Title: PSI 2018/2019 - Strong Field Gravity - Lecture 13

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Collection: PSI 2018/2019 - Strong Field Gravity (East)

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Perturbation of Kerr black holes

$$h_{\mu\nu} = \sum_m e^{im\phi} H_{m,\mu\nu}(t, r, \theta)$$

N-P formalism

$$l^\mu = \left[\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right], \quad n^\mu = \left[r^2+a^2, -\Delta, 0, a\right]/2\Sigma$$

$$m^\mu = \left[ia\sin\theta, 0, 1, i\sin\theta\right]/\left(\sqrt{2}(r+ia\cos\theta)\right)$$

$\{l, n, m, m^*\}$, null

$\ell \cdot n = 1, \quad m \cdot m^* = -1$

Weyl tensor (10 free components) C_{abcd}

$$\Psi_4 = C_{abcd}\, n^a \bar{m}^b n^c \bar{m}^d \sim \Psi_4^0/r + \dots \longrightarrow \text{radiation at infinity}$$

$$\Psi_3 = C_{abcd}\, n^a \ell^b n^c \bar{m}^d \sim \Psi_3^0/r^2 + \dots$$

$$\Psi_2 = C_{abcd}\, m^a \ell^b n^c \bar{m}^d \sim \Psi_2^0/r^3 + \dots$$

$$\Psi_1 = C_{abcd}\, \ell^a m^b \ell^c n^d \sim \Psi_1^0/r^4 + \dots$$

$$\Psi_0 = C_{abcd}\, \ell^a m^b \ell^c m^d \sim \Psi_0^0/r^5 + \dots$$

N-P formalism

$l^\mu = \left[\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right]$, $n^\mu = \left[r^2+a^2, -\Delta, 0, a\right]/\Sigma$

$m^\mu = [ia\sin\theta, 0, 1, i\sin\theta]/(\sqrt{2}(r+ia\cos\theta))$

$\{l, n, m, m^*\}$, null

$\Delta = r^2 + M^2 - 2Mr$, $\Sigma = r^2 + a^2\cos^2\theta$

Kerr: $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$

$$\psi_2 = \frac{M}{(r + ia\cos\theta)^3} \quad \text{for Kerr}$$

→ radiation at infinity

$\underline{\psi_0 \text{ or } \psi_4}$ $\quad \psi = \begin{cases} \psi_0 & S=2 \\ \psi_4 \, (r - ia\cos\theta)^4 & S=-2 \end{cases}$

Teukolsky equation (ApJ 1973)

$$\left[\frac{(r^2+a^2)}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2}$$

$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin\theta}\right]\frac{\partial\psi}{\partial\phi}$$

$$-2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T \quad (=0 \text{ for vacuum})$$

$$\psi = e^{-i\omega t} e^{im\phi} S(\theta) R(r)$$

Radial eqn:

$$\Delta^{-s} \frac{d}{dr}\left(\Delta^{s+1} \frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4i\omega s r - \lambda\right) R = 0$$

$$K = (r^2+a^2)\omega - am, \quad \lambda = A + a^2\omega^2 - 2am\omega$$

Angular eqn: $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dS(\theta)}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right) S(\theta) = 0$$

Angular Teukolsky eqn $\rightarrow$ Spin-weighted Spheroidal harmonics

Radial: $u = \Delta^{s/2}(r^2+a^2)^{1/2} R$, $\frac{dr_*}{dr} = (r^2+a^2)/\Delta$

$$u_{,r_*r_*} + \left\{ \left[K^2 - 2is(r-M)K + \Delta(4ir\omega s - \lambda) \right]/(r^2+a^2)^2 - G^2 - G_{,r_*} \right\} u = 0$$

$$G \equiv \frac{s(r-M)}{r^2+a^2} + \frac{r\Delta}{r^2+a^2}$$

$r \to \infty \quad r_* \to \infty$

$$u_{,r_*r_*} + (\omega^2 + 2i\omega s/r)\, u \approx 0$$

$$u \sim r^{\pm s} e^{\mp i\omega r_*}$$

$$\Rightarrow R = \begin{cases} \sim e^{-i\omega r_*}/r & \text{outgoing} \\ \sim e^{i\omega r_*}/r^{2s+1} & \text{ingoing} \end{cases}$$

$r \to r_+$

$$u_{,r_*r_*} + \left[k^2 - \frac{2is(r_+ - M)k}{2Mr_+} - \frac{s^2(r_+ - M)^2}{(2Mr_+)^2} \right] u = 0$$

$$k = \omega - m\omega_+, \quad \omega_+ = \frac{a}{2Mr_+}$$

$$\Rightarrow u \sim e^{\pm i\left[k - is\frac{(r_+ - M)}{2Mr_+}\right] r_*} \sim \Delta^{\pm s/2} e^{\pm ikr_*}$$

$$\Rightarrow R \sim \begin{cases} e^{ikr_*} & \text{outgoing} \\ \Delta^{-s} e^{-ikr_*} & \text{ingoing} \end{cases}$$

$r \to r_+$

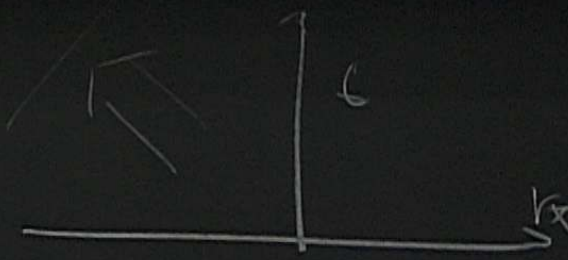
$$u_{,r_* r_*} + \left[k^2 - \frac{2is(r_+ - M)k}{2Mr_+} - \frac{s^2(r_+ - M)^2}{(2Mr_+)^2} \right] u = 0$$

$$k = \omega - m\omega_+, \quad \omega_+ = \frac{a}{2Mr_+}$$

$$\Rightarrow u \sim e^{\pm i\left[k - is\frac{(r_+ - M)}{2Mr_+}\right] r_*} \sim \Delta^{\pm s/2} e^{\pm ikr_*}$$

$$\Rightarrow R \sim \begin{cases} e^{ikr_*} & \text{outgoing} \\ \Delta^{-s} \cdot e^{-ikr_*} & \text{ingoing} \end{cases}$$

The group velocity



$$v_{group} = -\frac{dk}{d\omega} = -1$$

$$v_{phase} = \frac{-k}{\omega} = -1 + \frac{m\omega_+}{\omega} = \begin{cases} < 0 & \omega > m\omega_+ \\ > 0 & \omega < m\omega_+ \end{cases}$$

$\dot{E} \propto v_{phase} > 0 \quad (m\omega_+ > \omega)$

Super-radiance ($\xleftarrow{\text{energy}}$ extraction from black hole)

WKB limit ($\ell >> 1$)

$u = e^{-i\omega t} e^{im\varphi} u_r u_\theta$, keep terms leading in ℓ

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{du_\theta}{d\theta}\right]+\left[a^2\omega^2\omega_1^2\theta-\frac{m^2}{\sin^2\theta}+A\right]u_\theta=0$$

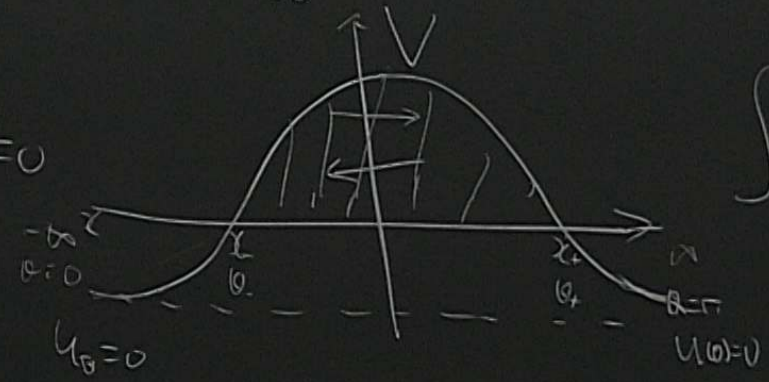
$$\Rightarrow \text{Bohr-Sommerfeld}$$

quantization condition

$$\frac{d}{dx}=\sin\theta\frac{d}{d\theta},$$

$$\chi\sin^2\theta\equiv V_\theta$$

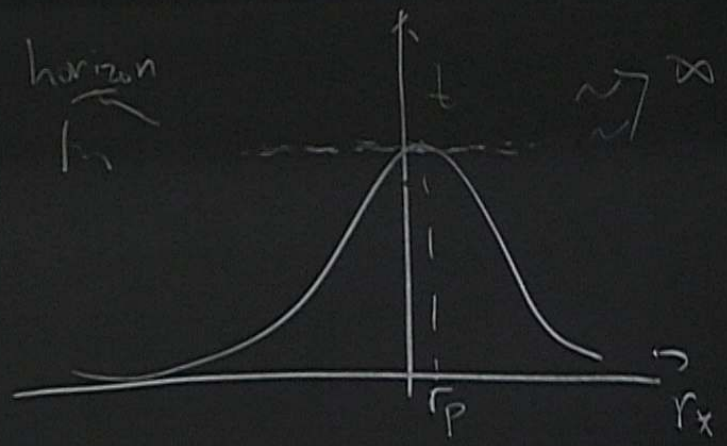
$$\frac{d^2u_\theta}{dx^2}+V(\theta)u_\theta=0$$



$$\int_{\theta_-}^{\theta_+}\frac{d\theta}{\sin\theta}\sqrt{V_\theta}=(L-|m|)\pi$$

$$L=l+\frac{1}{2}$$

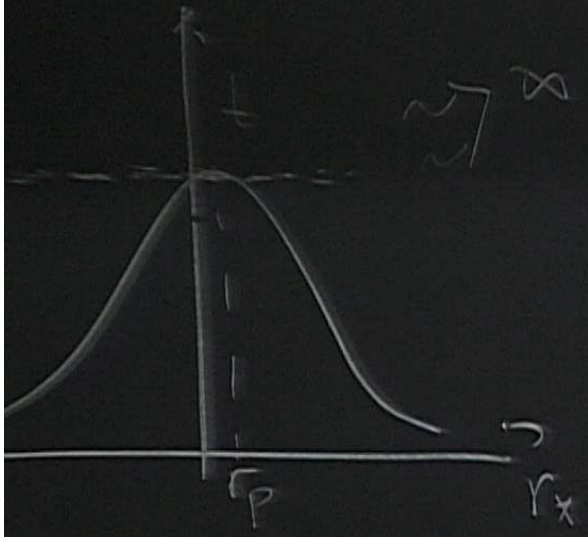
$$\frac{d^2 u_r}{dr_*^2} + \frac{K^2 - \Delta\lambda}{(r^2+a^2)^2} u_r = 0$$



$$\begin{cases} K = am - \omega(r^2+a^2) \\ \lambda = A + a^2\omega^2 - 2am\omega \\ \Delta = r^2 - 2Mr + a^2 \end{cases}$$

$$\Rightarrow \begin{cases} \left.\frac{\partial V_r}{\partial r}\right|_{r_p, \omega_R} = 0 \\ V_r(r_p, \omega_R) = 0 \end{cases}$$

$\omega \propto l$, $m \propto l$, $A \propto l^2$



$$= 0 \Big|_{r_p, \omega_R}$$

$$(r_p, \omega_R) = 0$$

Geometric picture

In the WKB limit, $u = A e^{iS}$

S varies much faster A

$$\nabla^\mu \nabla_H u = 0$$

$$\Rightarrow$$

leading order

$$g^{\mu\nu}_{Kerr} (\partial_\mu S)(\partial_\nu S) = 0$$

Hamilton-Jacobi equation

$$S = S_\theta(\theta) + L_z\phi + S_r(r) - Et$$

$$S_r = \int^r \frac{\sqrt{R(r')}}{\Delta(r')}\,dr', \qquad S_\theta(\theta) = \int^\theta \sqrt{\Theta(\theta')}\,d\theta'$$

$$R(r) = \left[E(r^2+a^2) - L_z a\right]^2 - \Delta\left[(L_z - aE)^2 + D\right]$$

$$\Theta(\theta) = D - \cos^2\theta\left(\frac{L_z^2}{\sin\theta} - a^2E^2\right)$$

↑
Carter

$u = \underline{A}\, e^{iS}$

...ch factor $\underline{A}$

$(\partial_\mu S)(\partial_\nu S) = 0$

...Hamilton-Jacobi equation

$$S = S_\theta(\theta) + L_z \phi + S_r(r) - Et$$

$$S_r = \int^r \frac{\sqrt{R(r')}}{\Delta(r')}\, dr', \quad S_\theta(\theta) = \int^\theta \sqrt{\Theta(\theta')}\, d\theta'$$

$$R(r) = \left[E(r^2+a^2) - L_z a\right]^2 - \Delta\left[(L_z - aE)^2 + D\right]$$

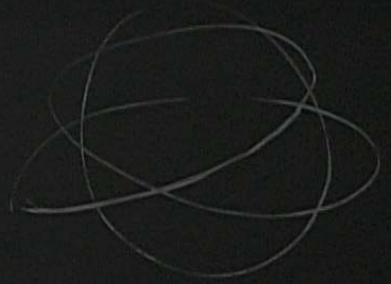
$$\Theta(\theta) = D - \cos^2\theta \left(\frac{L_z^2}{\sin^2\theta} - a^2E^2\right)$$

$\Updownarrow$ $V(\theta)$

D: Carter

$$u = e^{iS} = e^{-iEt} e^{iL_z\phi} e^{\pm i S_\theta} e^{\pm i S_r}$$

$$u = u_r u_\theta e^{im\phi} e^{-i\omega t}$$



$$\begin{cases} \omega_R \longleftrightarrow E \\ m \longleftrightarrow L_z \\ A^{-m^2} \longleftrightarrow D \end{cases}$$

$e^{-i\cdot Sr}$ ω_R phase is invariant for null rays

$\theta_- \longrightarrow \theta_+$

$\frac{T_\theta}{2}$

$$\Delta S = 0 = -\omega_R \frac{T_\theta}{2} + S_\theta \Big|_{\theta_-}^{\theta_+} + m\left(\frac{\Delta\phi}{2}\right)$$

Phase is invariant for null rays

$\theta_- \longrightarrow \theta_+$

$\frac{T_\theta}{2}$

$$\Delta S = 0 = -\omega_R \frac{T_\theta}{2} + S_\theta \Big|_{\theta_-}^{\theta_+} + m\left(\frac{\Delta\phi}{2}\right)$$

$$= -\omega_R \frac{T_\theta}{2} + (L - |m|)\pi + m\left(\frac{\Delta\phi}{2}\right)$$

$$\omega_R = L\,\Omega_\theta\,(m/L) + m\,\Omega_{prec}(m/L)$$

$$\Omega_\theta = \frac{2\pi}{T_\theta}, \quad \Omega_{prec} = \left(\Delta\phi - 2\pi\,\mathrm{Sign}(m)\right)/T_\theta$$