Title: PSI 2018/2019 - Quantum Information Review - Lecture 12

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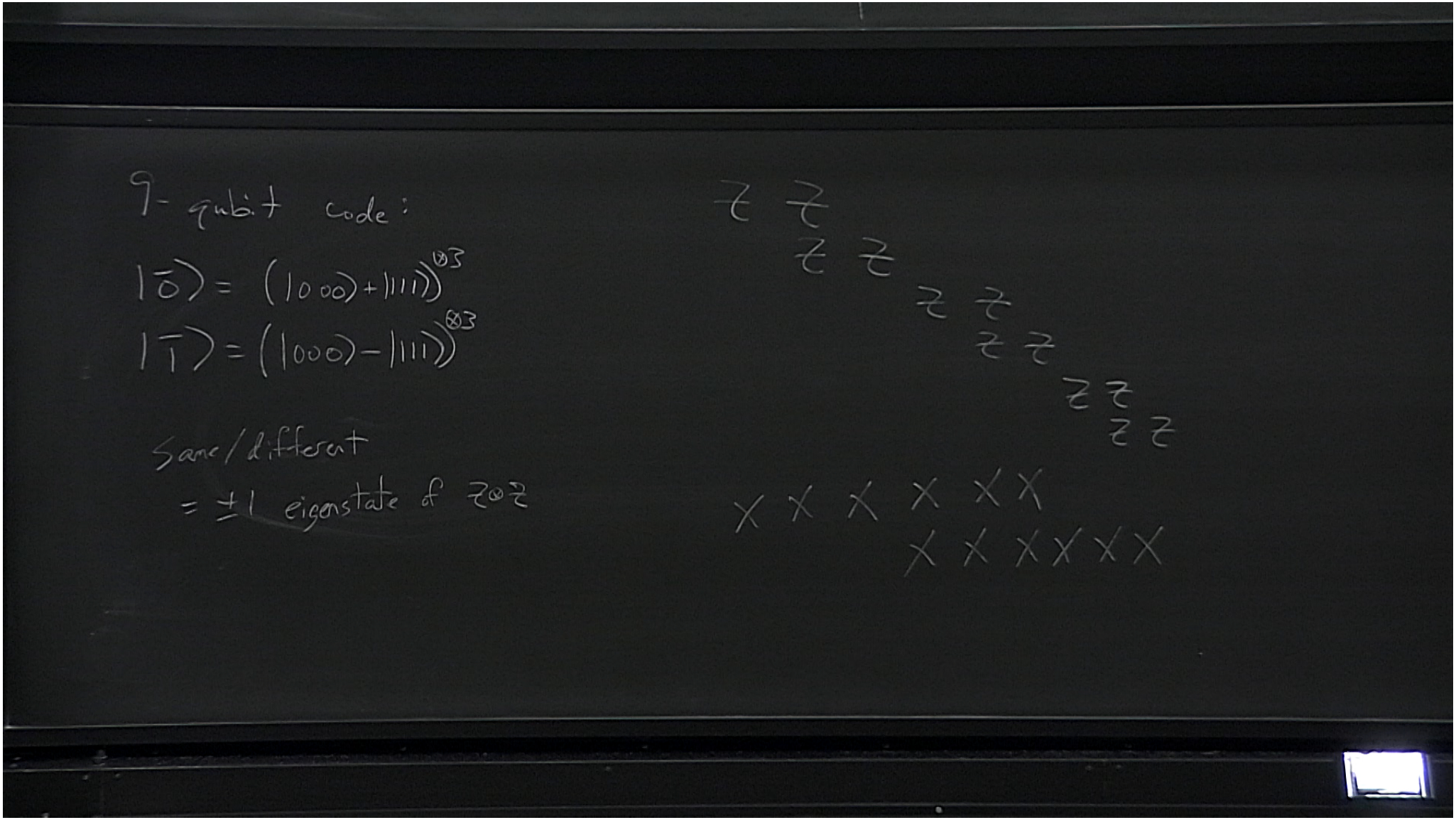
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$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = iXZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli group $\mathcal{P}_n = \{$ tensor products of I, X, Y, Z on n qubits w/ overall phase $\pm 1, \pm i \}$

- Group (closed under multiplication)
- Any element of $\mathcal{P}_n$ squares either to I or to $-I$.
- Any two $P, Q \in \mathcal{P}_n$ either commute or anticommute.


9-qubit code:
$|\bar{0}\rangle = (|000\rangle + |111\rangle)^{\otimes 3}$
$|\bar{1}\rangle = (|000\rangle - |111\rangle)^{\otimes 3}$
same/different
$= \pm 1$ eigenstate of $Z \otimes Z$
Z Z
Z Z
Z Z
Z Z
Z Z
Z Z
X X X X X X
X X X X X X

Stabilizer of a code T (a subspace of the n-qubit Hilbert space)

is $S(T) = \{ M \in \mathcal{P}_n \mid M|\psi\rangle = |\psi\rangle \ \forall |\psi\rangle \in T \}$

- Stabilizer is always an Abelian group:

  $M, N \in S(T)$, then $MN|\psi\rangle = M|\psi\rangle = |\psi\rangle \quad \forall |\psi\rangle \in T$

  $NM|\psi\rangle = |\psi\rangle$

  $\Rightarrow [M,N]|\psi\rangle = 0 \Rightarrow (M,N) = 0$ since $M \otimes N \in \mathcal{P}_n$.

- $-I \notin S(T)$.

Given $S \subseteq \mathcal{P}_n$, S an Abelian subgroup, $-I \notin S$.

Let $T(S) = \{ |\psi\rangle \mid M|\psi\rangle = |\psi\rangle \ \forall M \in S \}$.

$T(S)$ is a stabilizer code.

Thm: When a stabilizer S has r generators, it has 2^r elements. Let $k = n - r$. Then $\dim T(S) = 2^k$, so the code encodes k qubits.

Let $M \in S$, $|\psi\rangle \in T(S)$.

Suppose $E \in P_n$ s.t.

$\{E, M\} = 0$: $M(E|\psi\rangle) = -EM|\psi\rangle = -(E|\psi\rangle)$

-1 eigenvector now.

$[E, M] = 0$: $M(E|\psi\rangle) = EM|\psi\rangle = E|\psi\rangle$

$+1$ eigenvector still.

Def: The *error syndrome* of $E \in \mathcal{P}_n$ is the vector which is 0/1 on bit i if E commutes/anticommutes w/ generator M_i of S.

$$N(S) = \{ N \in \mathcal{P}_n \mid [M, N] = 0 \ \forall M \in S \}$$

Note $S \subseteq N(S)$.

Thm: The set of errors (in $\mathcal{P}_n$) that cannot be detected by S is $N(S) \setminus S$.

If $E \in S \Rightarrow E|\psi\rangle = |\psi\rangle$

$E \in N(S) \setminus S \Rightarrow \exists |\psi\rangle \in T(S)$ s.t. $E|\psi\rangle \neq |\psi\rangle$.

Thm: S can correct set $\mathcal{C}$ of errors $\subseteq \mathcal{P}_n$

iff $\forall E, F \in \mathcal{C}$. $E^\dagger F \notin N(S) \setminus S$

Proof: E & F have different error syndromes

$\Leftrightarrow \exists M \in S$ that commutes w/ one & anticommutes w/ other

$\Leftrightarrow \{E^\dagger F, M\} = 0 \Leftrightarrow E^\dagger F \notin N(S)$.

other

If $\epsilon^\dagger f \in S$, then

$\epsilon^\dagger f |\psi\rangle = |\psi\rangle$

$\Leftrightarrow F|\psi\rangle = \epsilon|\psi\rangle$.

Def: The *distance* of a stabilizer code is minimum weight of an element of $N(S) \setminus S$. A QECC is *degenerate* if S contains a non-trivial element w/ weight less than distance.

A stabilizer code w/ n physical qubits, k logical qubits, & distance d is an $[[n,k,d]]$.

Distance $2t+1$ code $\Rightarrow$ corrects t errors.

code is

A

non-trivial

2 distance d

5-qubit code $[[5,1,3]]$

$XZZXI$

$IXZZX$

$XIXZZ$

$ZXIXZ$

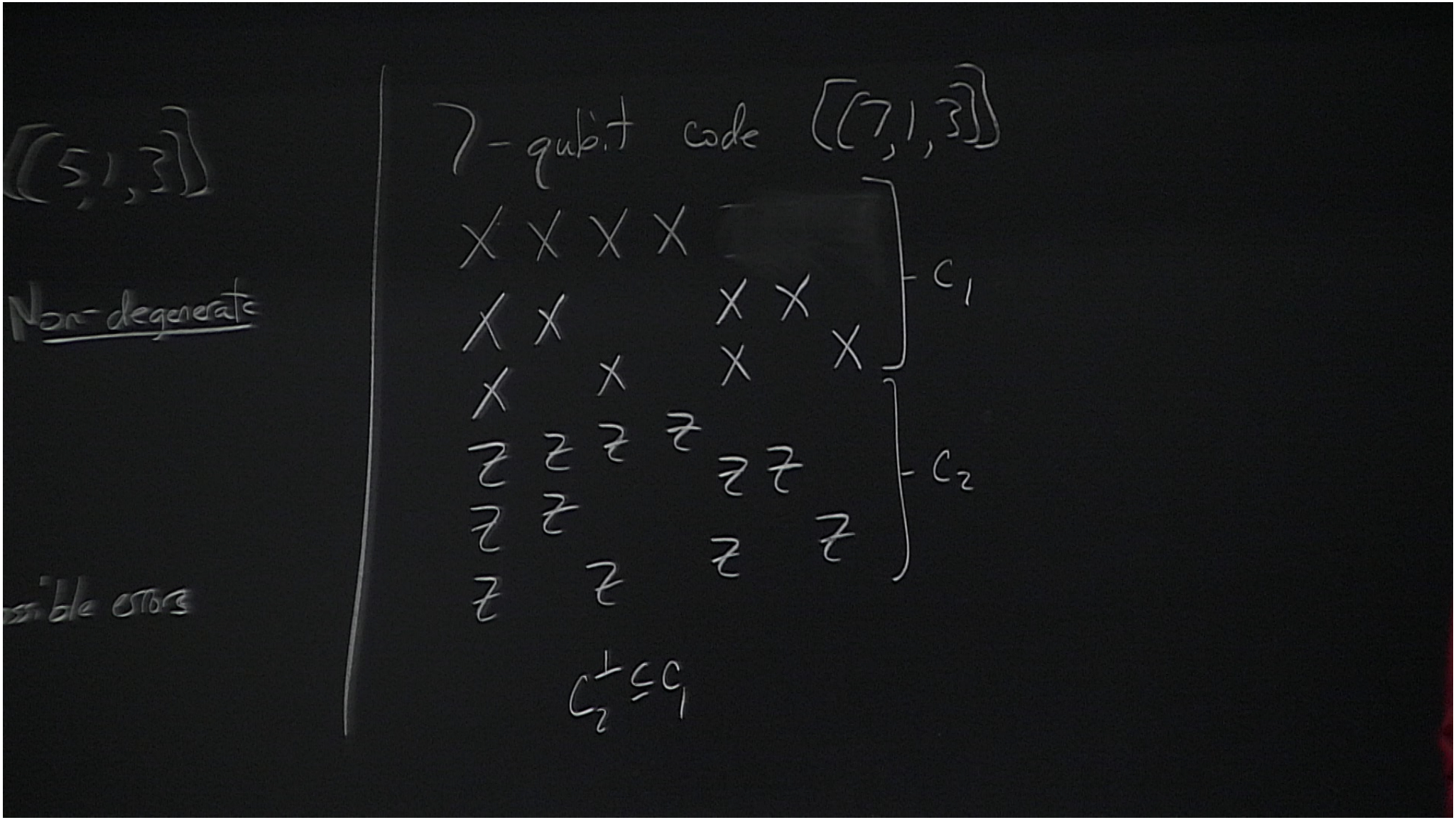
Non-degenerate

$5 \times 3 + 1 = 16$ possible errors

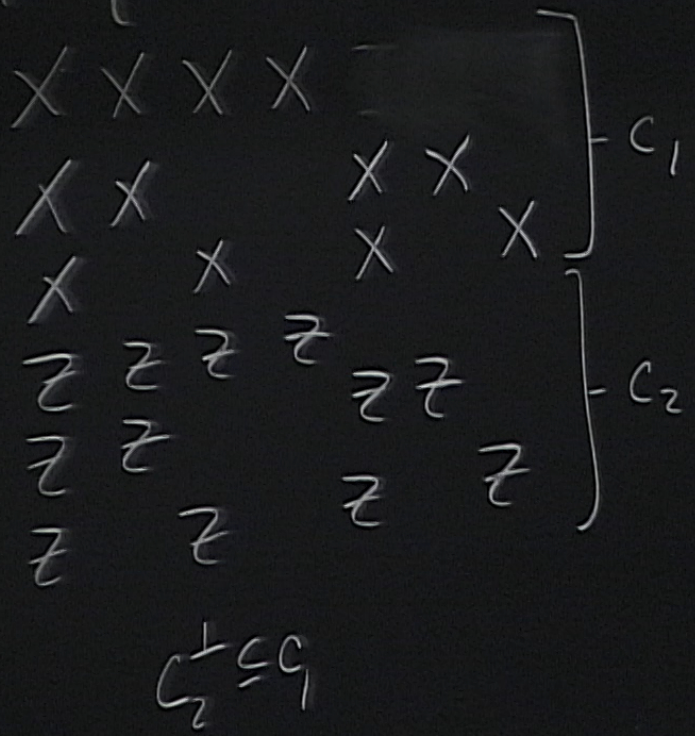
2^4 error syndromes

7-qubit

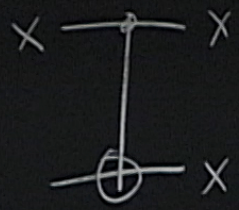
XXX

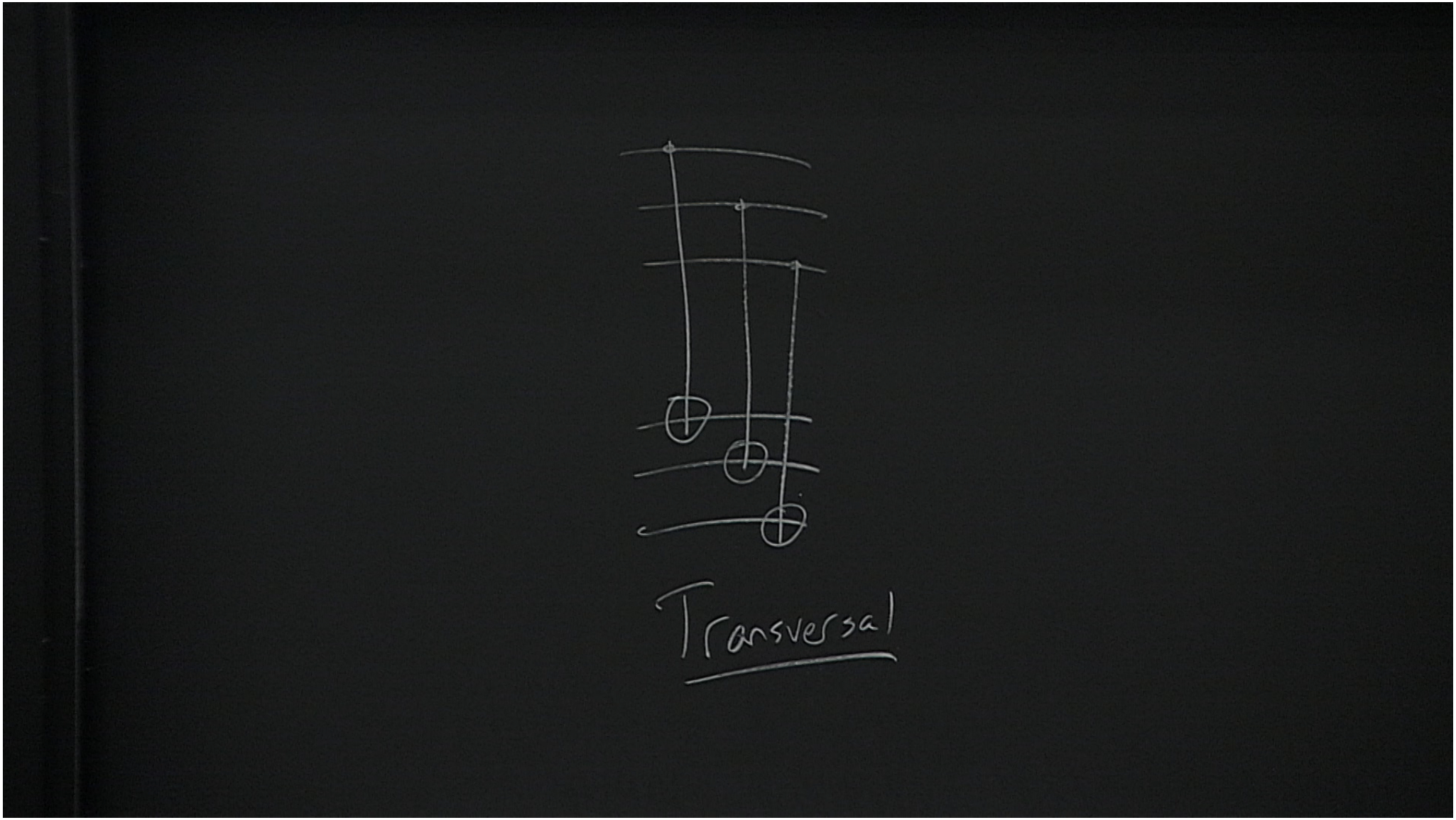

[[5,1,3]]
Non-degenerate
ssible errors
7-qubit code [[7,1,3]]
X X X X
X X X X
X X X X
X X X
Z Z Z Z
Z Z Z Z
Z Z Z Z
Z Z Z
C_1
C_2
$C_2^\perp \subseteq C_1$

7-qubit code $[[7,1,3]]$



$C_2^{\perp} \subseteq C_1$




Transversal

Threshold thm: ∃ threshold error rate p_T such that if error rate per gate & time step p is less than p_T, ∃ FT protocol for arbitrarily long quantum computations w/ arbitrarily low logical error rate ε. Overhead is polylog(T/ε)

length T