Title: PSI 2018/2019 - Strong Field Gravity - Lecture 8

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Collection: PSI 2018/2019 - Strong Field Gravity (East)

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$$u_{,t} = \sum_j A_j D_j u \equiv P(D)\, u$$

Ex $u_{,t} = \lambda u$

with $u(x,t) = e^{i\vec{\omega}\vec{x}}\, \hat{u}(\omega, t)$

$$\hookrightarrow \hat{u}_t = i|\omega| \sum A_j \omega_j \hat{u} \equiv i|\omega|\, P(\omega')\, \hat{u}$$

$$\Rightarrow \hat{u}_t = i|\omega|\left(\lambda \frac{\omega}{|\omega|}\right)\hat{u} \Rightarrow \hat{u} = e^{i|\omega|\left(\lambda \frac{\omega}{|\omega|}\right)t}$$

$\underline{Ex}$ $\quad u_{,t} = \lambda u_{,x} \Rightarrow \hat{u}_{,t} = i|\omega|\left(\lambda \frac{\omega}{|\omega|}\right)\hat{u} \Rightarrow \hat{u} = e^{i|\omega|\left(\lambda \frac{\omega}{|\omega|}\right)t}$

$\lambda \to \underline{\text{Real}}$

$\Rightarrow e^{\pm \omega t}$

$u_{,t} = A^x \partial_x u$

Ex $u_{,t} = \lambda u_{,x} \Rightarrow \hat{u}_{,t} = i|\omega|\left(\lambda \frac{\omega}{|\omega|}\right)\hat{u} \Rightarrow \hat{u} = e^{i|\omega|\left(\lambda \frac{\omega}{|\omega|}\right)t}$

$\lambda \to$ Real

$\pm \omega t$

$\Rightarrow e^{\pm \omega t}$

$T^{-1} u_{,t} = \sqrt{A}\, \partial_x u$

$\sqrt{A} = T T^{-1}$

$\exists\, T \ni T^{-1} A T = D$

$T^{-1} T = \mathbb{1}$

$T^{-1} u_{,t} = T^{-1} A T\, T^{-1} \partial_x u$

$T^{-1} u_{,t} = D\, T^{-1} u_{,x}$

$T^{-1} u_{,a} = (T^{-1} u)_{,a} - (T^{-1})_{,a} u$

$$V \equiv T^{-1} U \qquad \boxed{V_{,t} = D\, V_{,x}} + \text{Undiff. } U \text{ terms}$$

$$\rightarrow \text{Collection of } v^{(i)}_{,t} = \lambda^{(i)} v^{(i)}_{,x}$$

$$u_{,tt} = P_0(u) + P_1 u_{,t}$$

with $P_0 \equiv \sum_{J,k} A_{Jk} \partial_J \partial_k$

$$P_1 \equiv \sum_J A_J \partial_J$$

$$u_{,tt} = u_{,xx}$$

$f \equiv u_{,x}$, $g \equiv u_{,t}$

$$\begin{cases} g_{,t} = f_{,x} \\ f_{,t} = g_{,x} \\ u_{,t} = g \end{cases}$$

$\boxed{v_{,t} = D\, v_{,x}} + \text{Diff. } O \text{ terms}$

$\rightarrow$ Collection of $v^{(i)}_{,t} = \lambda^{(i)} v^{(i)}_{,x}$

ADM $\quad \partial_t \gamma_{ij} \; - -$

$\partial_t K_{ij} = -R_{ij}$

with $P_0 = \sum_{j,k} A_{jk}\, \partial_j \partial_k$

$P_1 = \sum_j A_j\, \partial_j$

$f = u_{,x}\,, \quad g = u_{,t}$

$$\begin{cases} g_{,t} = f_{,x} \\ f_{,t} = g_{,x} \\ u_{,t} = g \end{cases}$$

$u(x,t) = e^{i\vec{\omega}\vec{x}}\, \hat{u}(\omega, t)$

$\hat{u}(x,t) \equiv -|\omega|^2 P_0(\omega')\, \hat{u} + i\,|\omega|\, P_1(\omega')\, \hat{u}_t \qquad$ with $\omega' = \frac{\omega}{|\omega|}$

$\longrightarrow$ A nec cond for well posedness is $\left[-k^2 I + P_1(\omega') k + P_0(\omega')\right]$ has all eigenvals real

define: $\hat{u}_t = i|\omega|\, \hat{v}$

$$\hookrightarrow \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}_{,t} = |\omega|\, i \begin{pmatrix} P_1 & P_0 \\ 1 & 0 \end{pmatrix} \equiv i|\omega|\, \hat{P} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$$

$\hat{P}$: $\begin{cases} \text{real eigenvals} \\ \& \\ \text{comp. set of linearly indep eigenvectors} \end{cases}$

$$u(x,t) = e^{i\vec{\omega}\vec{x}}\,\hat{u}(\omega,t)$$

$$\hat{u}(x,t) \equiv -|\omega|^2 P_2(\omega')\,\hat{u} + i\,|\omega|\,P_1(\omega')\,\hat{u}_t \qquad \text{with } \omega' =$$

$\longrightarrow$ A nec. cond for well posedness is $\left[-k^2 I + P_1(\omega')k + P_0\right.$

<u>define</u>: $\hat{u}_t = i|\omega|\,\hat{v}$

$$\hookrightarrow \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}_t = |\omega|\, i \begin{pmatrix} P_1 & P_0 \\ \mathbb{1} & 0 \end{pmatrix} \equiv$$

$\frac{w}{|w|}$

$s(w)]]$ has all eigenvals real

$i|w| \hat{P}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix}$

$\hat{P}$: $\begin{cases} \text{real eigenvals} \\ \& \\ \text{comp. set of linearly indep eigenvectors} \end{cases}$

If $P_1 = 0$ $\rightarrow$ strong hyperbolicity when eigenvalues of P_0 are positive & real and $\exists$. complete set of eigenvectors.

$$u_{,tt} = \lambda u_{,xx}$$

$$= \lambda (i\omega)^2 u$$

$$u_{,tt} = -\lambda \omega^2 u$$

$$\pm i\sqrt{\lambda}\omega$$

$$u = e^{st}$$

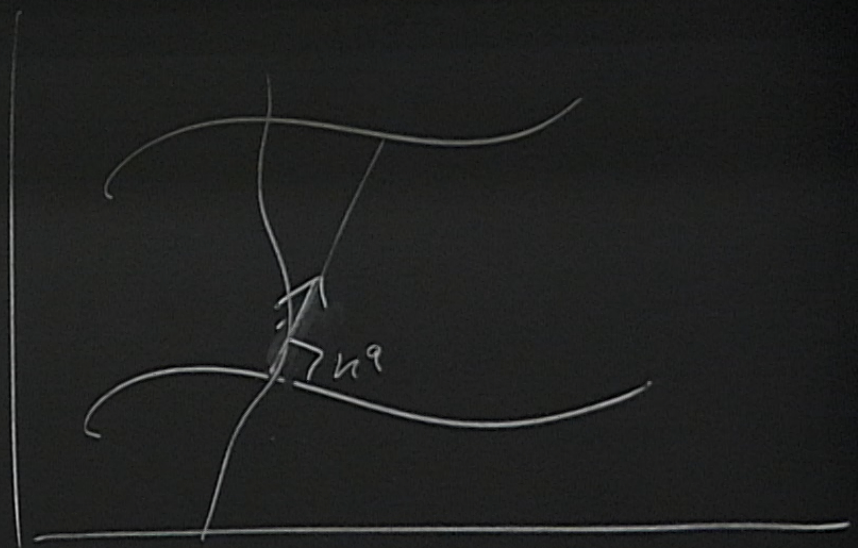
$$s^2 = -\lambda \omega^2$$

s will be purely imaginary if $\lambda > 0$

has all eigenvals real

$\hat{P}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix}$

$\hat{P}$: $\left\{\begin{array}{l}\text{real eigenvals}\\ \quad \& \\ \text{comp. set of linearly indep eigenvectors}\end{array}\right.$



$$\partial_t \gamma_{ij} = \gamma^{lm}\left(\partial_l \partial_m \gamma_{ij} + \partial_i \partial_j \gamma_{lm} - \partial_i \partial_l \gamma_{mj} - \partial_j \partial_l \gamma_{mi}\right)$$

$$\partial_t h_{ij} = \delta^{lm}\left(\partial_l \partial_m h_{ij} + \partial_i \partial_j h_{lm} - \partial_i \partial_l h_{mj} - \partial_j \partial_l h_{mi}\right)$$

$$= \Delta h_{ij} + \partial_j\left(\partial_i H - \delta^{lm} \partial_l h_{mi}\right) - \partial_i \partial_j H$$

$\pm i\sqrt{\lambda\omega}$

$$\partial_{tt}\gamma_{ij} = \gamma^{lm}\left(\partial_l\partial_m\gamma_{ij} + \partial_i\partial_j\gamma_{lm} - \partial_i\partial_l\gamma_{mj} - \partial_j\partial_l\gamma_{mi}\right)$$

$$\partial_{tt}h_{ij} = \delta^{lm}\left(\partial_l\partial_m h_{ij} + \partial_i\partial_j h_{lm} - \partial_i\partial_l h_{mj} - \partial_j\partial_l h_{mi}\right)$$

$$\boxed{\partial_{tt}h_{ij} = \partial^k\partial_k h_{ij}} + \partial_j\left(\partial_i H - \cancel{\delta^{lm}\partial_l h_{mi}}\right) - \partial_i\partial_j H \qquad \text{with } H = \delta^{lm}h_{lm}$$

$$0$$

$\rightarrow P_0$

$\lambda_{(1)} = 0$

$\lambda_{(5)} = 1 \rightarrow$ degenerate

with $H = \delta^{lm} \partial_{l} \partial_{m}$

$\alpha = 1$ $\qquad \gamma_{ij} = \delta_{ij} + \varepsilon h_{ij}$ $\qquad K_{ij} \propto \partial_t \gamma_{ij} + \mathcal{L}_\beta(\gamma_{ij})$

$\beta^i = 0$

$$\partial_{tt} \gamma_{ij} = \gamma^{lm} \left(\partial_l \partial_m \gamma_{ij} + \partial_i \partial_j \gamma_{lm} - \partial_i \partial_l \gamma_{mj} - \partial_j \partial_l \gamma_{mi} \right)$$

$$\partial_{tt} h_{ij} = \delta^{lm} \left(\partial_l \partial_m h_{ij} + \partial_i \partial_j h_{lm} - \partial_i \partial_l h_{mj} - \partial_j \partial_l h_{mi} \right)$$

$$\boxed{\partial_{tt} h_{ij} = \partial_k \partial_k h_{ij}} + \partial_j \left(\partial_i H - \cancel{\delta^{lm} \partial_l h_{mi}} \right) - \partial_i \partial_j H$$

with $H = \delta^{lm} h_{lm}$

0

3+1 decomposition : $\nabla_a F^{ab} = -4\pi J^b$

$$F_{ab} = \left(\cancel{F_{cd} n^c n^d}\right)^{\to 0} n_a n_b + F_{dc} h^d_a n^c n_b + F_{dc} h^c_b n^d n_a + F_{cd} h^c_a h^d_b$$

$$= -E_a n_b + E_b n_a + {}^{(3)}\epsilon_{abc} B^c$$

where $E_a \equiv F_{ab} n^b$

$B_a = {}^*F_{ab} n^b$

$$n_b \nabla_a F^{ab} = \nabla_a (F^{ab} n_b) - F^{ab} \nabla_a n_b \qquad -n_a J^a \equiv \rho.$$

$$= \nabla_a (E^a) + F^{ba} (K_{ab} - a_b n_a).$$

$$= \nabla_a E^a - E^b a_b \longrightarrow \boxed{\nabla_a E^a = 4\pi\rho}$$

$$\nabla_a E^a \equiv g^{ab} \nabla_a E_b$$

$$\equiv (\gamma^{ab} - n^a n^b) \nabla_a E_b = \nabla_a E^a - n^a n^b \nabla_a E_b$$

$$= \nabla_a E^a - n^a \nabla_a (\cancel{E_b n^b}) + E_b \overbrace{n^a \nabla_a n^b}^{a^b}$$

$$\partial_t K_{ij} = R_{ij}$$

$$u(x,t) = e^{i\vec{\omega}\vec{x}} \hat{u}(\omega, t)$$

ρ.

$$= 4\pi\rho$$

$$h^{c}_{b}\left(\nabla_a F^{ab} = -4\pi j^{b} \right)$$

n^b