

Title: PSI 2018/2019 - Strong Field Gravity - Lecture 13

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Collection: PSI 2018/2019 - Strong Field Gravity (East)

Date: March 20, 2019 - 9:00 AM

URL: <http://pirsa.org/19030020>

Perturbation of Kerr black holes

$$h_{\mu\nu} = \sum_m e^{im\phi} H_{m,\mu\nu}(t, r, \theta)$$

N-P formalism

$$\ell^H = \left[\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right], \quad n^H = \left[\frac{r^2 + a^2}{\Delta}, -\Delta, 0, a \right] / 2\Sigma$$

$$m^H = [ia \sin\theta, 0, 1, i \sin\theta] / (\sqrt{2} (r + i a \cos\theta))$$

$$\{\ell, n, m, m^*\}, \quad \text{null}$$

$$\ell \cdot n = 1, m \cdot \bar{m}^* = -1$$

Weyl tensor (10 free components) C_{abcd}

$$\left\{ \begin{array}{l} \psi_4 = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d \sim \psi_4^0 / r \rightarrow \text{radiation at infinity} \\ \psi_3 = C_{abcd} n^a \ell^b n^c \bar{m}^d \sim \psi_3^0 / r^2 + \dots \\ \psi_2 = C_{abcd} m^a \ell^b n^c \bar{m}^d \sim \psi_2^0 / r^3 + \dots \\ \psi_1 = C_{abcd} \ell^a m^b \ell^c n^d \sim \psi_1^0 / r^4 + \dots \\ \psi_0 = C_{abcd} \ell^a m^b \ell^c m^d \sim \psi_0^0 / r^5 + \dots \end{array} \right.$$

N=4 formalism

$$l^H = \left[\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right], n^H = \left[\frac{r^2 + a^2}{\Delta}, -\Delta, 0, a \right] / \Delta$$

$$m^H = [ia \sin \theta, 0, 1, i \sin \theta] / (\sqrt{2} (r + ia \cos \theta))$$

$$\{l, n, m, m^*\}, \text{ null}$$

$$\Delta = r^2 + M^2 - 2Mr, \quad \Sigma = r^2 + a^2 \cos^2 \theta$$

$$\text{Kerr: } \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$$

$$\psi_2 = \frac{M}{(r + i a \cos \theta)^3} \quad \text{for Kerr}$$

$$\psi_0 \text{ or } \psi_4 \quad \psi = \begin{cases} \psi_0 & S=2 \\ \psi_4 (r - i a \cos \theta)^4 & S=-2 \end{cases}$$

Teukolsky equation (A.P.J 1973)

→ radiation
at infinity

$$\left[\frac{(r^2 + a^2)}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \frac{\partial^2 \psi}{\partial \phi^2}$$

$$- \Delta^{-s} \frac{\partial}{\partial r} \left(\Delta^{s+1} \frac{\partial \psi}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) - 2s \left[\frac{a(r-M)}{\Delta} + \frac{ia \cos \theta}{\sin \theta} \right] \frac{\partial \psi}{\partial \phi}$$

$$- 2s \left[\frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right] \frac{\partial \psi}{\partial t} + (s^2 \cot^2 \theta - s) \psi = 4\pi \Sigma T \quad (=0 \text{ for Vacuum})$$

$$\psi = e^{-i\omega t} e^{im\phi} S(\theta) R(r)$$

Radial eqn:

$$\Delta S \frac{d}{dr} \left(\Delta S \frac{dR}{dr} \right) + \left(\frac{K^2 - 2is(r-m)\epsilon}{\Delta} + 4i\omega sr - \lambda \right) R = 0$$

$$K = (r^2 + a^2)\omega - am, \quad \lambda = A + a^2\omega^2 - 2am\omega$$

$$\text{Angular eqn: } \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dS(\theta)}{d\theta} \right) + \left(a^2\omega^2 \cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s \cos\theta - \frac{2ms \cos\theta}{\sin^2\theta} - s^2 \cot^2\theta + S + A \right) S(\theta) = 0$$

Angular Teukolsky eqn $\rightarrow$ Spin-weighted Spheroidal harmonics

Radial: $u = \Delta^{s/2} (r^2 + a^2)^{1/2} R$, $\frac{dr_x}{dr} = (r^2 + a^2) / \Delta$

$$u_{,rr} + \left\{ [K^2 - 2is(r-m)K + \Delta(4ir\omega s - \lambda)] / (r^2 + a^2)^2 - G^2 - G_{,r} \right\} u = 0$$

$$G \equiv \frac{s(r-m)}{r^2 + a^2} + \frac{r\Delta}{r^2 + a^2}$$

$$r \rightarrow \infty \quad x \rightarrow \infty$$

$$u_{r \neq r_+} + (\omega^2 + 2i\omega s/r) u \approx 0$$

$$u \sim r^{\pm s} e^{\mp i\omega r_*$$

$$\Rightarrow R = \begin{cases} \sim e^{-i\omega r_*} / r & \text{outgoing} \\ \sim e^{i\omega r_*} / r^{2s+1} & \text{ingoing} \end{cases}$$

$$r \rightarrow r_+$$

$$u_{1/r \neq r_+} + \left[k^2 - \frac{2is(r_+ - M)k}{2Mr_+} - \frac{s^2(r_+ - M)^2}{(2Mr_+)^2} \right] u = 0$$

$$k = \omega - m\omega_+, \quad \omega_+ = \frac{a}{2Mr_+}$$

$$\Rightarrow u \sim e^{\pm i \left[k - is \frac{(r_+ - M)}{2Mr_+} \right] r_*} \sim \Delta^{\pm s/2} e^{\pm ikr_*}$$

$$\Rightarrow R \sim \begin{cases} e^{ikr_*} & \text{outgoing} \\ \Delta^{-s} e^{-ikr_*} & \text{ingoing} \end{cases}$$

$$r \rightarrow r_+$$

$$U_{\text{eff}} + \left[k^2 - \frac{2is(r_+ - M)k}{2Mr_+} - \frac{s^2(r_+ - M)^2}{(2Mr_+)^2} \right] u = 0$$

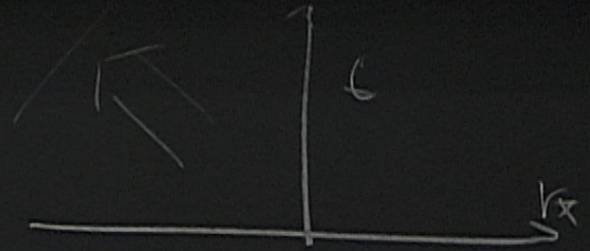


$$k = \omega - m\omega_+, \quad \omega_+ = \frac{a}{2Mr_+}$$

$$\Rightarrow u \sim e^{\pm i \left[k - is \frac{(r_+ - M)}{2Mr_+} \right] r_*} \sim \Delta^{\pm s/2} e^{\pm i k r_*}$$

$$\Rightarrow R \sim \begin{cases} e^{ikr_*} & \text{outgoing} \\ \Delta^{-s} e^{-ikr_*} & \text{ingoing} \end{cases}$$

The group velocity



$$V_{\text{group}} = -\frac{dk}{d\omega} = -1$$

$$V_{\text{phase}} = \frac{-k}{\omega} = -1 + \frac{m\omega_+}{\omega} = \begin{cases} < 0 & \omega > m\omega_+ \\ > 0 & \omega < m\omega_+ \end{cases}$$

$$E \propto v_{\text{phase}} > 0 \quad (m\omega_+ > \omega)$$

Super-radiance ^{energy} (extraction from black hole)

WKB limit ($\ell \gg 1$)

$$u = e^{-i\omega t} e^{im\phi} u_r u_\theta, \quad \text{keep terms leading in } \ell$$

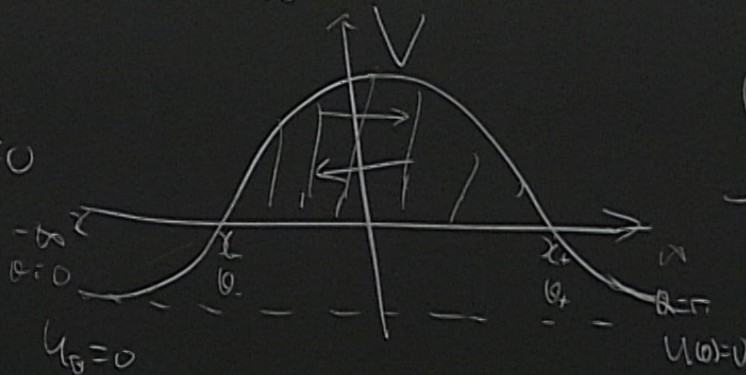
$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left[\sin\theta \frac{d u_\theta}{d\theta} \right] + \left[a^2 \omega^2 \omega^2 \theta - \frac{m^2}{\sin^2\theta} + A \right] u_\theta = 0$$

$$\frac{d}{dx} = \sin\theta \frac{d}{d\theta}$$

$$X \sin^2\theta = V_\theta$$

⇒ Bohr-Sommerfeld
quantization condition

$$\frac{d^2 u_\theta}{dx^2} + V(x) u_\theta = 0$$



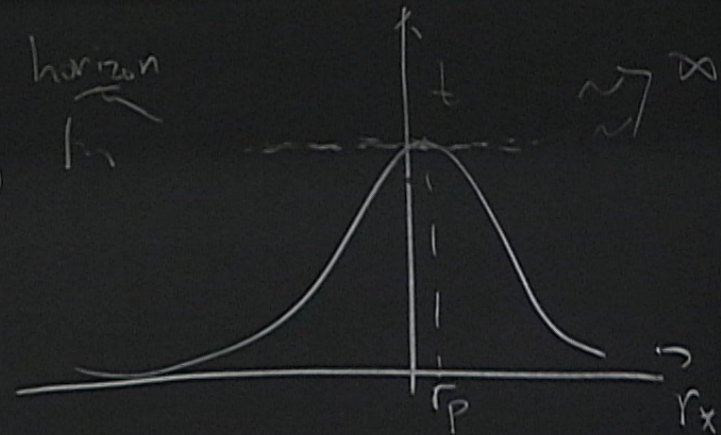
$$\int_{\theta_-}^{\theta_+} \frac{d\theta}{\sin\theta} \sqrt{V_\theta} = (L - |m|) \pi$$

$$L = l + \frac{1}{2}$$

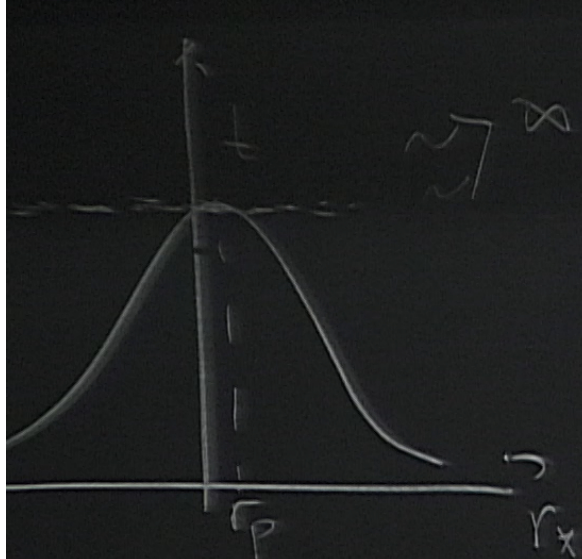
$$\frac{d^2 u_r}{dr_*^2} + \frac{K^2 - \Delta \lambda}{(r^2 + a^2)^2} u_r = 0$$

$$\begin{cases} K = am - \omega(r^2 + a^2) \\ \lambda = A + a^2 \omega^2 - 2am\omega \\ \Delta = r^2 - 2Mr + a^2 \end{cases}$$

$$\omega \propto l, \quad m \propto l, \quad A \propto l^2$$



$$\Rightarrow \begin{cases} \frac{\partial V_r}{\partial r} \Big|_{r_p, \omega_R} = 0 \\ V_r(r_p, \omega_R) = 0 \end{cases}$$



$$\left. \frac{u}{x} \right|_{x_p, 0} = 0$$

$$(x_p, \omega_R) = 0$$

Geometric picture

In the WKB limit, $\underline{u} = \underline{A} e^{iS}$

S varies much faster $\underline{A}$

$$\nabla^u \nabla_H u = 0 \leftarrow$$

$\Rightarrow$

leading order

$$\frac{g^{uv}(\partial_u S)(\partial_v S)}{\text{Kerr}} = 0$$

Hamilton-Jacobi equation

$$S = S_0(\theta) + L_z \phi + S_r(r) - Et$$

$$S_r = \int^r \frac{\sqrt{R(r')}}{\Delta(r')} dr', \quad S_0(\theta) = \int^\theta \sqrt{H(\theta')} d\theta'$$

$$R(r) = [Ecr^2 + a^2 - L_z a]^2 - \Delta[(L_z - aE)^2 + D]$$

$$H(\theta) = \underset{\substack{\uparrow \\ \text{ Carter}}}{D} - \omega^2 \theta \left(\frac{L_z^2}{\sin^2 \theta} - a^2 E^2 \right)$$

mit $\psi = \underline{A} e^{iS}$
 (factor $\underline{A}$)

$$\partial_\mu S \partial_\mu S = 0$$

Hamilton-Jacobi equation

$$S = S_\theta(\theta) + L_z \phi + S_r(r) - Et$$

$$S_r = \int^r \frac{\sqrt{R(r')}}{\Delta(r')} dr', \quad S_\theta(\theta) = \int^\theta \sqrt{H(\theta')} d\theta'$$

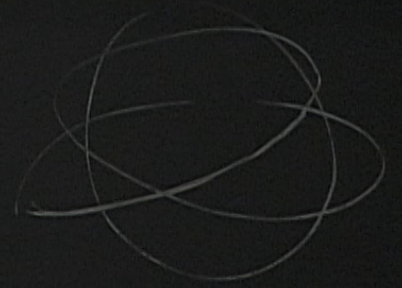
$$R(r) = [E(r^2 + a^2) - L_z a]^2 - \Delta [(L_z - aE)^2 + D]$$

$$H(\theta) = D - \cos^2 \theta \left(\frac{L_z^2}{\sin^2 \theta} - a^2 E^2 \right)$$

$\updownarrow$
 $V(\theta)$ (Carter)

$$u = e^{iS} = e^{-iEt} e^{iL_z \phi} e^{\pm iS_{\theta}} e^{\pm iS_r}$$

$$u = u_r u_{\theta} e^{im\phi} e^{-i\omega t}$$

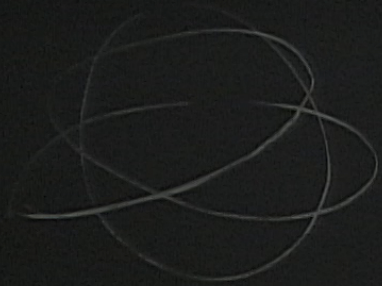


$$\left\{ \begin{array}{ll} \omega_r \longleftrightarrow E \\ m \longleftrightarrow L_z \\ A^{-m^2} \longleftrightarrow D \end{array} \right.$$

$$g^{\pm} e^{\pm i S r}$$

ω_R

Phase is invariant for null rays



$$\int_{\theta_-}^{\theta_+} \frac{T_\theta}{2} d\theta$$

$$\Delta S = 0 = -\omega_R \frac{T_\theta}{2} + S_\theta \Big|_{\theta_-}^{\theta_+} + m \left(\frac{\Delta \phi}{2} \right)$$

ω_R Phase is invariant for null rays

$$\theta_- \rightarrow \theta_+ \\ \frac{T_0}{2}$$

$$\Delta S = 0 = -\omega_R \frac{T_0}{2} + S_\theta \Big|_{\theta_-}^{\theta_+} + m \left(\frac{\Delta \phi}{2} \right)$$

$$= -\omega_R \frac{T_0}{2} + (L - |m|)\pi + m \left(\frac{\Delta \phi}{2} \right)$$

$$\omega_R = L \Omega_\theta (m/L) + m \Omega_{\text{prec}} (m/L)$$

$$\Omega_\theta = \frac{2\pi}{T_\theta}, \quad \Omega_{\text{prec}} = (\Delta\phi - 2\pi \text{Sign}(m_1)) / T_\theta$$