nu} d^3x$
$\omega \longrightarrow A$
$S = \int \dot{e}^d_b \partial_a A_{aj} \tilde{\mathcal{E}}^{ab} + e_{aj} \frac{1}{2} F^j_{ab} \tilde{\mathcal{E}}^{ab} + A^d_a \left(\partial_a e_{bd} + \varepsilon_{jlm} A^l_a e^m_b \right) \tilde{\mathcal{E}}^{ab} \, d^3x \, dt$
we have an additional SO(3) symmetry
$\hookrightarrow e^i_\mu(x) = R^i_j(x) e^j_\mu(x) \Longrightarrow$

$\delta_N^T \omega_\mu^I = 0$

Canonical analysis

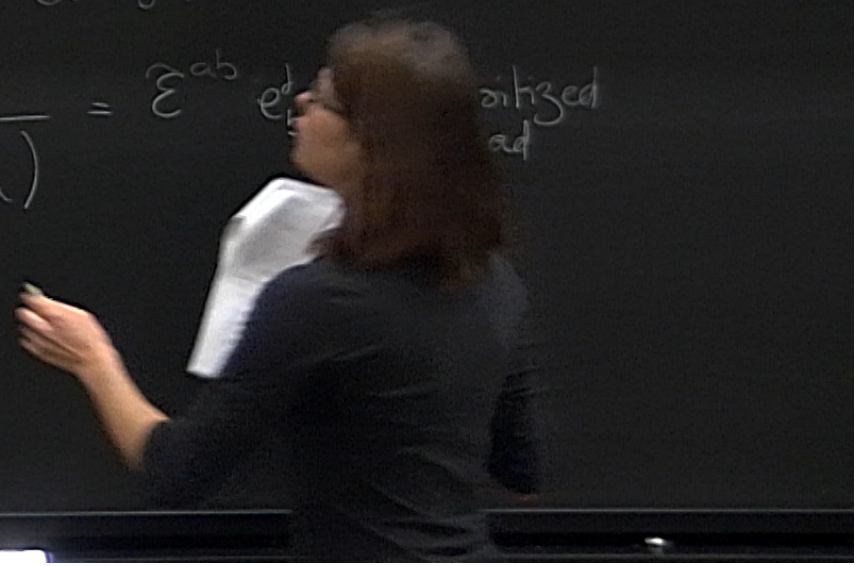
Hamiltonian analysis: $\mathcal{M} = \mathbb{R} \times \Sigma$; $\mu \to (0, a)$ → spatial coordinates

$$S = \frac{1}{2} \int e_{IJ} F^{IJ}_{\mu\nu} \tilde{\varepsilon}^{\mu\nu} d^3x$$

$$S = \int e^I_b \partial_0 A_a^J \tilde{\varepsilon}^{ab} + e_a^I \frac{1}{2} F^J_{ab} \tilde{\varepsilon}^{ab} + A_0^I \left(\partial_a e_b^J + \varepsilon_{jlm} A_a^l e_b^m \right) \tilde{\varepsilon}^{ab} \, d^3x \, dt$$

Canonical pair: $\{ A_a^I(x), E_k^b(y) \} = \delta_k^I \delta_a^b \delta(x,y)$

$$\hookrightarrow E^a_J = \frac{\delta S}{\delta(\partial_0 A_a^J)} = \tilde{\varepsilon}^{ab} e^J_b$$



* translation $\longrightarrow$ N^{μ} $\quad \delta_N e^I_\mu = D_\mu N^I$ $\quad$ action is invariant.

$$\delta^T_N \omega^{IJ}_\mu = 0$$

cal analysis

Hamiltonian analysis: $\mathcal{M} = \mathbb{R} \times \Sigma$, $\mu \longrightarrow (0, a)$ $\longrightarrow$ spatial coordinates

$$S = \frac{1}{2} \int e_{IJ} e^I_\mu F^{IJ}_{\mu\nu} \tilde{\mathcal{E}}^{\mu\nu} d^3x$$

$$S = \int e^I_b \partial_0 A^I_a \tilde{\mathcal{E}}^{ab} + e_0^I \boxed{\frac{1}{2} F^I_{ab} \tilde{\mathcal{E}}^{ab}} + \underline{A^I_0} \boxed{\left(\partial_a e^I_b + \varepsilon_{IJK} A^J_a e^K_b\right) \tilde{\mathcal{E}}^{ab}} \, d^3x \, dt$$

Lagrange multipliers

Canonical pair: $\{A^I_a(x), E^b_K(y)\} = \delta^I_K \delta^b_a \delta(x,y)$

$\hookrightarrow E^a_I = \dfrac{\delta S}{\delta(\partial_0 A^I_a)} = \tilde{\mathcal{E}}^{ab} e^I_b$ = densitized triad

Constraints: $\mathcal{F}^I = \frac{1}{2} F^I_{ab} \tilde{\mathcal{E}}^{ab}$ $\quad \mathcal{F}^I = \tilde{\mathcal{E}}_{ab} \left(\partial_a A^I_b + \frac{1}{2} \varepsilon^I{}_{KL} A^K_a A^L_b\right)$: Flatness constraint

$\mathcal{G}^I = \frac{1}{2} T^I_{ab} \tilde{\mathcal{E}}^{ab}$ $\quad \mathcal{G}^I = \partial_a E^a_I + \varepsilon_{IJK} A^J_a E^{aK}$: Gauss constraint

Constraints: $\mathcal{G}^j = \frac{1}{2} T^j_{ab} \hat{\mathcal{E}}^{ab}$

$\mathcal{F}^j = \mathcal{E}_{ab} (\partial_a A^j_b + \frac{1}{2} C^j{}_{kl} A^k_a A^l_b)$

$\mathcal{G}^j = \partial_a E^a_j + \mathcal{E}_{jmk} A^k_a E^a_m$

Hamiltonian: $H = -\int d^3x \left(N^j \mathcal{F}_j + \Lambda^j \mathcal{G}_j \right)$

↳ totally constrained system.

$\mathcal{G}^j = \partial_a E^a_j + \varepsilon_{jme} A^l_a E^{am}$. Gauss constraint

$\Lambda^j \mathcal{G}_j$)

system.

see : ROMANO's article

