Title: PSI 2018/2019 - Explorations in Quantum Gravity - Lecture 3

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Collection: PSI 2018/2019 - Explorations in Quantum Gravity (Dupuis)

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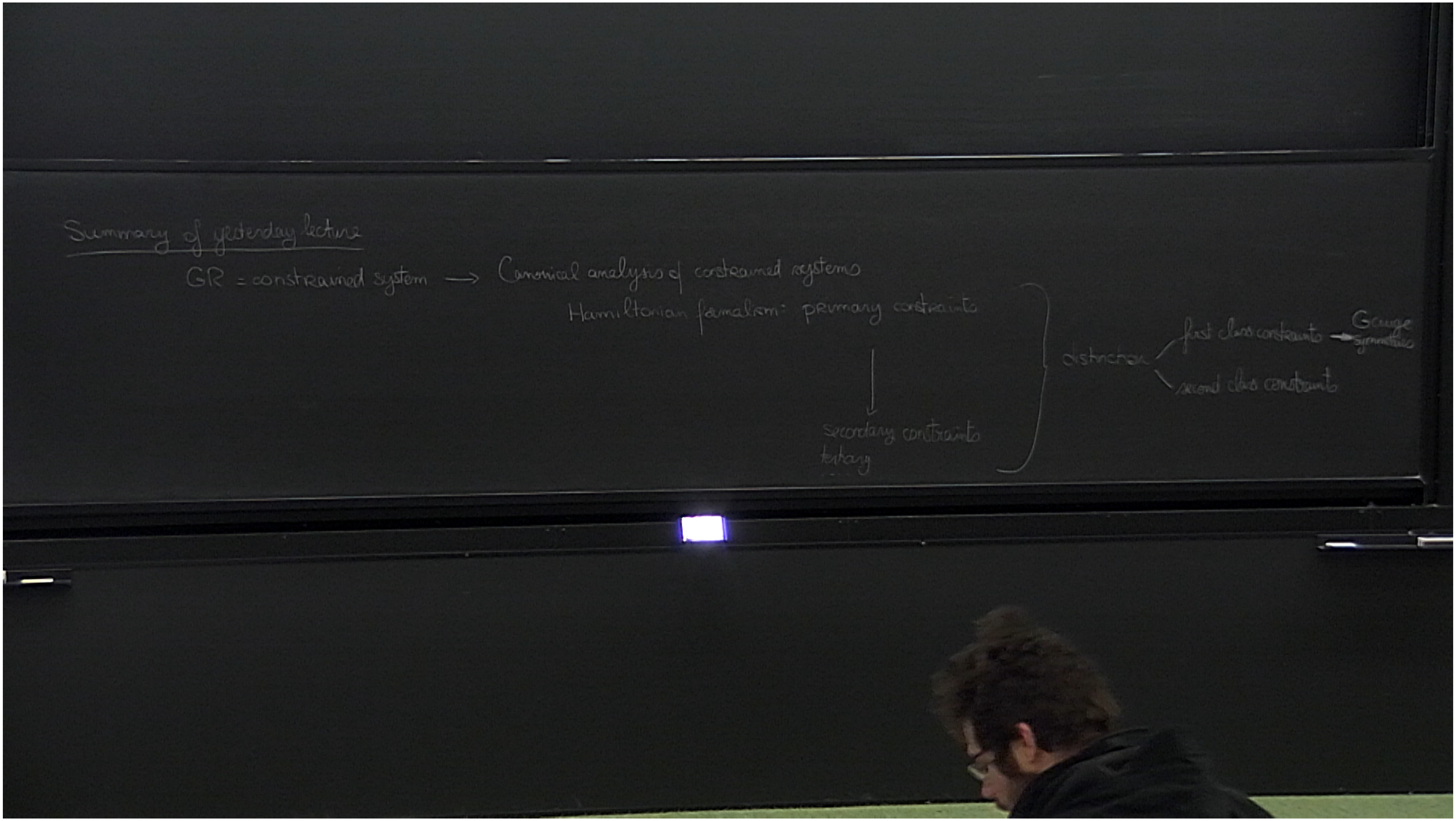

Summary of yesterday lecture
GR = constrained system → Canonical analysis of constrained systems
Hamiltonian formalism: primary constraints
secondary constraints
tertiary
distinction
first class constraints → Gauge symmetries
second class constraints

Summary of yesterday lecture

GR = constrained system $\longrightarrow$ Canonical analysis of constrained systems

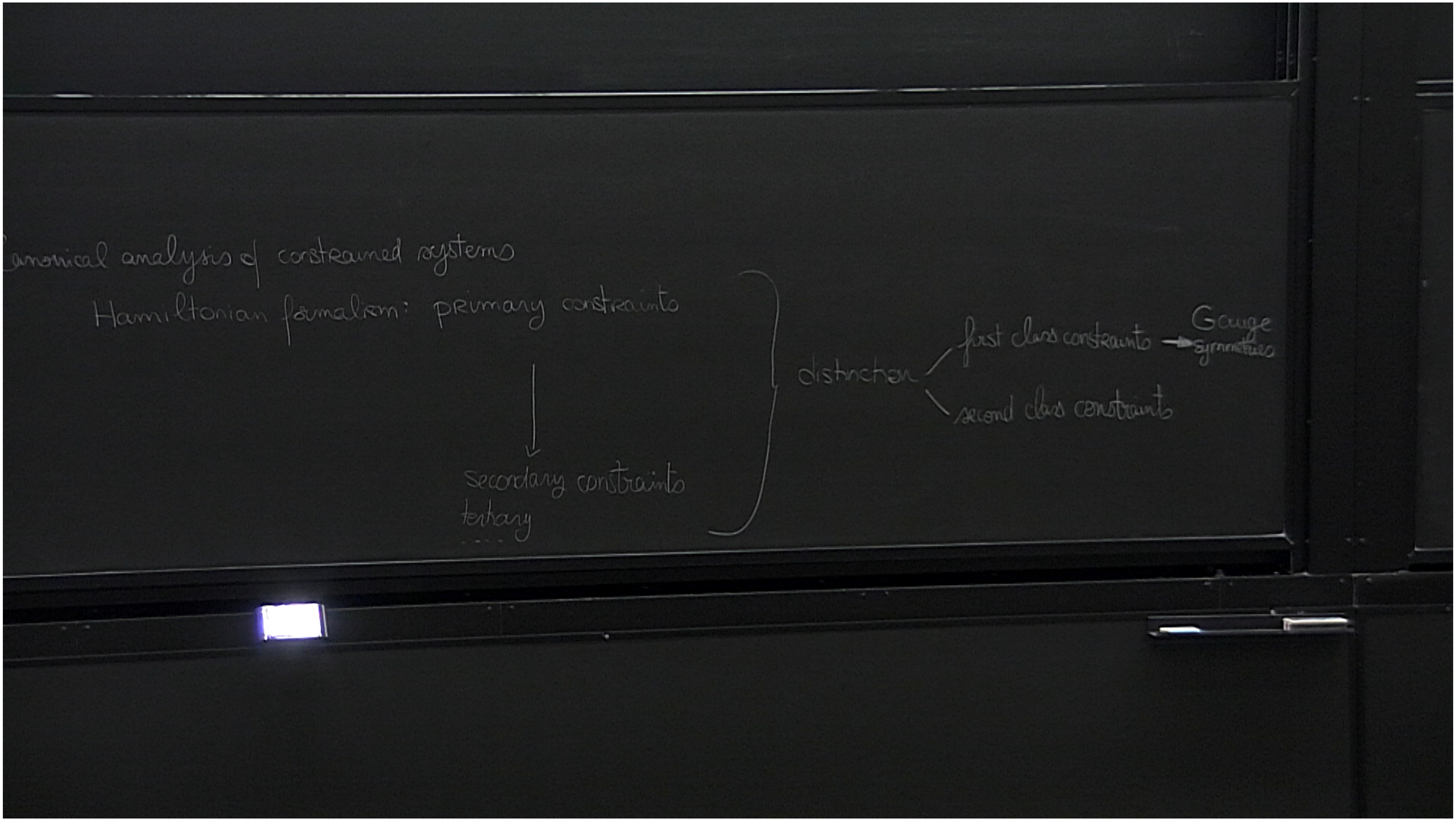
Hamiltonian formalism: primary constraints

$\downarrow$

secondary contraints

tertiary

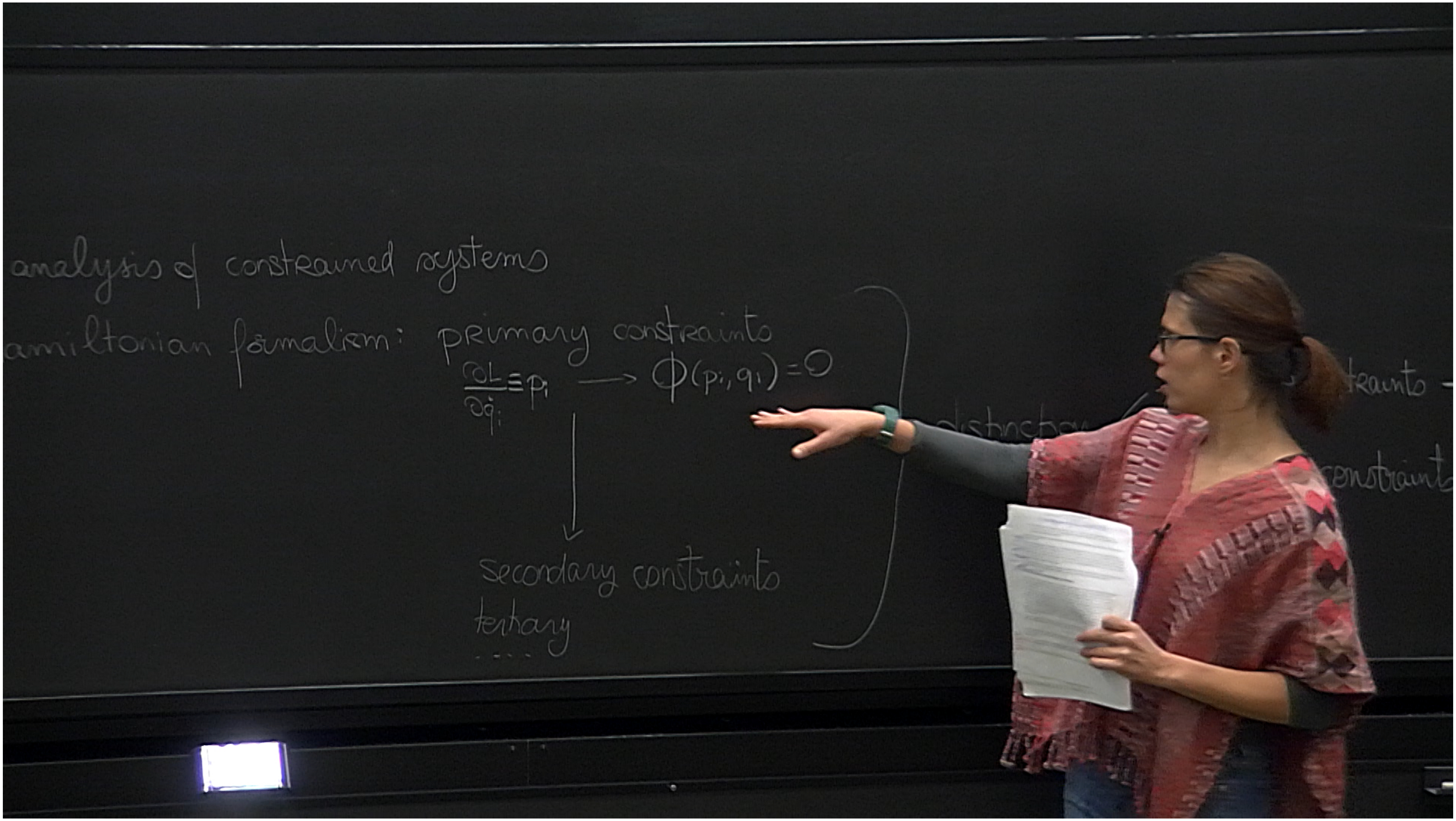
....

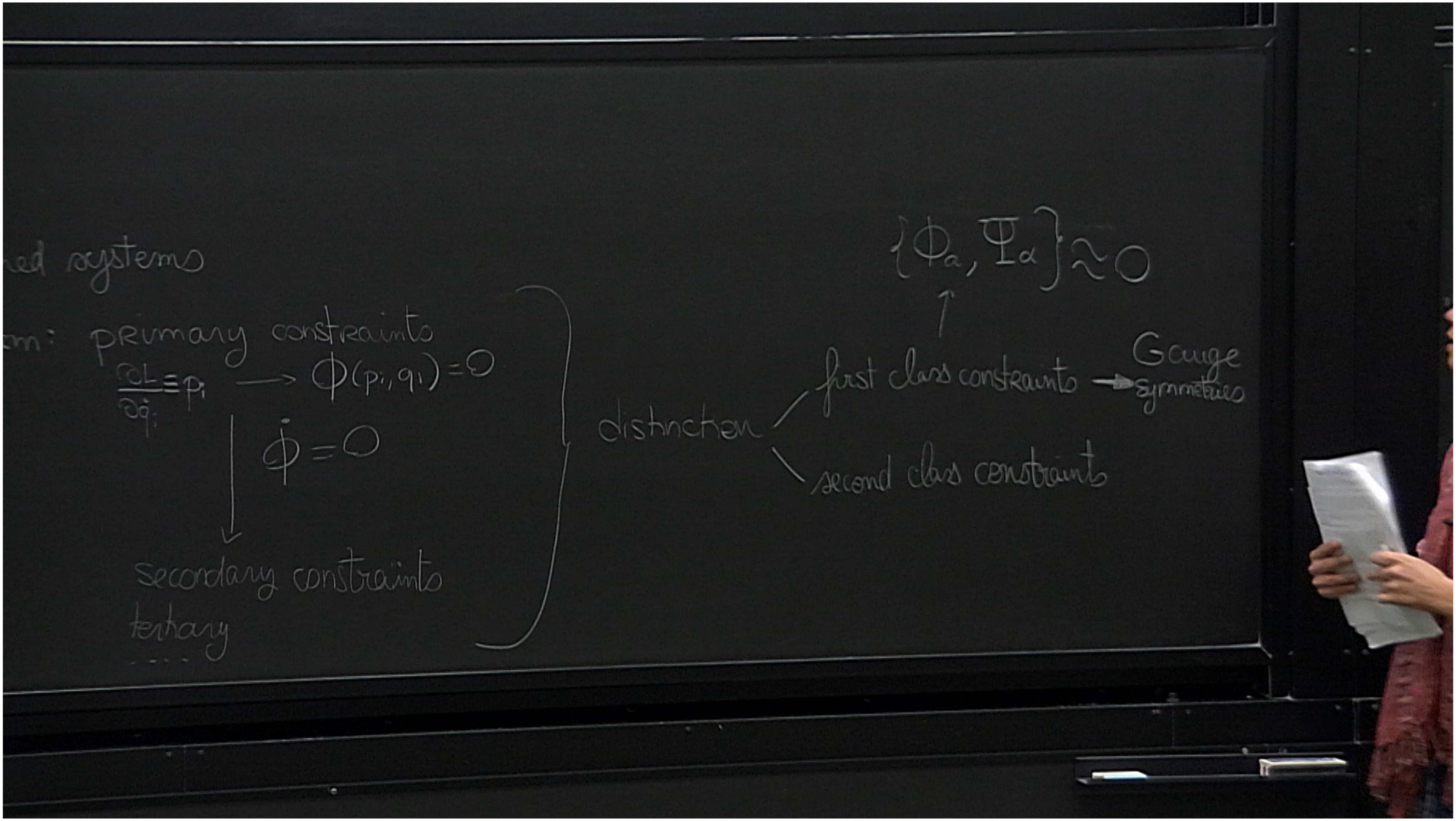

anonical analysis of constrained systems
Hamiltonian formalism: primary constraints
secondary constraints
tertiary
distinction
first class constraints
Gauge symmetries
second class constraints

3D Euclidean Gravity with a zero cosmological constant.

**TODAY**

- A totally constrained system
  - ↳ Canonical analysis
  - ↳ Physical phase space / Dirac observables
- The Dirac program –


analysis of constrained systems
amiltonian formalism: primary constraints
$\frac{\partial L}{\partial \dot{q}_i} = p_i \longrightarrow \phi(p_i, q_i) = 0$
secondary constraints
tertiary
. . . .


ned systems
on: primary constraints
$\frac{\partial L}{\partial \dot{q}_i} = p_i \longrightarrow \phi(p_i, q_i) = 0$
$\dot{\phi} = 0$
secondary constraints
tertiary
distinction
first class constraints → Gauge symmetries
$\{\Phi_a, \Psi_\alpha\} \approx 0$
second class constraints

A totally constrained system - example of the parametrized particle.

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free non-relativistic particle in 1d $\quad S[q] = \int_{t_1}^{t_2} dt \, \frac{1}{2} m \dot{q}^2 \qquad \dot{q} = \frac{dq}{dt}$

cle.

$\dot{q} = \frac{dq}{dt}$ $\longrightarrow$ $q(t) = q(\tau) + \frac{p(\tau)}{m}(t-\tau)$

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. $t \rightarrow$ independent variable $\quad S_p[q(s), t(s)] = \int_{s_1}^{s_2} \left( \frac{1}{2} m \frac{q'^2}{t'} \right) ds$

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$$S[q] = \int_{t_1}^{t_2} dt \; \frac{1}{2} m \dot{q}^2 \qquad \dot{q} = \frac{dq}{dt} \qquad \longrightarrow \quad q(t) = q(\tau) + \frac{p(\tau)}{m}(t-\tau).$$

$$S_0[q(s), t(s)] = \int_{s_1}^{s_2} \left( \frac{1}{2} m \frac{q'^2}{t'} \right) ds$$

' derivative with respect to s.

ystem – example of the parametrized particle.

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variable $\quad S_p[q(s), t(s)] = \int_{s_1}^{s_2} \left( \frac{1}{2} m \frac{q'^2}{t'} \right) ds \quad$ ' derivative

invariant under $s \longrightarrow \hat{s} := f(s)$

Canonical analysis: $p_t = \frac{\delta S}{\delta t'} = -m\frac{q'^2}{2t'^2}$ , $p_q = \frac{\delta S}{\delta q'} = m\frac{q'}{t'}$

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$\Rightarrow$ primary constraint $C = p_t + \frac{p_q^2}{2m} = 0$

Hamiltonian: $H = p_t t' + p_q q' - L = t' C$

Canonical analysis: $p_t = \frac{\delta S}{\delta t'} = -m\frac{q'^2}{2t'^2}$ , $p_q = \frac{\delta S}{\delta q'} = m\frac{q'}{t'}$

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Hamiltonian: $H = p_t t' + p_q q' - L = t' C$

$H = N(s)\, C$

flow generated by H? $q' = \{q, H\} = N\frac{p_q}{m}$ , $p_q' = 0$

$t' = \{t, H\} = N$ , $p_q' = 0$

## Physical phase space, Dirac observables

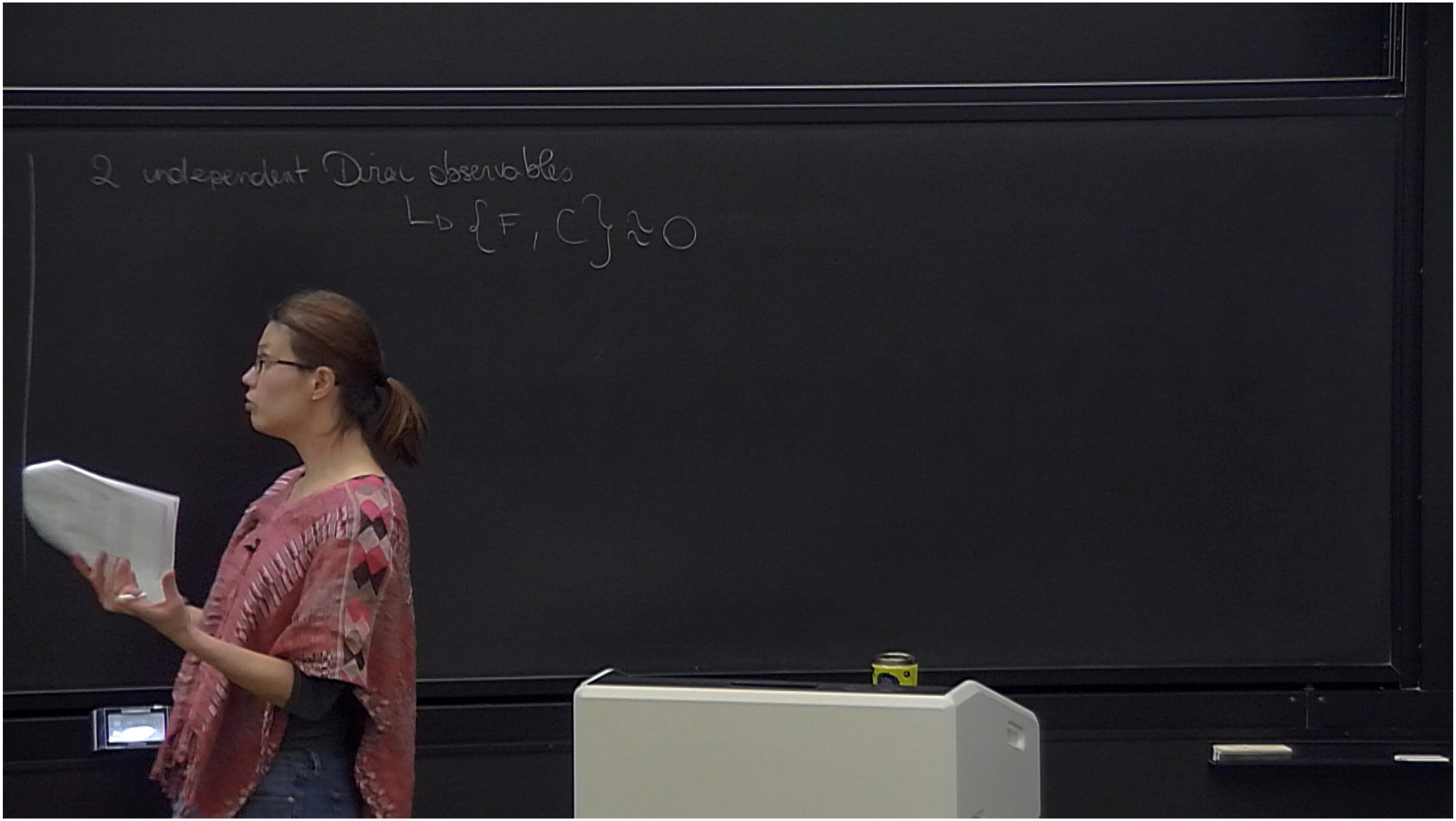
Gauge orbits of the phase space variables generated by $C$.

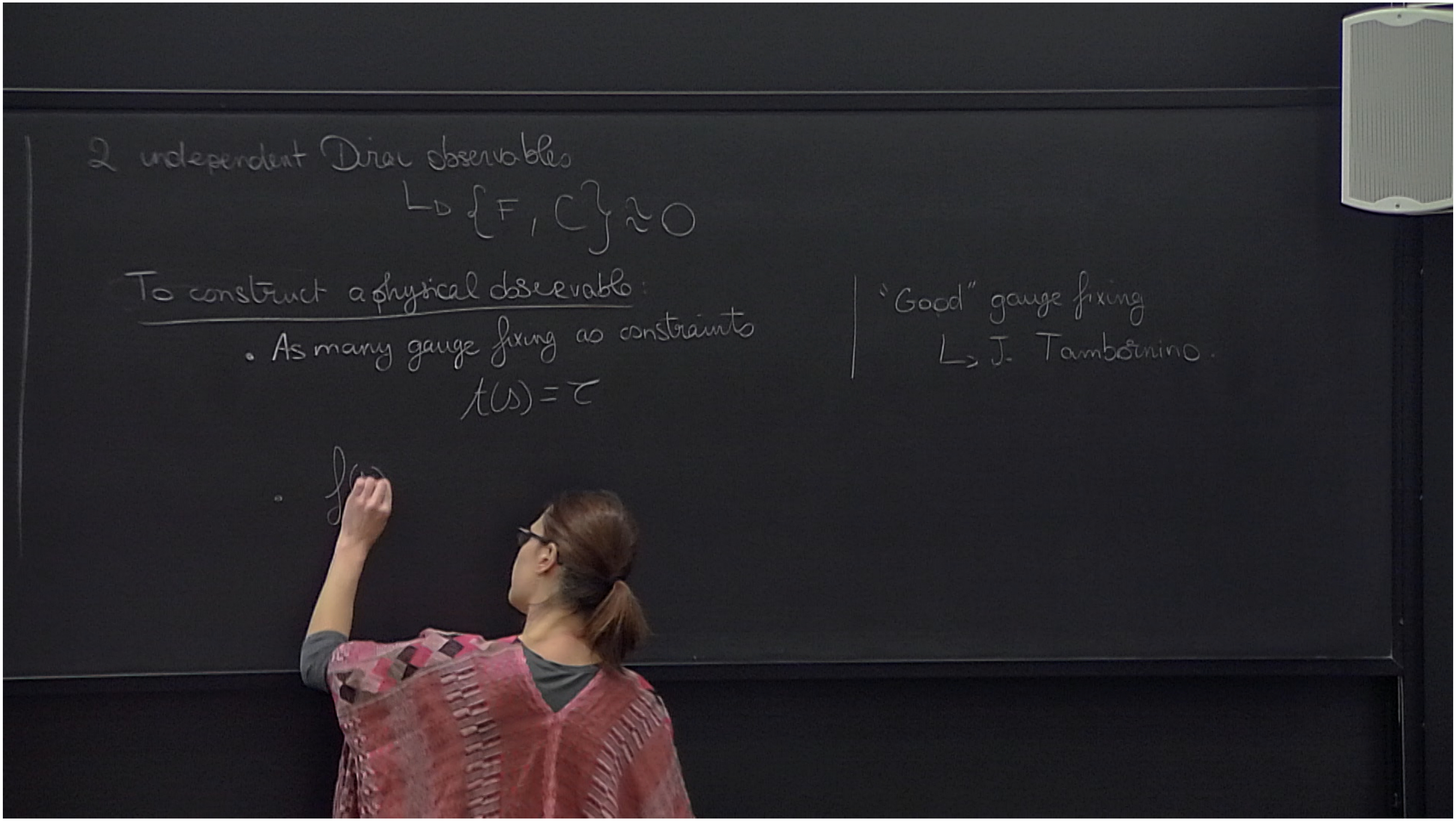
$$\frac{dq}{ds} = \{q, C\} = \frac{p_q}{m} \longrightarrow q(s) = q + \frac{p_q}{m} s.$$

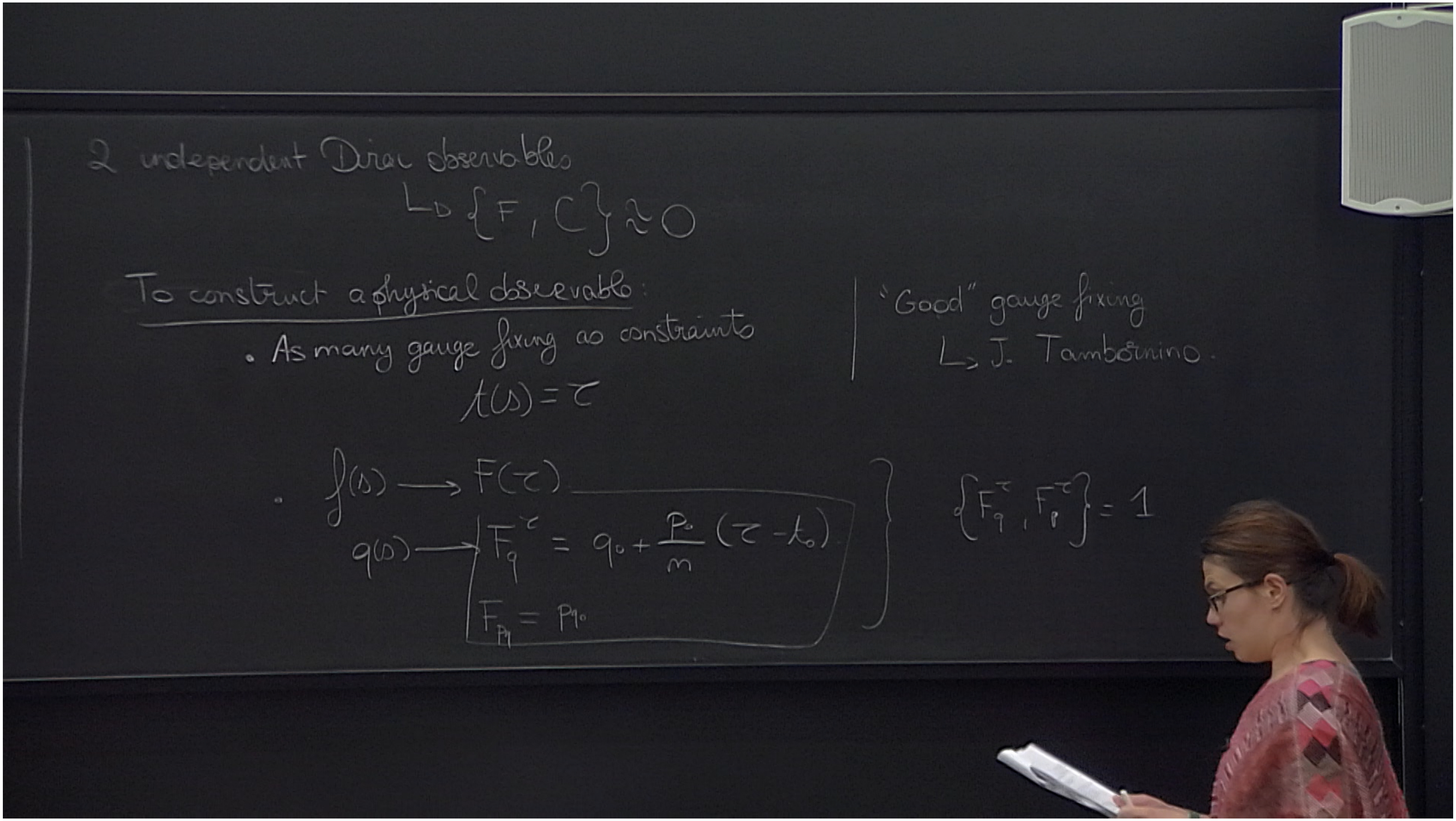
$$\frac{dt}{ds} = \{t, C\} = 1 \longrightarrow t(s) = s + t$$

$$\frac{dp_q}{ds} = \{p_q, C\} = 0 \Rightarrow p_q = p_{q0}$$

$$\frac{dp_t}{ds} = \{p_t, C\} = 0 \Rightarrow p_t = -\frac{p_q^2}{2m}$$


2 independent Dirac observables
$L_D\{F, C\} \approx 0$


2 independent Dirac observables
↳ {F, C} ≈ 0
To construct a physical observable:
• As many gauge fixing as constraints
t(s) = τ
"Good" gauge fixing
↳ J. Tambornino.


2 independent Dirac observables
$L_D \{F, C\} \approx 0$
To construct a physical observable:
• As many gauge fixing as constraints
$t(s) = \tau$
• $f(s) \longrightarrow F(\tau)$
$q(s) \longrightarrow F_q^{\tau} = q_0 + \frac{p_0}{m}(\tau - t_0)$
$F_{p_q} = p_{q_0}$
$\{F_q^{\tau}, F_p^{\tau}\} = 1$
"Good" gauge fixing
$L_>$ J. Tambornino.

Dirac program

a). Find a representation of the phase space variables as operators acting in $\mathcal{H}_{kin}$. satisfying the standard commutation relations.

$$\{\cdot,\cdot\} \longrightarrow -i/\hbar\, [\cdot,\cdot]$$

b) Promote the constraints to (self-adjoint) operators in $\mathcal{H}_{kin}$

c) Characterize the space of solutions of the constraints $\rightarrow$ inner product

$\mathcal{H}_{phys}$.

d) Find a complete set of gauge inv observables

$\mathcal{H}_{phys}$

d) Find a complete set of gauge inv observables

Case of the parametrized particle

a) $\mathcal{H}_{kin} = \mathcal{L}^2(\mathbb{R}^2)$ , $\psi(q,t)$ , $\langle \psi, \varphi \rangle = \int dq\,dt\, \overline{\psi(q,t)}\, \varphi(q,t)$

$$q \longrightarrow \hat{q}\, \psi(q,t) =$$
$$t \longrightarrow \hat{t}$$
$$p_q \longrightarrow \hat{p}_q$$
$$p_t \longrightarrow \hat{p}_t$$

$\mathcal{H}_{phys}$

d) Find a complete set of gauge inv observables

Case of the parametrized particle

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$$
\begin{aligned}
q &\longrightarrow \hat{q}\, \psi(q,t) = q\, \psi(q,t) \\
t &\longrightarrow \hat{t}\, \psi(q,t) = t\, \psi(q,t) \\
p_q &\longrightarrow \hat{p}_q\, \psi(q,t) = -i\hbar \frac{\partial}{\partial q} \psi(q,t) \\
p_t &\longrightarrow \hat{p}_t\, \psi(q,t) = -i\hbar \frac{\partial}{\partial t} \psi(q,t)
\end{aligned}
$$

d) Find a complete set of gauge inv observables

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$$p_t \longrightarrow \hat{p}_t\, \psi(q,t) = -i\hbar \frac{\partial}{\partial t} \psi(q,t)$$

b) $C = p_t + \frac{p_q^2}{2m} \longrightarrow \hat{C} = -i\hbar \frac{\partial}{\partial t} +$

d) Find a complete set of gauge inv observables

Case of the parametrized particle

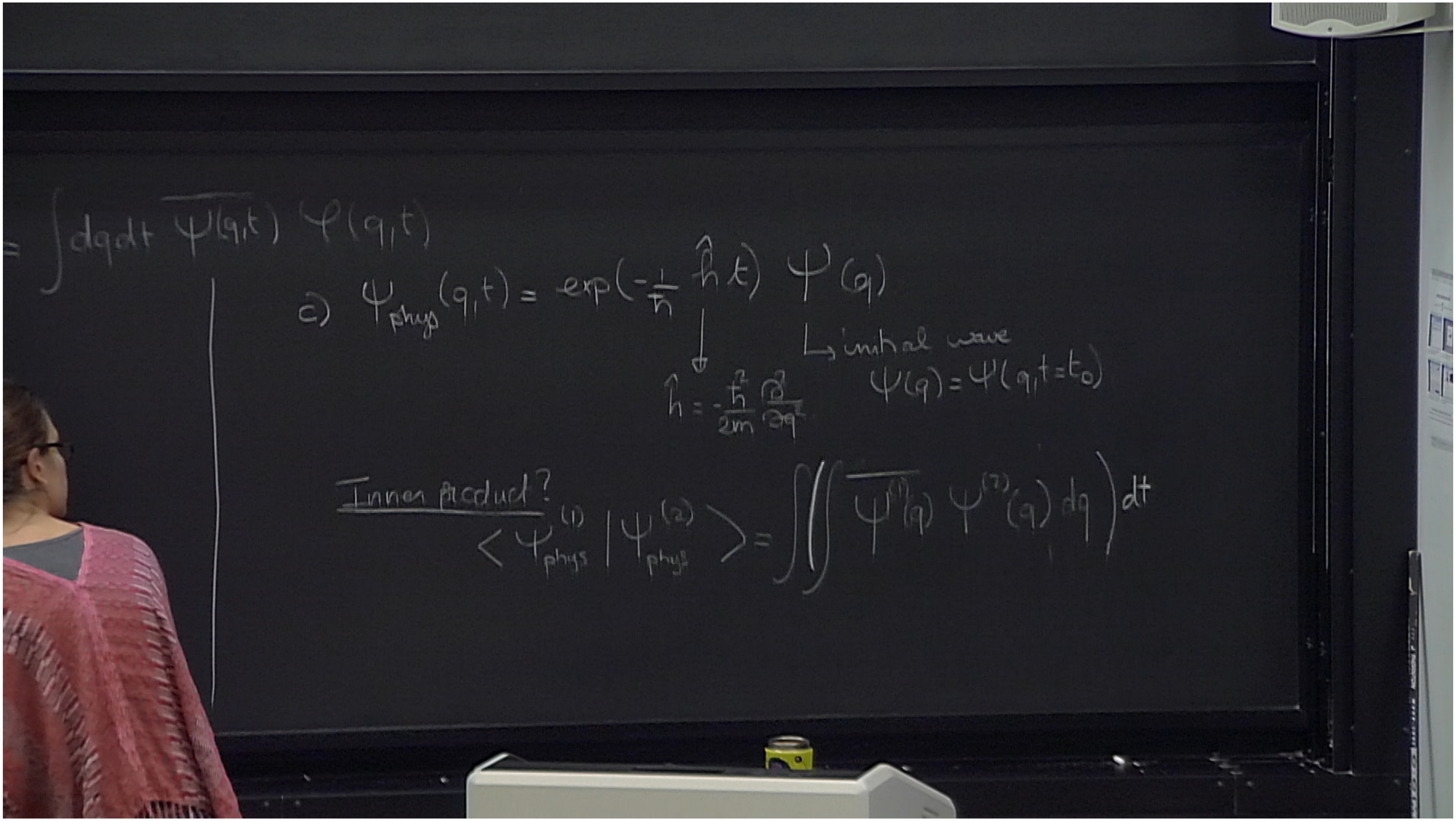
a) $\mathcal{H}_{kin} = \mathcal{L}^2(\mathbb{R}^2)$ , $\Psi(q,t)$ , $\langle \Psi, \varphi \rangle = \int dq\,dt\, \overline{\Psi(q,t)}\, \varphi(q,t)$

$$q \longrightarrow \hat{q}\, \Psi(q,t) = q\, \Psi(q,t)$$
$$t \longrightarrow \hat{t}\, \Psi(q,t) = t\, \Psi(q,t)$$
$$p_q \longrightarrow \hat{p}_q\, \Psi(q,t) = -i\hbar \frac{\partial}{\partial q} \Psi(q,t)$$
$$p_t \longrightarrow \hat{p}_t\, \Psi(q,t) = -i\hbar \frac{\partial}{\partial t} \Psi(q,t)$$

b) $C = p_t + \frac{p_q^2}{2m} \longrightarrow \hat{C} = -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2}$

$\hat{C} |\Psi\rangle = 0$

↳ Schrödinger eq!



$= \int dq\,dt\, \overline{\psi(q,t)}\, \varphi(q,t)$
c) $\psi_{phys}(q,t) = \exp(-\frac{i}{\hbar} \hat{h} t)\, \psi(q)$
↳ initial wave
$\psi(q) = \psi(q, t=t_0)$
$\hat{h} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2}$
Inner product?
$\langle \psi^{(1)}_{phys} | \psi^{(2)}_{phys} \rangle = \int \left( \int \overline{\psi^{(1)}(q)}\, \psi^{(2)}(q)\, dq \right) dt$

$\rangle = 0$

hrödingen
eq!

c) $\Psi_{phys}(q,t) = \exp\left(-\frac{i}{\hbar}\hat{h}\,t\right)\Psi(q)$

↳ initial wave $\Psi(q) = \Psi(q, t=t_0)$

↳ fixed

$$\hat{h} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}$$

Inner product?

$$\langle \Psi^{(1)}_{phys} | \Psi^{(2)}_{phys} \rangle_{kin} = \int\left(\int \overline{\Psi^{(1)}(q)}\,\Psi^{(2)}(q)\,dq\right)dt$$

$$\rightarrow \mathcal{H}_{phys} = L^2(\mathbb{R}) \qquad \langle \Psi^{(1)}_{phy} | \Psi^{(2)}_{phys} \rangle_{phy} = \int \overline{\Psi^{(1)}(q)}\,\Psi^{(2)}(q)\,dq$$

$(t_{fixed})$

$p'_q = 0$

$p'_q = 0$

d) Two independent observables

$$\hat{O}_1 = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t} - t_0)$$

$$\hat{O}_2 = \hat{p}_q$$

$[\hat{O}_i, C] = 0$ : to check