Title: PSI 2018/2019 - Strong Field Gravity - Lecture 12

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Collection: PSI 2018/2019 - Strong Field Gravity (East)

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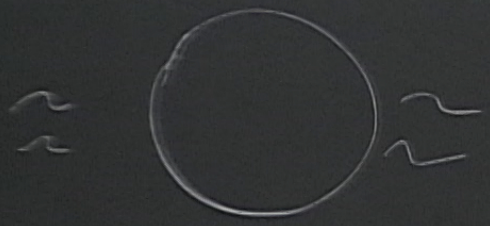
Black hole perturbation Theory



Post-Newtonian formalism

= Newtonian theory + $\left(\frac{1}{c}\right)^{\#}$

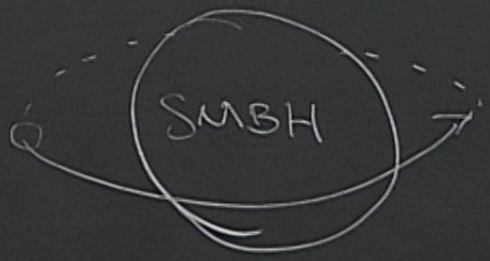
PN fails: gravity strong. Speed is comparable to c



binary black hole

ringdown

LIGO already detected



extreme mass-ratio inspirals in galactic centres.

LISA

The Kerr solution

$g = g_{bac,\mu\nu} + \epsilon h_{\mu\nu}$

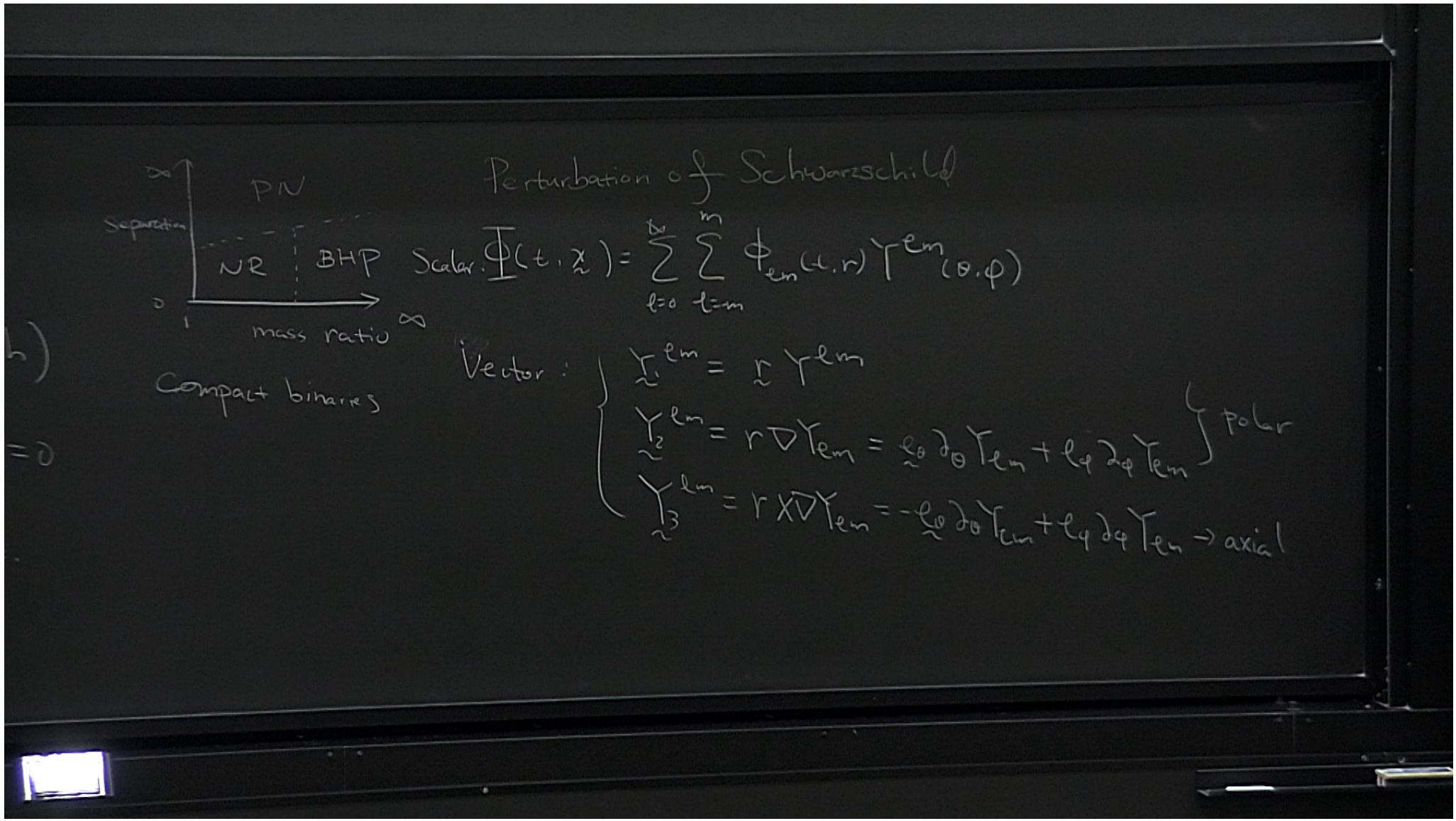
Einstein equation in expansion of ϵ:

$G_{\mu\nu}(g_{bac}) = 0$ 0th

1st order in ϵ:

$$\nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a} \nabla^c h_{b)c}$$
$$+ 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c \nabla^d h_{cd} - \nabla^2 h)$$
$$- 2R^c{}_{(a} h_{b)c} + R h_{ab} - g_{ab} R^{cd} h_{cd} = 0$$

∇, R - background metric, $h = h_{ab} g^{ab}_{bac}$.


PN
Perturbation of Schwarzschild
separation
∞
0
NR
BHP
1
mass ratio
∞
Scalar: $\Phi(t,\vec{x}) = \sum_{\ell=0}^{\infty} \sum_{l=-m}^{m} \phi_{\ell m}(t,r) Y^{\ell m}(\theta,\phi)$
Compact binaries
Vector:
$\vec{Y}_1^{\ell m} = \vec{r}\, Y^{\ell m}$
$\vec{Y}_2^{\ell m} = r \vec{\nabla} Y_{\ell m} = \vec{e}_\theta \partial_\theta Y_{\ell m} + \vec{e}_\varphi \partial_\varphi Y_{\ell m}$ } Polar
$\vec{Y}_3^{\ell m} = r \vec{X} \vec{\nabla} Y_{\ell m} = -\vec{e}_\theta \partial_\theta Y_{\ell m} + \vec{e}_\varphi \partial_\varphi Y_{\ell m}$ → axial

e.g. electric field

$$\underset{\sim}{E} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[E^{r}_{\ell m}(t,r)\, \underset{\sim}{Y}^{\ell m}_{1} + E^{1}_{\ell m}\, \underset{\sim}{Y}^{\ell m}_{2} + E^{2}_{\ell m}\, \underset{\sim}{Y}^{\ell m}_{3} \right]$$

$Y^{\ell m}_{A} \equiv \partial_A Y^{\ell m}$, $(A = \theta, \phi)$: polar vector on S^2

$X^{\ell m}_{A} \equiv \epsilon_A{}^B \partial_B Y^{\ell m}$ (: axial . . . . .

$\epsilon_A{}^B$: Levi-Civita tensor on S^2

$$d\Omega = \sin^2\theta\, d\varphi^2 + d\theta^2$$

$$= \Omega_{AB}\, d\theta^A d\theta^B$$

tensor (3-D symmetric)

V: Scalar on S^2 : V_{rr}

~~Vector~~ on S^2 : V_{rA}

tensor on S^2 : V_{AB}

$$V_{rr} = \sum_{lm} b_{lm} Y^{lm}$$

$$V_{rA} = \sum_{lm} \left(b'_{lm} Y_A^{lm} + c_{lm} X_A^{lm} \right)$$

$$V_{AB} = \sum_{lm} \left(d_{lm} \Omega_{AB} Y^{lm} + e_{lm} Y_{AB}^{lm} + f_{lm} X_{AB}^{lm} \right)$$

Tensor harmonics:

$$Y_{AB} = D_A D_B Y^{lm} + \frac{1}{2} l(l+1) \Omega_{AB} Y^{lm} \quad : \text{polar}$$

$$X_{AB} = D_A X_B^{lm} + D_B X_A^{lm} \quad : \text{axial}$$

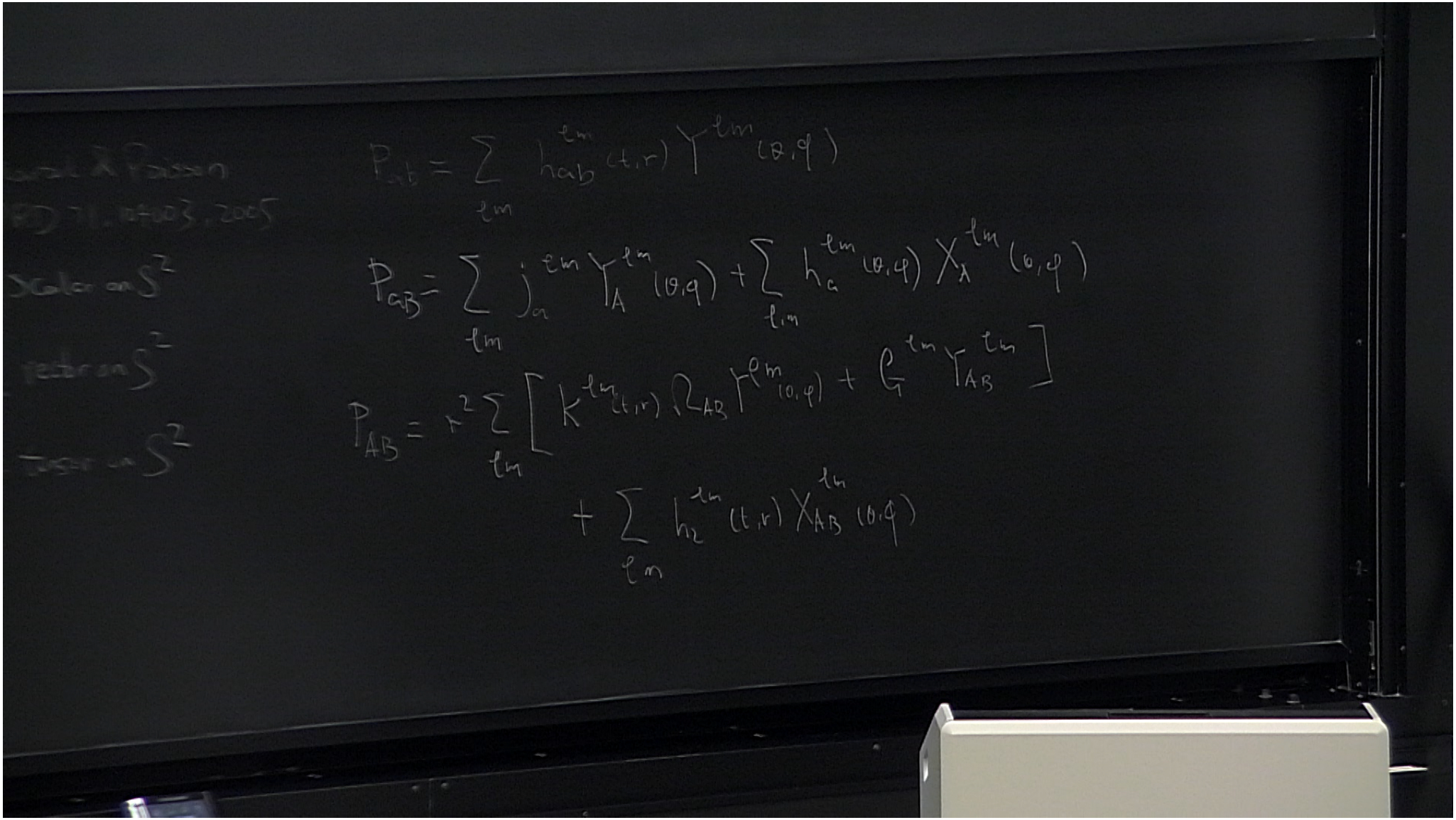
D : : respect to Ω_{AB}

Metric perturbations: Martel & Poisson
PRD 71, 104003, 2005

$P_{ab} : (P_{tt}, P_{tr}, P_{rr})$: Scalar on S^2

$P_{aA} : (P_{t\theta}, P_{t\varphi}, P_{r\theta}, P_{r\theta})$: vector on S^2

$P_{AB} : (P_{\theta\theta}, P_{\theta\varphi}, P_{\varphi\varphi})$: tensor on S^2


$P_{ab} = \sum_{lm} h_{ab}^{lm}(t,r)\, Y^{lm}(\theta,\varphi)$
$P_{aB} = \sum_{lm} j_a^{lm}\, Y_A^{lm}(\theta,\varphi) + \sum_{lm} h_a^{lm}(\theta,\varphi)\, X_A^{lm}(\theta,\varphi)$
$P_{AB} = r^2 \sum_{lm} \left[K^{lm}(t,r)\, \Omega_{AB}\, Y^{lm}(\theta,\varphi) + G^{lm}\, Y_{AB}^{lm} \right]$
$+ \sum_{lm} h_2^{lm}(t,r)\, X_{AB}^{lm}(\theta,\varphi)$

2 types of Master equations:

Γ^{em}, Γ_A^{em}, Γ_{AB}^{em} : even parity

X_A^{em}, X_{AB}^{em} : odd parity

even parity:

$$\left[-\frac{\partial^2}{\partial t^2}+f\frac{\partial}{\partial r}f\frac{\partial}{\partial r}-V_{even(r)}\right]\psi_{even(t,r)}=S_{even}$$

$= 0$ Vacuum

Regge-Wheeler gauge

$$J_\mu^{tm}=G^{tm}=0$$

$$\psi_{even}\equiv\frac{2r}{\ell(\ell+1)}\left[K+\frac{2f}{E}(f h_{rr}-r\partial_r K)\right]$$

$$V_{even}=\frac{f}{k^2}\left[\left[(\ell-1)(\ell+2)^2\left(\frac{\ell(\ell+1)}{r^2}+\frac{6M}{r^3}\right)+\frac{36M^2}{r^2}\left((\ell-1)(\ell+2)+\frac{2M}{r}\right)\right]\right.$$

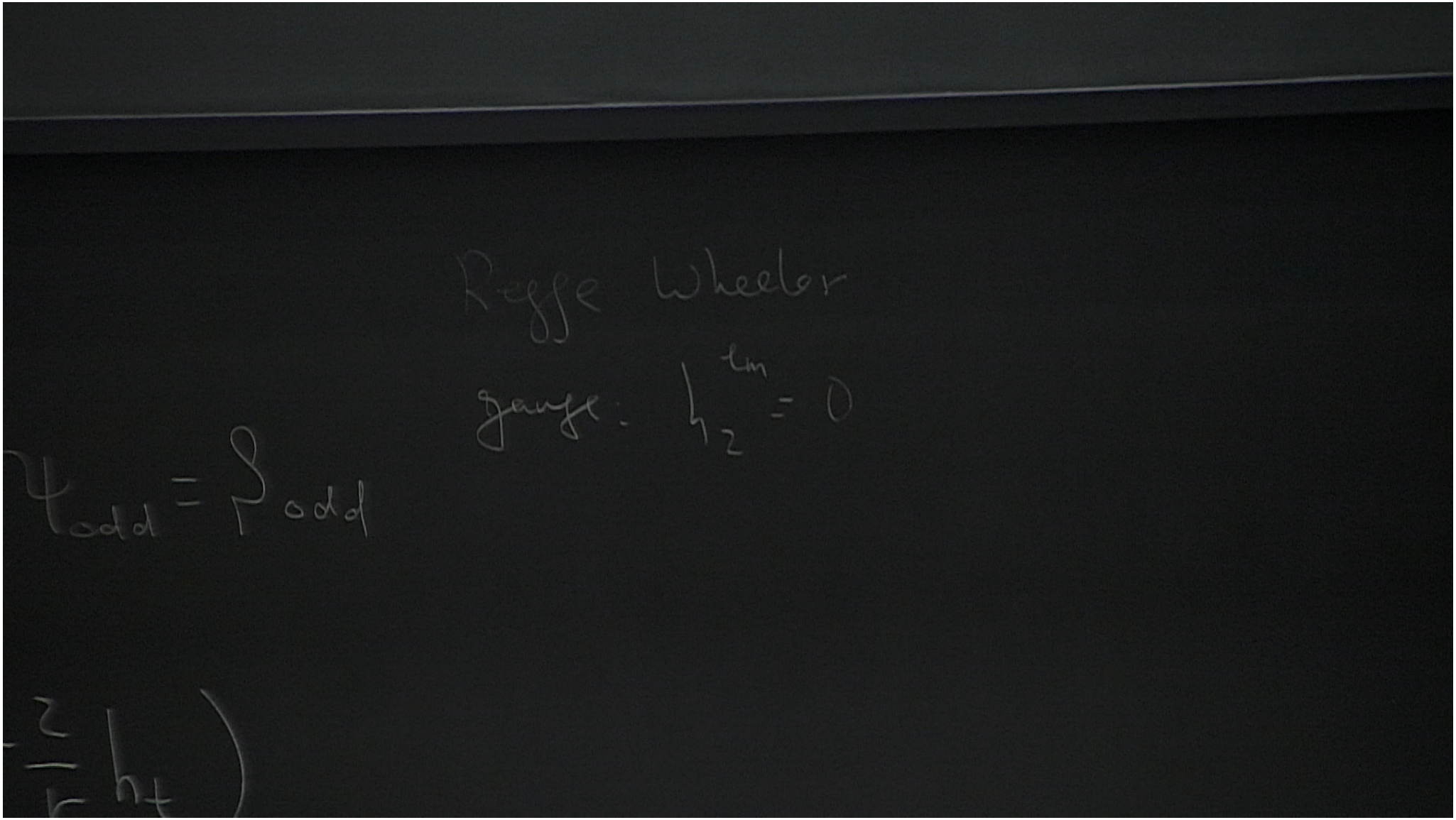
$$f=1-\frac{2M}{r},\quad k=(\ell-1)(\ell+2)+\frac{6M}{r}$$

odd parity:

$$\left[-\frac{\partial^2}{\partial t^2}+f\frac{2}{\partial r}f\frac{\partial}{\partial r}-V_{odd}(r)\right]\psi_{odd}=S_{odd}$$

$$\psi_{odd}=\frac{2r}{(\ell-1)(\ell+2)}\left(\partial_r h_t-\partial_t h_r-\frac{2}{r}h_t\right)$$

$$V_{odd}=f\left[\frac{\ell(\ell+1)}{r^2}+\frac{6M}{r^3}\right)$$


Regge Wheeler
gauge: $h_2^{lm} = 0$
$\psi_{odd} = \rho_{odd}$

equations:

: even parity

ity

even parity:

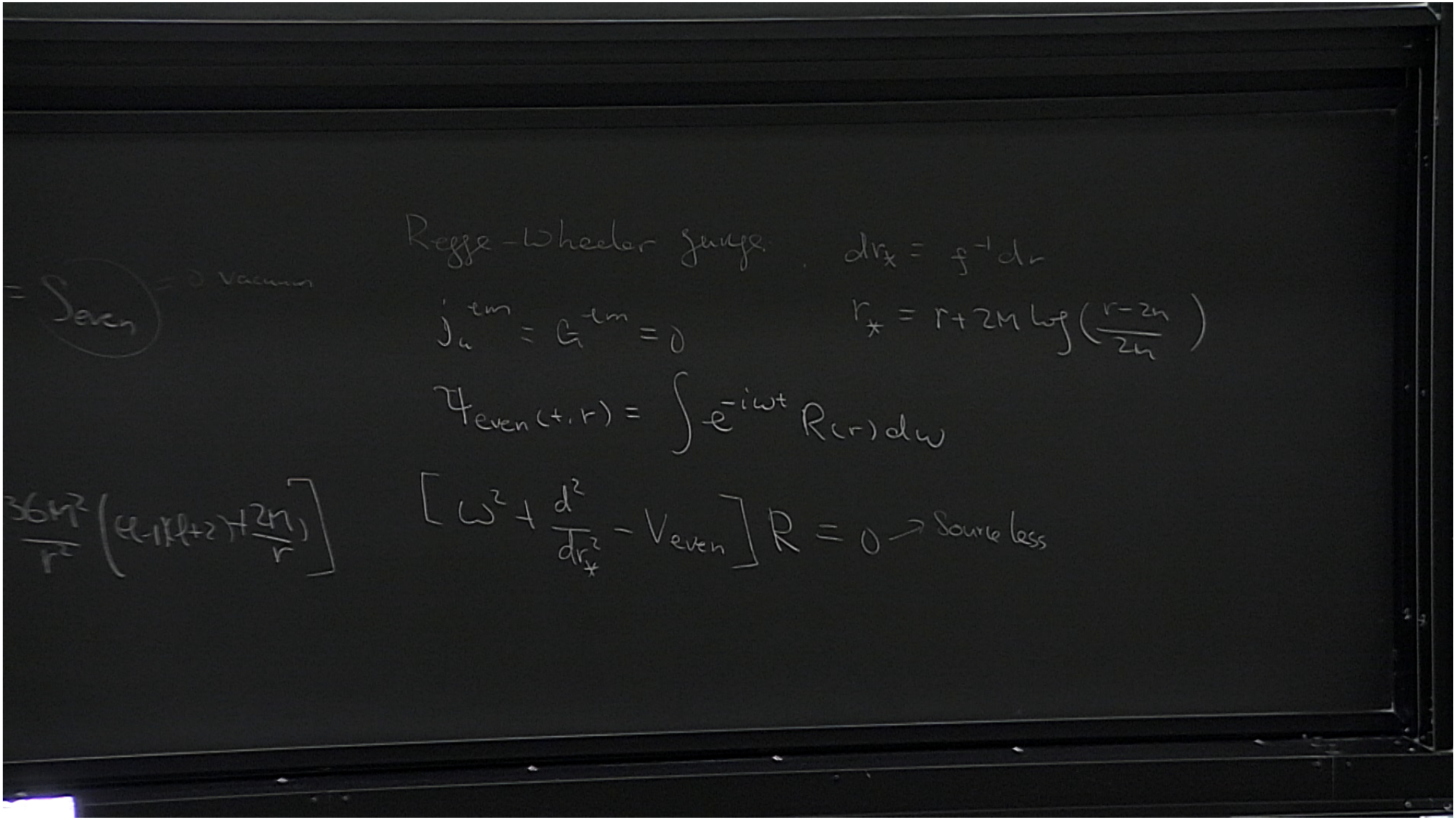
$$\left[-\frac{\partial^2}{\partial t^2} + f\frac{\partial}{\partial r}f\frac{\partial}{\partial r} - V_{even(r)}\right]\Psi_{even}(t,r) = S_{even}$$

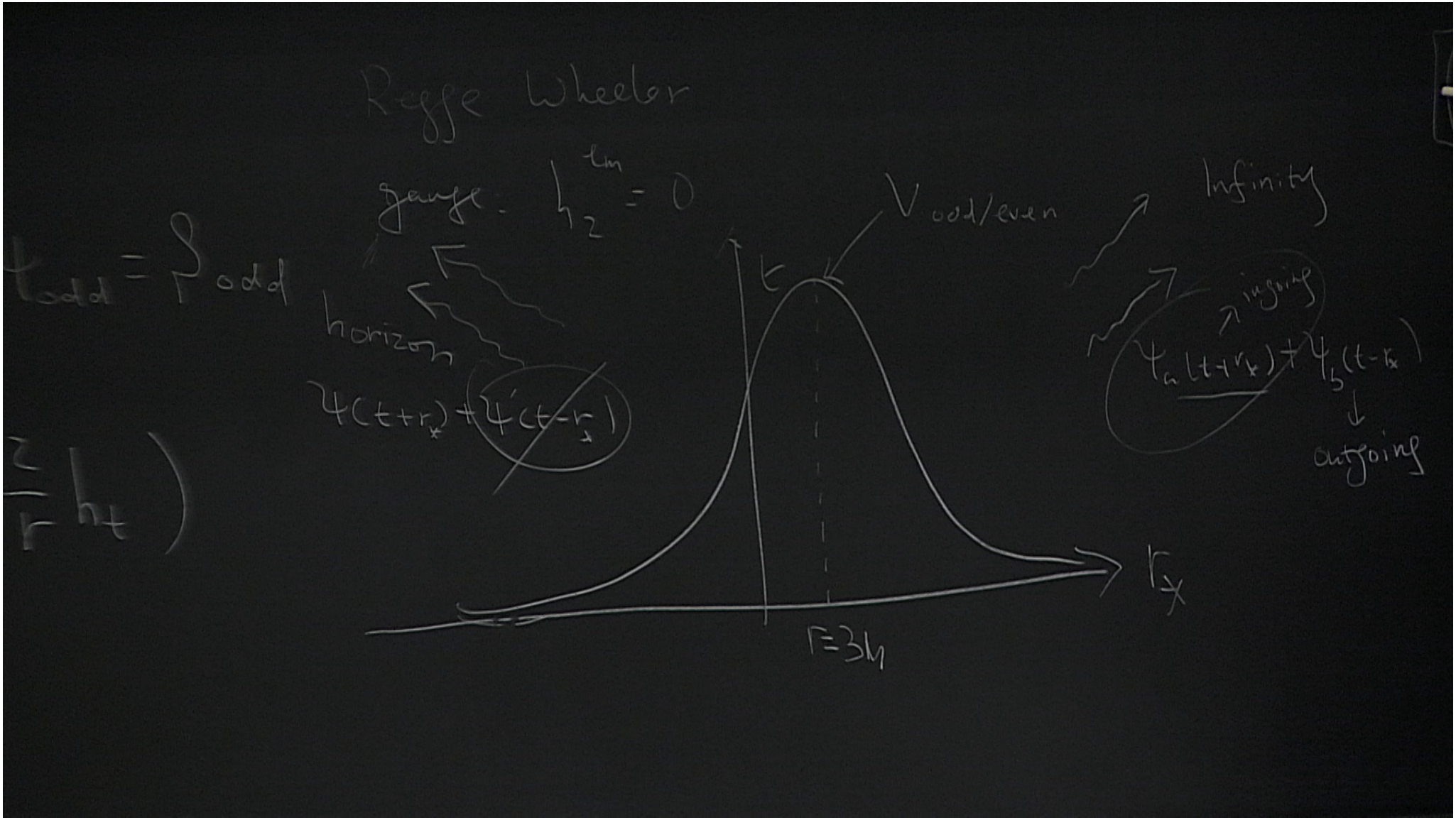
$\frac{\partial}{\partial r} \to \frac{\partial^2}{\partial r_*^2}$

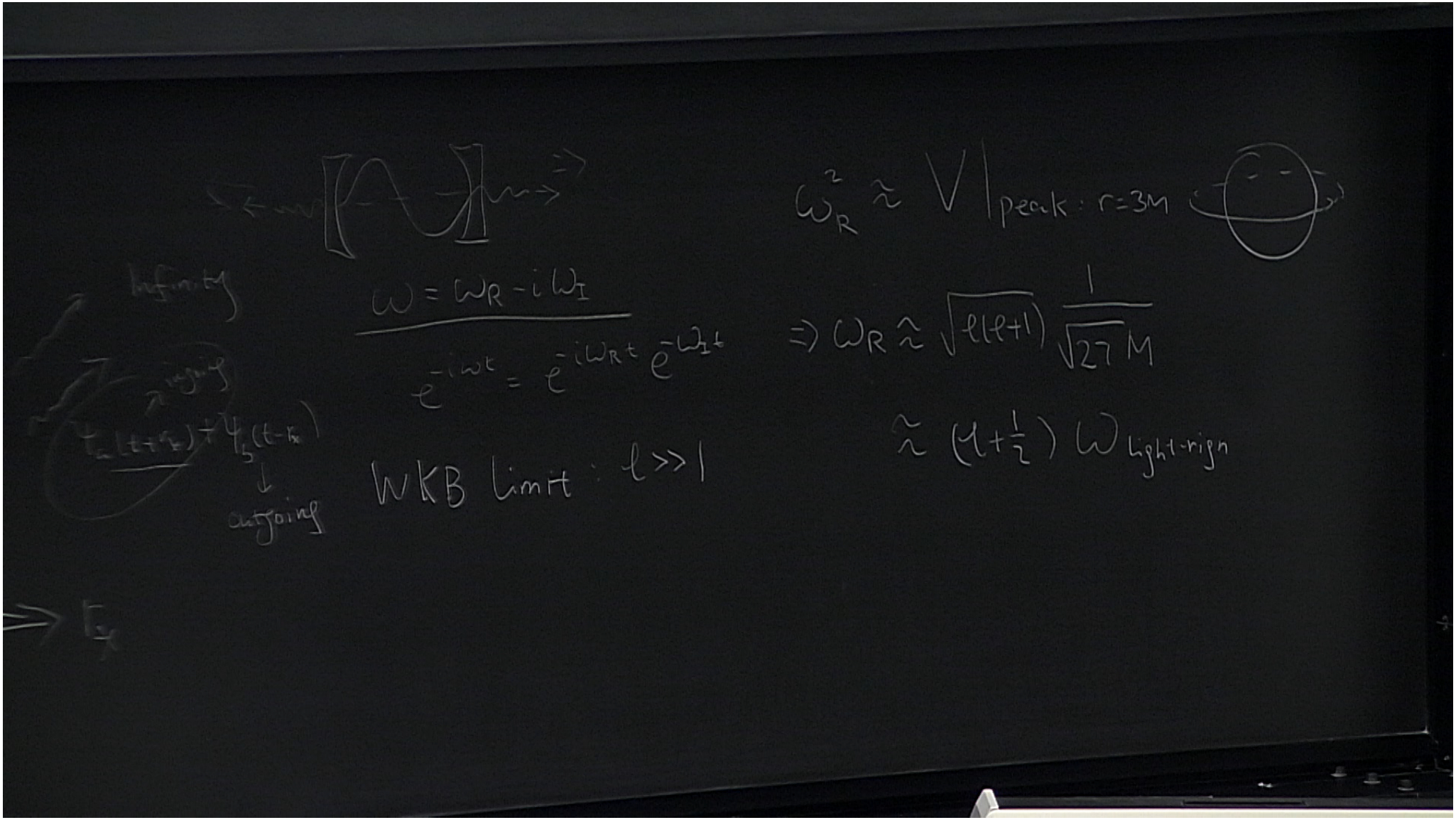
$$\Psi_{even} \equiv \frac{2r}{\ell(\ell+1)}\left[K + \frac{2f}{k}\left(f\, h_{rr} - r\partial_r K\right)\right]$$

$$V_{even} = \frac{f}{k^2}\left[\left[(\ell-1)(\ell+2)^2\left(\frac{\ell(\ell+1)}{r^2} + \frac{6M}{r^3}\right) + \frac{36M^2}{r^2}\left((\ell-1)(\ell+2)+\right.\right.\right.$$

$$f = 1 - \frac{2M}{r}, \quad k = (\ell-1)(\ell+2) + \frac{6M}{r}$$

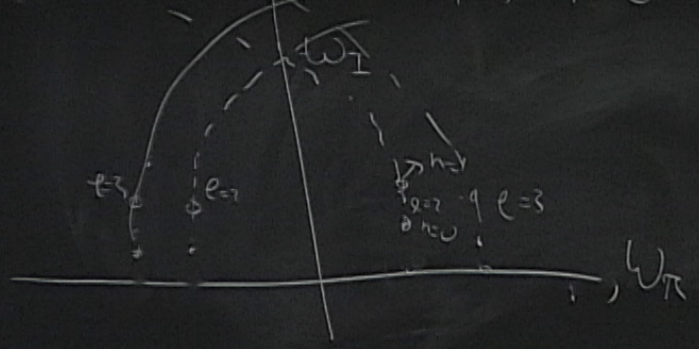

Regge-Wheeler gauge
$dr_* = f^{-1} dr$
$r_* = r + 2M \log\left(\frac{r-2M}{2M}\right)$
$j_\mu^{lm} = G^{lm} = 0$
$\Psi_{even}(t,r) = \int e^{-i\omega t} R(r)\, d\omega$
$\left[\omega^2 + \frac{d^2}{dr_*^2} - V_{even}\right] R = 0 \rightarrow$ Source less
$= S_{even}$
$= 0$ Vacuum
$\frac{36M^2}{r^2}\left(\ell(\ell+1)(\ell+2) + \frac{2M}{r}\right)\Big]$


Regge Wheeler
gauge: $h_2^{tm} = 0$
$V_{odd/even}$
Infinity
horizon
$\psi(t+r_*) + \psi(t-r_*)$
ingoing
$\psi_a(t+r_*) + \psi_b(t-r_*)$
outgoing
r_*
r=3M


$\omega = \omega_R - i\omega_I$
$e^{-i\omega t} = e^{-i\omega_R t} e^{-\omega_I t}$
WKB limit : $l \gg 1$
$\omega_R^2 \approx V|_{\text{peak}: r=3M}$
$\Rightarrow \omega_R \approx \sqrt{l(l+1)}\,\frac{1}{\sqrt{27}M}$
$\approx (l+\frac{1}{2})\,\omega_{\text{light-ring}}$

$$\omega_I = -\left(n+\tfrac{1}{2}\right)\frac{\sqrt{2\frac{d^2V}{dx^2}\Big|_{peak}}}{\frac{\partial V}{\partial \omega}\Big|_{peak}} \quad \text{(WKB)}$$

Leaver PRD 34, 384, 1986



Regge Wheeler gauge. $h_2^{lm} = 0$

