Title: PSI 2018/2019 - Strong Field Gravity - Lecture 11

Speakers: William East

Collection: PSI 2018/2019 - Strong Field Gravity (East)

Date: March 18, 2019 - 9:00 AM

URL: http://pirsa.org/19030018

Gravitational Waves

Weak field regime

$$g_{ab} = \eta_{ab} + h_{ab} \qquad |h_{ab}| \ll |\eta_{ab}|$$

flatspace

$$g^{ab} = \eta^{ab} - h^{ab} \qquad \text{to linear order in } h_{ab}$$

Use harmonic gauge: $\Box x^a = 0$

$$\underbrace{\eta^{cd}\partial_c\partial_d}_{\Box} h_{ab} = -16\pi\left(T_{ab} - \frac{1}{2} T g_{ab}\right)$$

$$\Box\underbrace{\left(h_{ab} - \frac{1}{2}\eta_{ab}\eta^{cd}h_{cd}\right)}_{\bar{h}_{ab}} = -16\pi T_{ab}$$

$\Box x^a = 0 \implies \partial_a \bar{h}^{ab} = 0$ (Lorenz gauge)

In E.M. $A_a \to A_a + \partial_a \phi$ $\quad \partial_a \partial^a \phi = 0$

leave $\partial_a A^a = 0$

Here : $x^a \to x^a + \xi^a$ $\quad \Box \xi^a = 0$ keeps harmonic gauge

Propagation of GWs

In the wavezone use 4 gauge DOF to set

$$h^{TT}_{0i} = 0 \quad \text{and} \quad \eta^{ab} h^{TT}_{ab} = 0 \ \text{(traceless)}$$

$$h^{TT}_{ab} = \bar{h}_{ab}$$

$$0 = \partial_a h^{ab}_{TT} = -\partial_0 h^{00}_{TT} + \cancel{\partial_i h^{i0}_{TT}}^{\,0} \qquad \partial_i \partial^i h^{00} = 0$$

Choose $h^{00} = 0$ at $t = 0$ $\Rightarrow$ $h^{00} = 0$ for all t

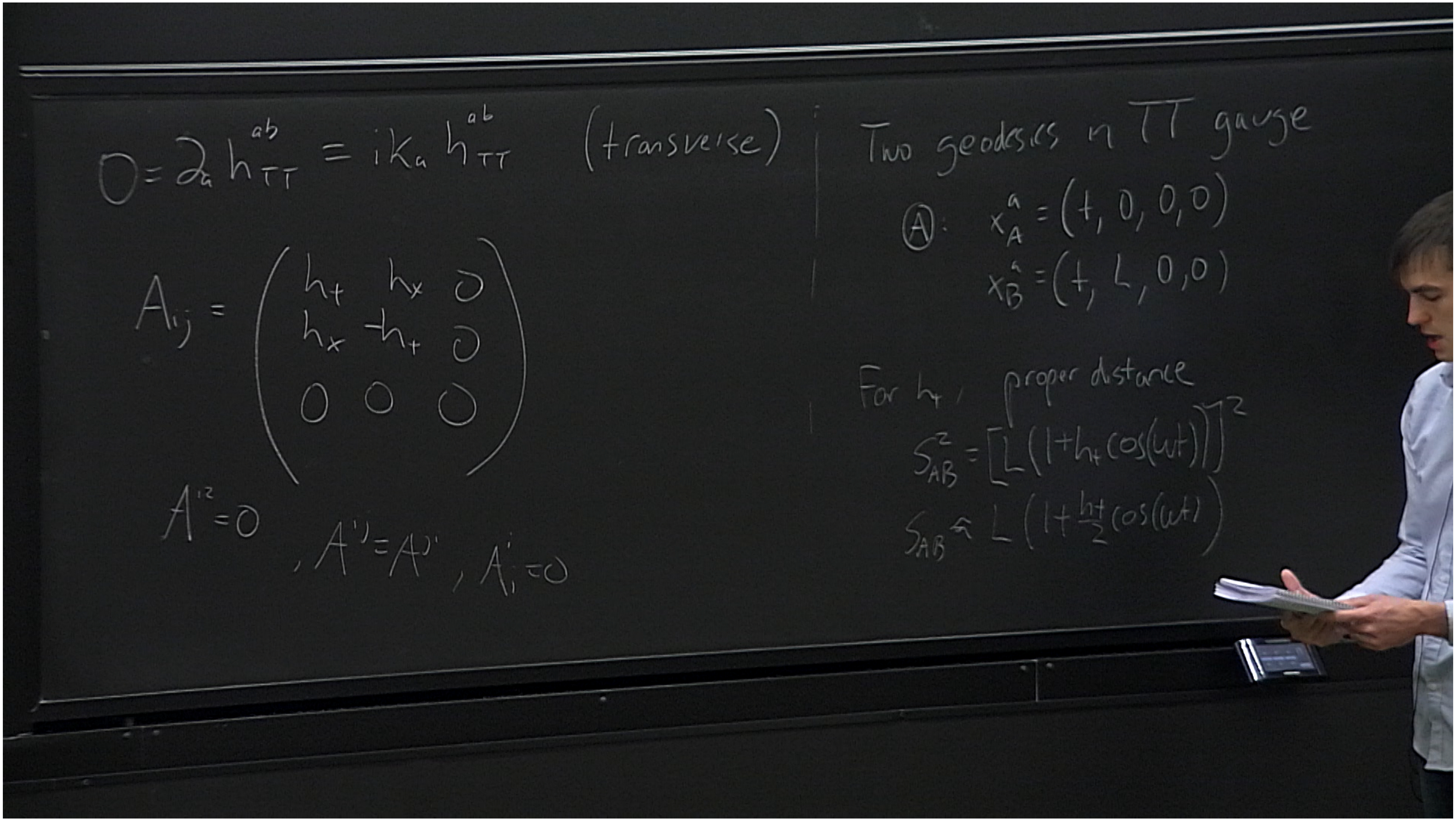
$h_{ij}^{TT} = A_{ij} \exp(i k_a x^a)$ (plane waves)

Constant

$$0 = \Box h_{ij}^{TT} = \eta^{ab}(ik_a)(ik_b)\, h_{ij}^{TT} = -k^a k_a\, h_{ij}^{TT}$$

Hence want $k^a k_a = 0$ (null)

Choose $k^a = (\omega, 0, 0, \omega)$ (propagates in z direction)


$0 = \partial_a h^{ab}_{TT} = i k_a h^{ab}_{TT}$ (transverse)
$A_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$A^{i2} = 0$, $A^{ij} = A^{ji}$, $A^i_i = 0$
Two geodesics in TT gauge
A: $x^a_A = (t, 0, 0, 0)$
$x^a_B = (t, L, 0, 0)$
For h_+, proper distance
$S^2_{AB} = [L(1 + h_+ \cos(\omega t))]^2$
$S_{AB} \approx L(1 + \frac{h_+}{2} \cos(\omega t))$

Rotate by ϕ about z-axis

$$h_\times \pm i h_+ \to (h_\times \pm i h_+) e^{\mp 2 i \phi}$$

Spin 2

Generation of GWs in the weak field regime

$$\Box \bar{h}_{ab} = -16\pi T_{ab}$$

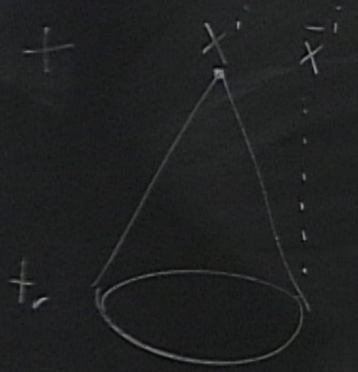
Green function: $\Box G(x^a - \bar{x}^a) = \delta^4(x^a - \bar{x}^a)$

$$G(x^a - \bar{x}^a) = \frac{-1}{4\pi |x^i - \bar{x}^i|} \delta\left[|x^i - \bar{x}^i| - (x^0 - \bar{x}^0) \right] \Theta(x^0 - \bar{x}^0)$$

$$\Theta(x^0 - \bar{x}^0) = \begin{cases} 1 & \text{when } x^0 - \bar{x}^0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{h}_{ab}(x^a) = -16\pi \int G(x^a - \bar{x}^a)\, T_{ab}(\bar{x}^a)\, d^4\bar{x}$$

$$= 4 \int \frac{T_{ab}\big(\underbrace{t - |x^i - \bar{x}^i|}_{t_r},\ \bar{x}^i\big)}{|x^i - \bar{x}^i|}\, d^3\bar{x}$$



Transform to fourier space

$d^3\bar{x}$

$$\tilde{\bar{h}}_{ab} = \int dt\, e^{i\omega t}\, \bar{h}_{ab}$$

$$= 4\int dt_r \int d^3\bar{x}\, \frac{T_{ab}(t_r, x')}{|x' - \bar{x}'|}\, e^{i\omega(t_r - |x' - \bar{x}'|)}$$

$$= 4\int d^3\bar{x}\, \frac{\tilde{T}_{ab}}{|x' - \bar{x}'|}\, e^{i\omega|x' - \bar{x}'|}$$

Lorenz: $\tilde{h}^{0a} = \frac{i}{\omega}\partial_i \tilde{h}^{ia}$

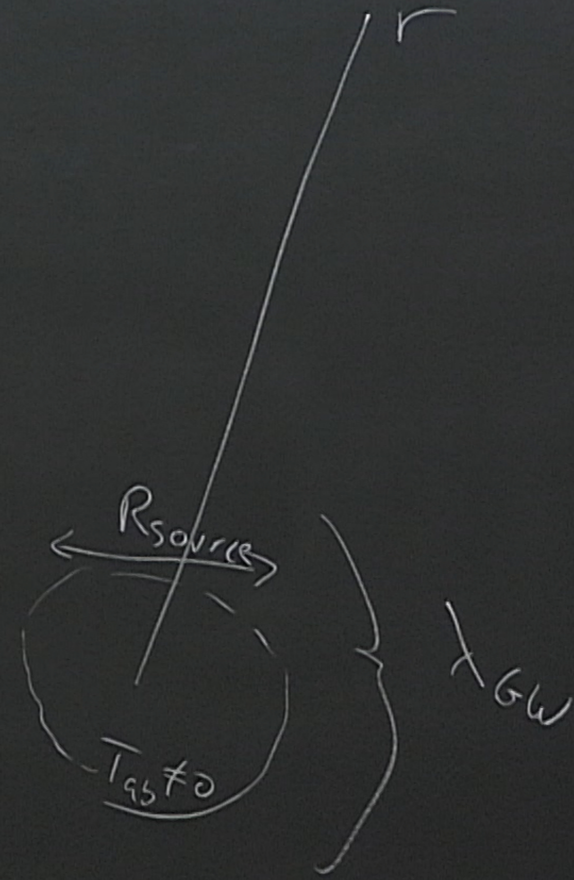
$$\frac{\ldots x')}{\ldots \bar{x}'|} e^{i\omega(t_r - |x' - \bar{x}'|)}$$

$$\ldots |x' - \bar{x}'|$$

Assume

$r \gg \frac{1}{\omega}$ (wavezone)

$R_{source} \ll \lambda_{GW} = \frac{1}{\omega}$



$$\frac{e^{i\omega|x'-\bar{x}'|}}{|x-\bar{x}'|} \approx \frac{e^{i\omega r}}{r}$$

Need to calculate

$$\int d^3\bar{x}\,\tilde{T}^{ij} = \int \Big[\underbrace{\cancel{\partial_k\left(\bar{x}^i\,\tilde{T}^{kj}\right)}}_{\text{total integral}}^{\to 0} - \underbrace{\partial_k\left(\tilde{T}^{kj}\right)}_{= i\omega \tilde{T}^{0j}}\bar{x}^i\Big]\, d^3\bar{x}$$

$$\int d^3\bar{x}\, \tilde{T}^{ij} = \frac{i\omega}{2}\int d^3\bar{x} \left(\tilde{T}^{0j}\bar{x}^i + \tilde{T}^{i0}\bar{x}^j \right)$$

$$= \frac{i\omega}{2}\int \left[\partial_l \left(\bar{x}^i \bar{x}^j \tilde{T}^{0l} \right) - \bar{x}^i \bar{x}^j \partial_l \left(\tilde{T}^{0l} \right) \right] d^3\bar{x}$$

$$= \frac{-\omega^2}{2} \underbrace{\int \bar{x}^i \bar{x}^j \tilde{T}^{00} d^3\bar{x}}_{\tilde{I}^{ij}}$$

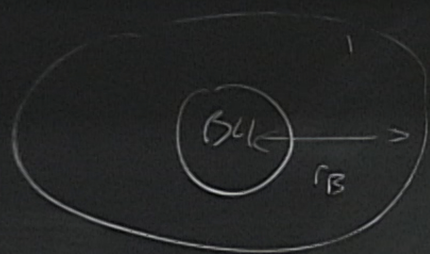
(quadrapole moment tensor)

$$\tilde{\bar{h}}_{ij} = -2\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}$$

$$\bar{h}_{ij} = \frac{2}{r} \ddot{I}_{ij}(t-r)$$

Monopole: $\int d^3\bar{x}\, T^{00}$

Dipole: $D^i(t) = \int d^3\bar{x}\, T^{00}\, \bar{x}^i$

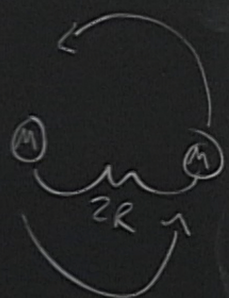


$$r_B = (\mu\alpha)^{-1} \qquad \alpha = M\mu \ll 1$$

$$\omega \simeq \mu$$

$$\omega_{GW} = 2\omega \simeq 2\mu$$

$$R_{source} \not\gg \frac{1}{\omega}$$



$$\omega_{GW} = 2\omega = \frac{4\pi}{T} = \frac{1}{R}\sqrt{\frac{M}{R}}$$

Energy and angular momentum in GWs

$$g_{ab} = \eta_{ab} + h^{(1)}_{ab} + h^{(2)}_{ab}$$

$$R_{ab} = \cancel{R^{(0)}_{ab}}^{\,\to 0} + R^{(1)}_{ab} + R^{(2)}_{ab}$$

First order: $G^{(1)}_{ab}[h^{(1)}_{ab}] = 0$

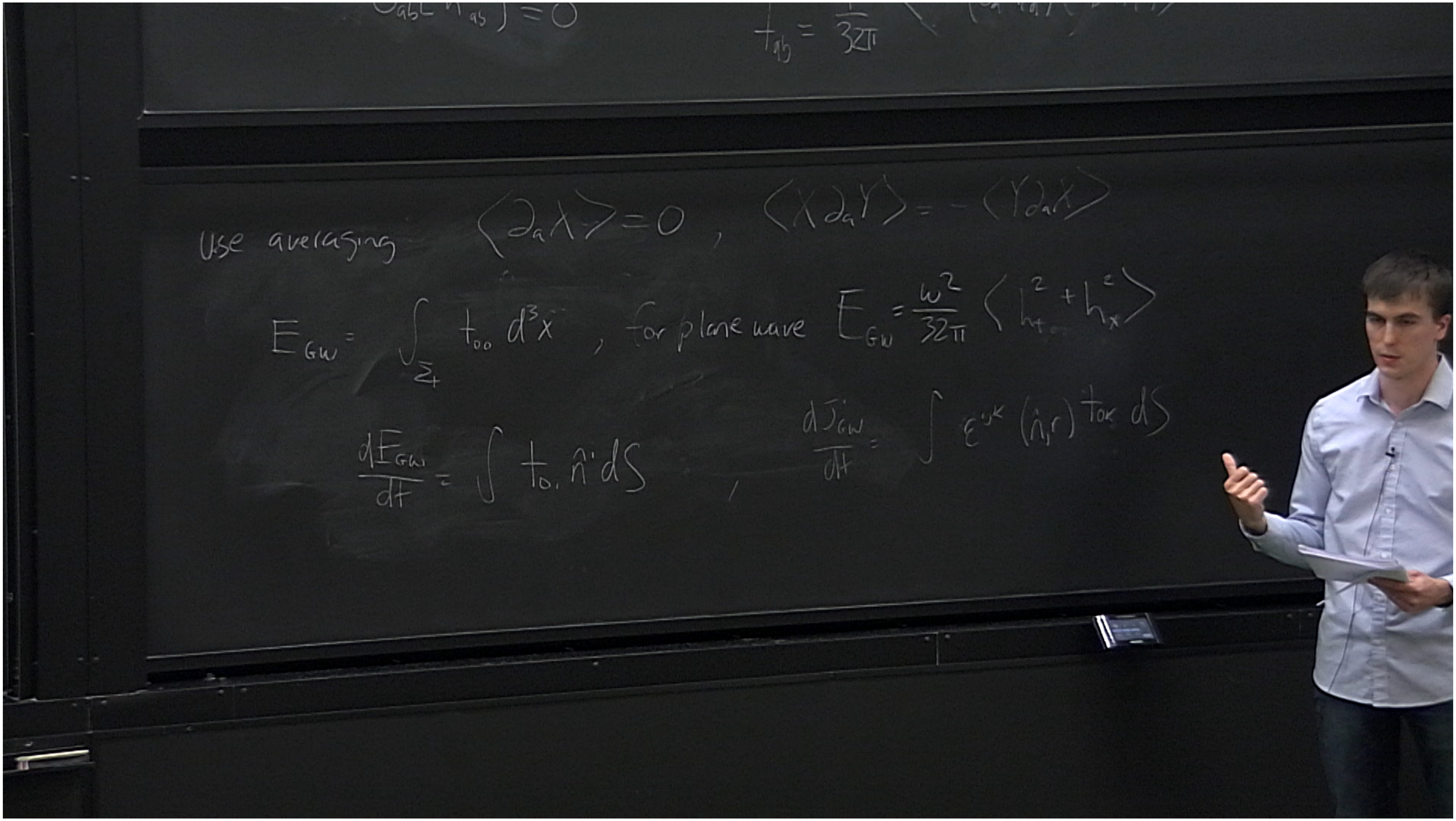
Second order $G^{(1)}_{ab}[h^{(2)}_{cd}] + G^{(2)}_{ab}[h^{(1)}_{ab}] = 0$

(linear) (quadratic)

$$G^{(1)}_{ab}[h^{(2)}_{cd}] = 8\pi t_{ab} := -G^{(2)}_{ab}[h^{(1)}_{cd}]$$

Caution: t_{ab} not gauge invariant

$$t_{ab} = \frac{1}{32\pi}\left\langle (\partial_a h^{TT}_{cd})(\partial_b h^{cd}_{TT})\right\rangle$$


$t_{ab} = \frac{1}{32\pi}$
Use averaging $\langle \partial_a X \rangle = 0$, $\langle X \partial_a Y \rangle = -\langle Y \partial_a X \rangle$
$E_{GW} = \int_{\Sigma_t} t_{00}\, d^3x$, for plane wave $E_{GW} = \frac{\omega^2}{32\pi} \langle h_+^2 + h_\times^2 \rangle$
$\frac{dE_{GW}}{dt} = \int t_{0i}\, \hat{n}^i\, dS$, $\frac{dJ^i_{GW}}{dt} = \int \varepsilon^{ijk} (\hat{n}_j r)\, t_{0k}\, dS$

$$\frac{dE_{GW}}{dt} = \frac{1}{5}\left\langle \left(\dddot{\bar{I}}_{ij}\, \dddot{\bar{I}}^{ij} \right) \right\rangle \Big|_{t_r}$$

$$\bar{I}_{ij} = I_{ij} - \frac{1}{3} I^{k\ell}\, \delta_{k\ell}\, \delta_{ij}$$

$$\frac{dJ^{i}_{GW}}{dt} = \frac{2}{5}\, \varepsilon^{ijk} \left\langle \dddot{\bar{I}}_{\ell j}\, \ddot{\bar{I}}_{k}{}^{\ell} \right\rangle$$