Title: PSI 2018/2019 - Strong Field Gravity - Lecture 7

Speakers: William East

Collection: PSI 2018/2019 - Strong Field Gravity (East)

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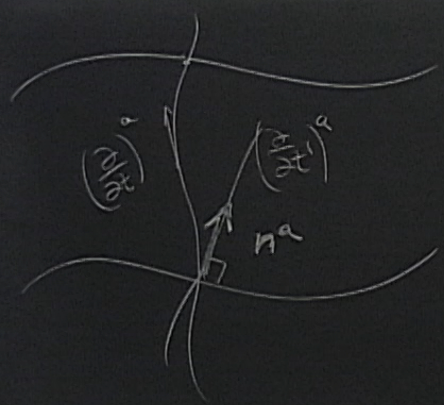


Ham Const. ${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho$

Mom Const. $D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi J^i$

$\rho = n^a n^b T_{ab}$

$J^i \equiv -\gamma^{ia} n^b T_{ab}$

Last Projection $\gamma^\alpha{}_a \gamma^\delta{}_b n^\beta n^\mu R_{\alpha\beta\delta\mu} = \mathcal{L}_{\vec{n}} K_{ab} + K_{ac}K^c{}_b + \frac{1}{\alpha} D_a D_b \alpha$

$$\gamma_\mu{}^\delta \gamma_\nu{}^\kappa \left[n^\lambda n^\sigma R_{\delta\lambda\kappa\sigma} + \underbrace{g^{\lambda\sigma} R_{\delta\lambda\kappa\sigma}}_{R_{\delta\kappa}} \right] = \gamma_\mu^\delta \gamma_\nu^\kappa \gamma^{\lambda\sigma} R_{\delta\lambda\kappa\sigma}$$

$$= {}^3R_{\mu\nu} + K K_{\mu\nu} - K_{\mu\lambda} K^\lambda{}_\sigma$$

$$\Updownarrow$$

$$8\pi \left(T_{\delta\kappa} - \frac{1}{2} g_{\delta\kappa} T \right)$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

$$= 8\pi T_{ab}$$

$$\rightarrow R_{ab} = \left(T_{ab} - \frac{1}{2} g_{ab} T \right) 8\pi$$

$$\mathcal{L}_t K_{ab} - \mathcal{L}_\beta K_{ab} = -D_\alpha D_\beta K$$

$$+ \alpha \left[-\gamma^\delta{}_a \gamma^\kappa{}_b R \right.$$

$$n^a = \frac{1}{\alpha}\left(\partial_t^a - \beta^a\right)$$

$$+ \underbrace{g^{\lambda\sigma} R_{\delta\lambda\kappa\sigma}}_{R_{\delta\kappa}} \Big] = \gamma^{\delta}_{\mu}\, \gamma^{\kappa}_{\nu}\, \gamma^{\lambda\sigma} R_{\delta\lambda\kappa\sigma}$$

$$R_{\delta\kappa} = {}^{3}R_{\mu\nu} + K K_{\mu\nu} - K_{\mu\lambda} K^{\lambda}{}_{\sigma}$$

$$\Updownarrow$$

$$8\pi\left(T_{\delta\kappa} - \frac{1}{2} g_{\delta\kappa} T\right)$$

$$\mathcal{L}_{\vec{t}} K_{ab} - \mathcal{L}_{\vec{\beta}} K_{ab} = - D_a D_\beta K + \alpha\left[-\gamma^{\delta}_{a}\gamma^{\kappa}_{b} R_{\delta\kappa} + {}^{(3)}R_{ab} + K K_{ab} - 2 K_{a\lambda} K^{\lambda}{}_{b}\right]$$

$$\gamma_\mu^{\ \delta}\, \gamma_\nu^{\ k}\, \gamma^{\lambda\sigma}\, R_{\delta\lambda k\sigma}$$

$$= {}^3R_{\mu\nu} + K K_{\mu\nu} - K_{\mu\lambda} K^\lambda{}_\sigma$$

$$\left(\ldots \partial_k{}^T\right)$$

$$\mathcal{L}_{\vec{t}} K_{ab} - \mathcal{L}_{\vec{\beta}} K_{ab} = - D_\alpha D_\beta K + \alpha\left[-\gamma^\delta{}_a \gamma^k{}_b R_{\delta k} + {}^{(3)}R_{ab} + K K_{ab} - 2 K_{a\lambda} K^\lambda{}_b\right]$$

$$\Rightarrow \mathcal{L}_{\vec{t}} K_{ab} - \mathcal{L}_{\vec{\beta}} K_{ab} = - D_\alpha D_\beta K + \alpha\left[{}^{(3)}R_{ab} + K K_{ab} - 2 K_{a\lambda} K^\lambda{}_b\right] + 4\pi\alpha\left[\gamma_{ab}(S-\rho) - 2 S_{ab}\right]$$

$$- \underbrace{g^{\lambda\sigma} R_{\delta\lambda\kappa\sigma}}_{R_{\delta\kappa}} \Big] = \gamma^{\delta}_{\mu} \gamma^{\kappa}_{\nu} \gamma^{\lambda\sigma} R_{\delta\lambda\kappa\sigma}$$

$$= {}^{3}R_{\mu\nu} + K K_{\mu\nu} - K_{\mu\lambda} K^{\lambda}_{\sigma}$$

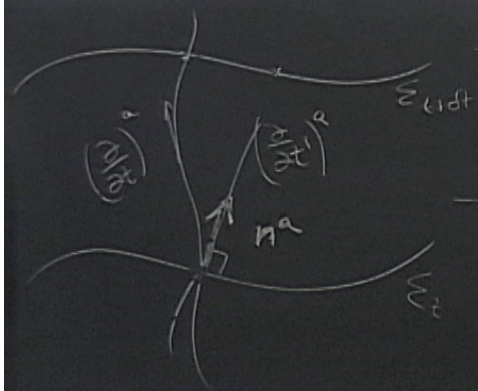
$$\Downarrow$$

$$8\pi \left(T_{\delta\kappa} - \frac{1}{2} g_{\delta\kappa} T \right)$$

$$\boxed{\begin{aligned} S_{ab} &\equiv \gamma^{\alpha}_{a} \gamma^{\beta}_{b} T_{\alpha\beta} \\ S &\equiv \gamma^{ab} S_{ab} \end{aligned}}$$

$$\mathcal{L}_{\vec{t}} K_{ab} - \mathcal{L}_{\vec{\beta}} K_{ab} = - D_{\alpha} D_{\beta} K + \alpha \left[- \gamma^{\delta}_{a} \gamma^{\kappa}_{b} R_{\delta\kappa} + {}^{(3)}R_{ab} + K K_{ab} - 2 K_{a\lambda} K^{\lambda}_{b} \right]$$

$$\Rightarrow \mathcal{L}_{\vec{t}} K_{ab} - \mathcal{L}_{\vec{\beta}} K_{ab} = - D_{\alpha} D_{\beta} K + \alpha \left[{}^{(3)}R_{ab} + K K_{ab} - 2 K_{a\lambda} K^{\lambda}_{b} \right] + 4\pi\alpha \left[\gamma_{ab} (S - \rho) - 2 S_{ab} \right]$$



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Last Projection $\;\gamma^\alpha_{\ a}\gamma^\delta_{\ b} n^\beta n^\mu R_{\alpha\beta\delta\mu} = \mathcal{L}_{\vec{n}} K_{ab} + K_{ac}K^c_{\ b} + \frac{1}{\alpha} D_a D_b \alpha$

$n^a = \frac{1}{\alpha}\left(\partial_t^a - \beta^a\right)$

$S_{ab} \equiv \gamma^\alpha_{\ a}\gamma^\beta_{\ b} T_{\alpha\beta}$

$S \equiv \gamma^{ab} S_{ab}$

$\Rightarrow \mathcal{L}_t K_{ab} - \mathcal{L}_\beta K_{ab} = -D_a D_b \alpha + \alpha\left[{}^{(3)}R_{ab} + K K_{ab} - 2K_{ac}K^c_{\ b}\right]$
$\quad + 4\pi\alpha\left[\gamma_{ab}(S-\rho) - 2S_{ab}\right]$

$$n^a = \frac{1}{2}\left(\partial_t^a - \beta^a\right) \qquad \left[\gamma_{ab}, K_{ab}, \alpha, \beta^i + \text{Sources}\right]$$

$$\Rightarrow \mathcal{L}_t K_{ab} - \mathcal{L}_{\vec{\beta}} K_{ab} = -D_a D_b \alpha + \alpha\left[{}^{(3)}R_{ab} + K K_{ab} - 2 K_{a\lambda} K^{\lambda}{}_b\right] + 4\pi\alpha\left[\gamma_{ab}(S-\rho) - 2 S_{ab}\right]$$

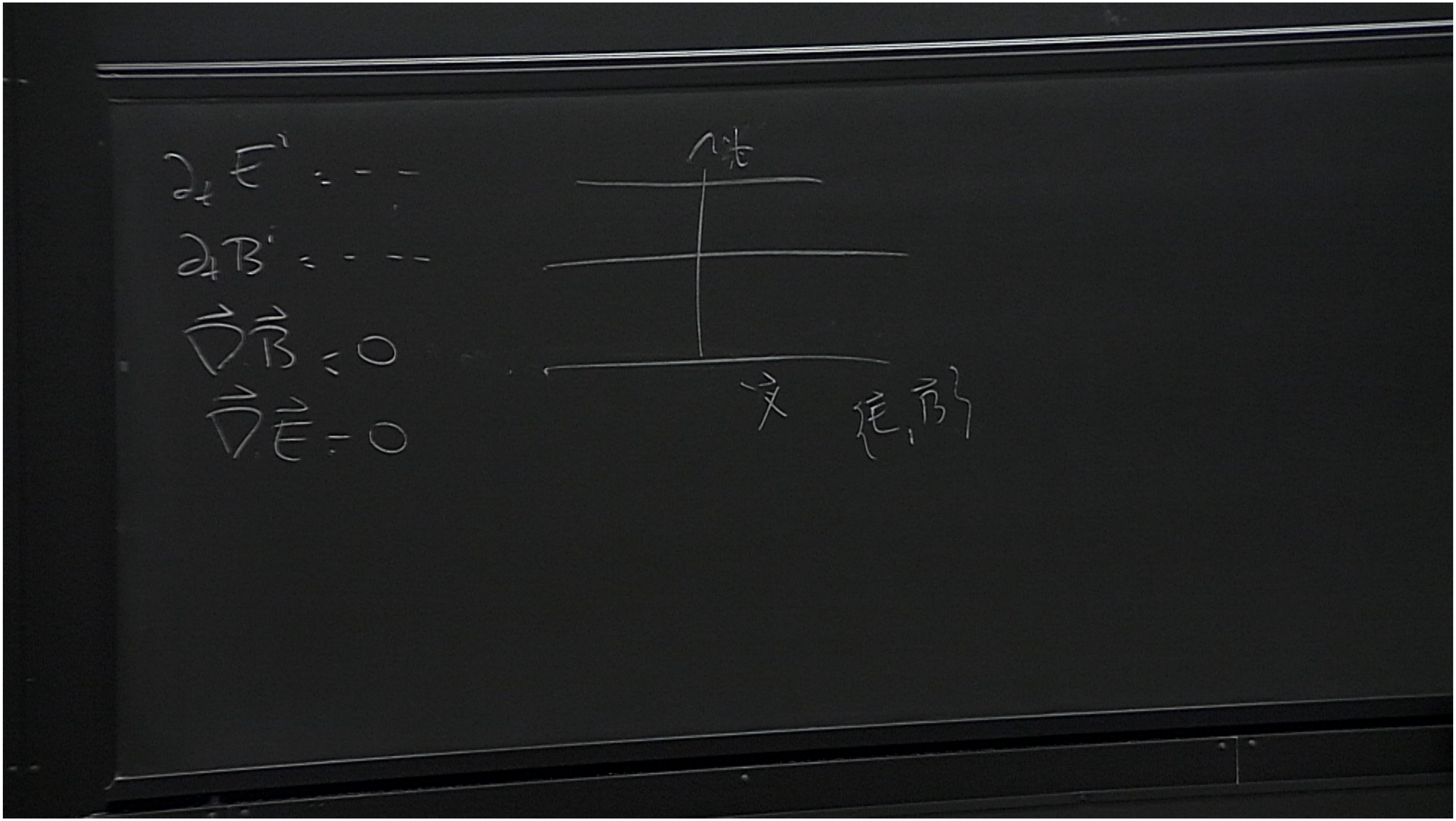
$$\boxed{\partial_t \gamma_{ab} = -2\alpha K_{ab} + D_a \beta_b + D_b \beta_a}$$

$$\to K_{ab} = \left(l_{ab} - \frac{1}{2} g_{ab} l \right)$$

$$S_{ab} \equiv \gamma^{\alpha}{}_{a} \gamma^{\beta}{}_{b} T_{\alpha\beta}$$

$$S \equiv \gamma^{ab} S_{ab}$$

$\Rightarrow$



$\partial_t E^i = - - -$
$\partial_t B^i = - - -$
$\vec{\nabla} \vec{B} = 0$
$\vec{\nabla} \vec{E} = 0$
t
x
$(\vec{E}, \vec{B})$

Σ_t

"$\boxed{\text{ADM}}$" equns

$$\boxed{\partial_t \gamma_{ab} = -2\alpha K_{ab} + D_a \beta_b + D_b \beta_a}$$

$$\gamma^{\delta}_{\mu}\, \gamma^{\kappa}_{\nu} \left[n^\lambda n^\sigma R_{\delta\lambda\kappa\sigma} + \underbrace{g^{\lambda\sigma} R_{\delta\lambda\kappa\sigma}}_{R_{\delta\kappa}} \right] = \gamma^{\delta}_{\mu}\, \gamma^{\kappa}_{\nu}$$

$$R_{\delta\kappa} = {}^3R_{\mu}$$

$$\Downarrow$$

$$8\pi \left(T_{\delta\kappa} - \frac{1}{2} g_{\delta\kappa} T \right)$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

$$= 8\pi T_{ab}$$

$$\rightarrow R_{ab} = \left(T_{ab} - \frac{1}{2} g_{ab} T \right) 8\pi$$

$$S_{ab} \equiv \gamma^{\alpha}_{a} \gamma^{\beta}_{b} T_{\alpha\beta}$$

$$S \equiv \gamma^{ab} S_{ab}$$

$\Rightarrow$

$\Rightarrow R_{ab} = \left(T_{ab} - \frac{1}{2} g_{ab} T\right) 8\pi$

$S_{ab} \equiv \gamma^{\alpha}{}_{a} \gamma^{\beta}{}_{b} T_{\alpha\beta}$

$S \equiv \gamma^{ab} S_{ab}$

$\Rightarrow \mathcal{L}_t K_{ab} - \mathcal{L}_\beta K_{ab} = -D_a D_b \alpha + \alpha \left[{}^{(3)}R_{ab} + \ldots \right.$

$+ 4\pi\alpha \left[\gamma \ldots \right.$

$K_{ij} \equiv \frac{1}{\sqrt{\gamma}} \left(\Pi_{ij} - \frac{1}{2} \gamma_{ij} \Pi \right)$

Π_{ij} can. conjugate to γ_{ij}

<u>ADM</u>

$\partial_t K_{ij} = \left(\qquad \right) - \frac{\alpha}{2} \gamma_{ij} \mathcal{H}$

$\mathcal{H} \equiv \frac{1}{2} \left({}^{(3)}R + K^2 - K_{ij} K^{ij} \right) - 8\pi\rho$

conjugate

γ_{IJ}

$$\mathcal{H} \equiv \frac{1}{2}\left({}^{(3)}R + K^2 - K_{IJ}K^{IJ}\right) - 8\pi\rho$$

– Off-shell

$$\partial_t E^i = \text{- - -} \qquad + \vec{B}(\vec{\nabla}\vec{B}^{+})$$

$$\partial_t B^i = \text{- - - -}$$

$$\vec{\nabla}\vec{B} = 0$$

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$G_{ab} - 8\pi T_{ab} = 0 \longrightarrow \mathcal{E}_{ab} + n_a M_b + n_b M_a + n_a n_b \mathcal{H}.$$

$$\mathcal{E}_{ab} = \gamma^{\alpha}_{\ a} \gamma^{\beta}_{\ b} \left[G_{ab} \right] - 8\pi S_{ab}$$

$$M_a = -n^{\alpha} \gamma^{\beta}_{\ a} G_{\alpha\beta} - 8\pi J_a$$

$$\mathcal{H} = n^{\alpha} n^{\beta} G_{\alpha\beta} - 8\pi \rho$$

$$+4\pi\alpha\left[\gamma_{ab}(S-\rho)-2S_{ab}\right]$$

ate

$$\mathcal{H}\equiv\frac{1}{2}\left({}^{(3)}R+K^2-K_{ij}K^{ij}\right)-8\pi\rho$$

- Off-shell

$$n^b\nabla^a\left(F_{ab}\right)$$

$$=\nabla^a\left(F^{ab}n_b\right)$$

$$-F^{ab}\left(\nabla_a n_b\right)$$

$$G_{ab} - 8\pi T_{ab} = 0 \Rightarrow \left(\mathcal{E}_{ab} + n_a M_b + n_b M_a + n_a n_b \mathcal{H}. \right)$$

$$\mathcal{E}_{ab} = \gamma^{\alpha}_{a} \gamma^{\beta}_{b} \left[G_{ab} \right] - 8\pi S_{ab}$$

$$M_a = -n^{\alpha} \gamma^{\beta}_{a} G_{\alpha\beta} - 8\pi J_a$$

$$\mathcal{H} = n^{\alpha} n^{\beta} G_{\alpha\beta} - 8\pi \rho$$

$$n^b \nabla^a \left(\qquad \right) = 0$$

$$= - D^c M_c - \nabla^a (n_a \mathcal{H}) + [\ ] (\nabla^a n^b)$$

$$(\mathcal{E}_{ab} + n_a n_b + n_b n_a + n_a n_b \mathcal{H}.)$$

$$\gamma^{\alpha}_{a}\gamma^{\beta}_{b}\left[G_{ab}\right] - 8\pi S_{ab} \qquad n^b\nabla^a(\qquad) = 0$$

$$\mathcal{E}_{ab} + \ldots$$

$$-n^{\alpha}\gamma^{\beta}_{a} G_{\alpha\beta} - 8\pi J_a \qquad = -D^c M_c - \nabla^a(n_a \mathcal{H}) - [\ ](\nabla^a n^b)$$

$$n^{\alpha}n^{\beta} G_{\alpha\beta} - 8\pi\rho \qquad h^b_c \nabla^a(\ ) = \nabla^a[h^b_c[\ ]) + [\ ]\ \nabla^a(h^b_c)$$

$$n\nabla^a(\qquad) = 0$$

$$= -D^c M_c - \nabla^a(n_a \mathcal{H}) + \Big[\underbrace{\ \ }_{\mathcal{E}_{ab} + \dots}\Big](\nabla^a n^b)$$

$$h^b{}_c = g^b{}_c + n^b n_c$$

$$h^b{}_c \nabla^a(\quad) = \nabla^a[h^b{}_c[\ \ ]) + [\mathcal{E}_{ab}]\ \nabla^a(h^b{}_c)$$

$$\rightsquigarrow n^a D_a M_b = -D^c \mathcal{E}_{bc} + [\quad](Dn)$$

$$n\nabla^a(\qquad) = 0$$

$$= -D^c M_c - \nabla^a(n_a \mathcal{H}) + [\overset{\mathcal{E}_{ab} + \dots}{\quad}](\nabla^a n^b)$$

$$h^b{}_c = g^b{}_c + n^b n_c$$

$$\gamma^b{}_c \nabla^a(\ ) = \nabla^a[\gamma^b{}_c[\ ]) + [\mathcal{E}_{ab}]\ \nabla^a(\gamma^b{}_c)$$

$$\rightsquigarrow \quad n^a D_a M_b = -D^c \mathcal{E}_{bc} + [\quad](Dn)$$

$$J_a \quad = -D^c M_c - \nabla^a(n_a \mathcal{H}) + [\ ](\nabla^a n^b)$$

$$h^b{}_c = g^b{}_c + n^b$$

$$\gamma^b{}_c \nabla^a(\ ) = \nabla^a[\gamma^b{}_c[\ ]) + [\mathcal{E}_{ab}]\ \nabla^a(\gamma^b{}_c)$$

$$\rightsquigarrow \quad n^a D_a M_b = -D^c \mathcal{E}_{bc} + [\mathcal{E}\ ](Dn)$$

$$n^a D_a \mathcal{H} = -D_c M^c + [\mathcal{E}\ ](Dn)$$

$$G_{ab} - 8\pi T_{ab} = 0 \rightarrow \boxed{\left(\mathcal{E}_{ab} + n_a M_b + n_b M_a + n_a n_b \mathcal{H}\right)}$$

$$\mathcal{E}_{ab} = \gamma^\alpha_a \gamma^\beta_b \left[G_{\alpha\beta}\right] - 8\pi S_{ab}$$

$$M_a = -n^\alpha \gamma^\beta_a G_{\alpha\beta} - 8\pi J_a$$

$$\mathcal{H} = n^\alpha n^\beta G_{\alpha\beta} - 8\pi \rho$$

$$n^b \nabla^a \left(\qquad \right) = 0$$

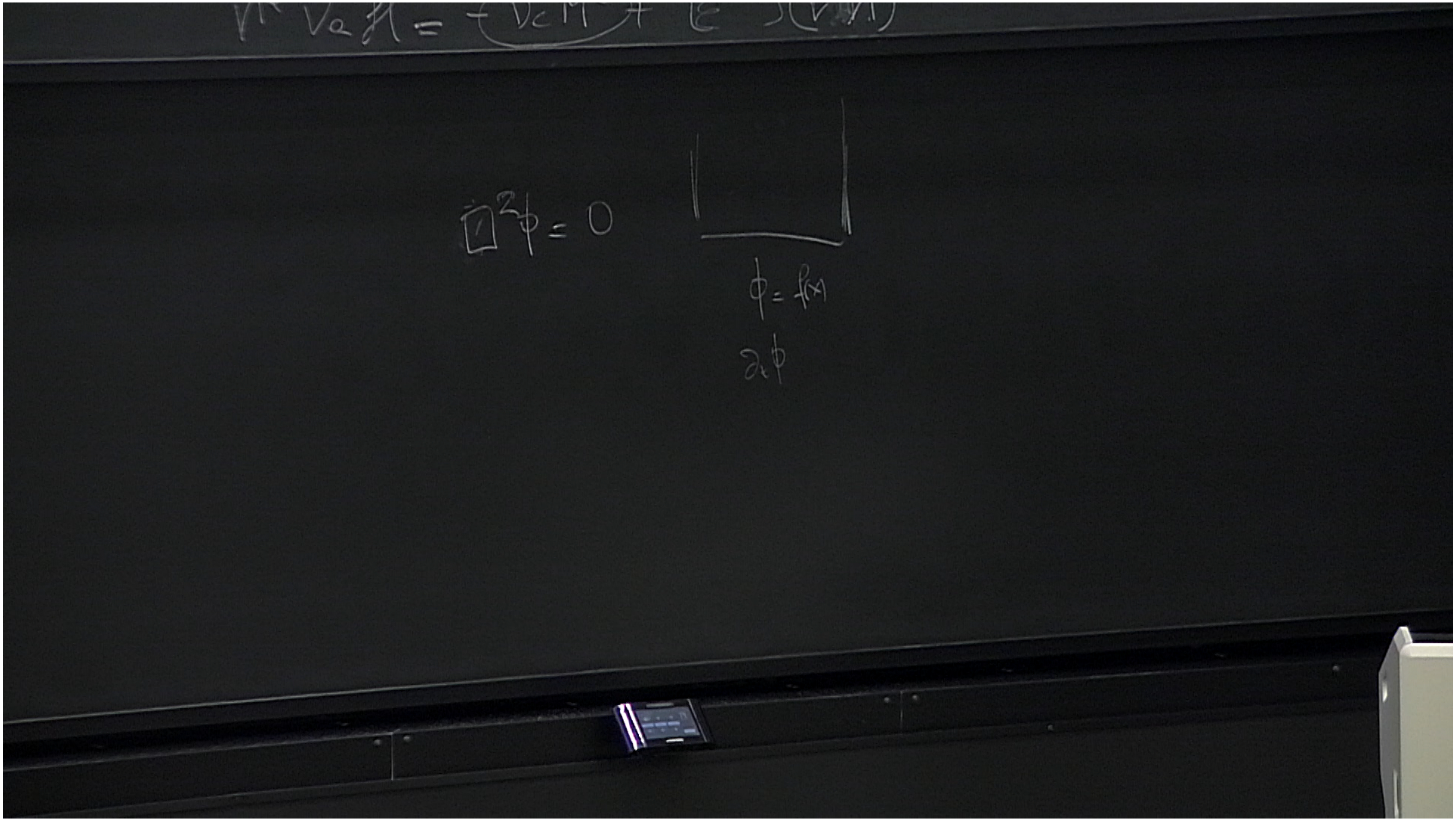
$$= -D^c M_c - \nabla^a (n_a \mathcal{H}) + [\mathcal{E}_{ab} + \cdots](\nabla^a n^b)$$

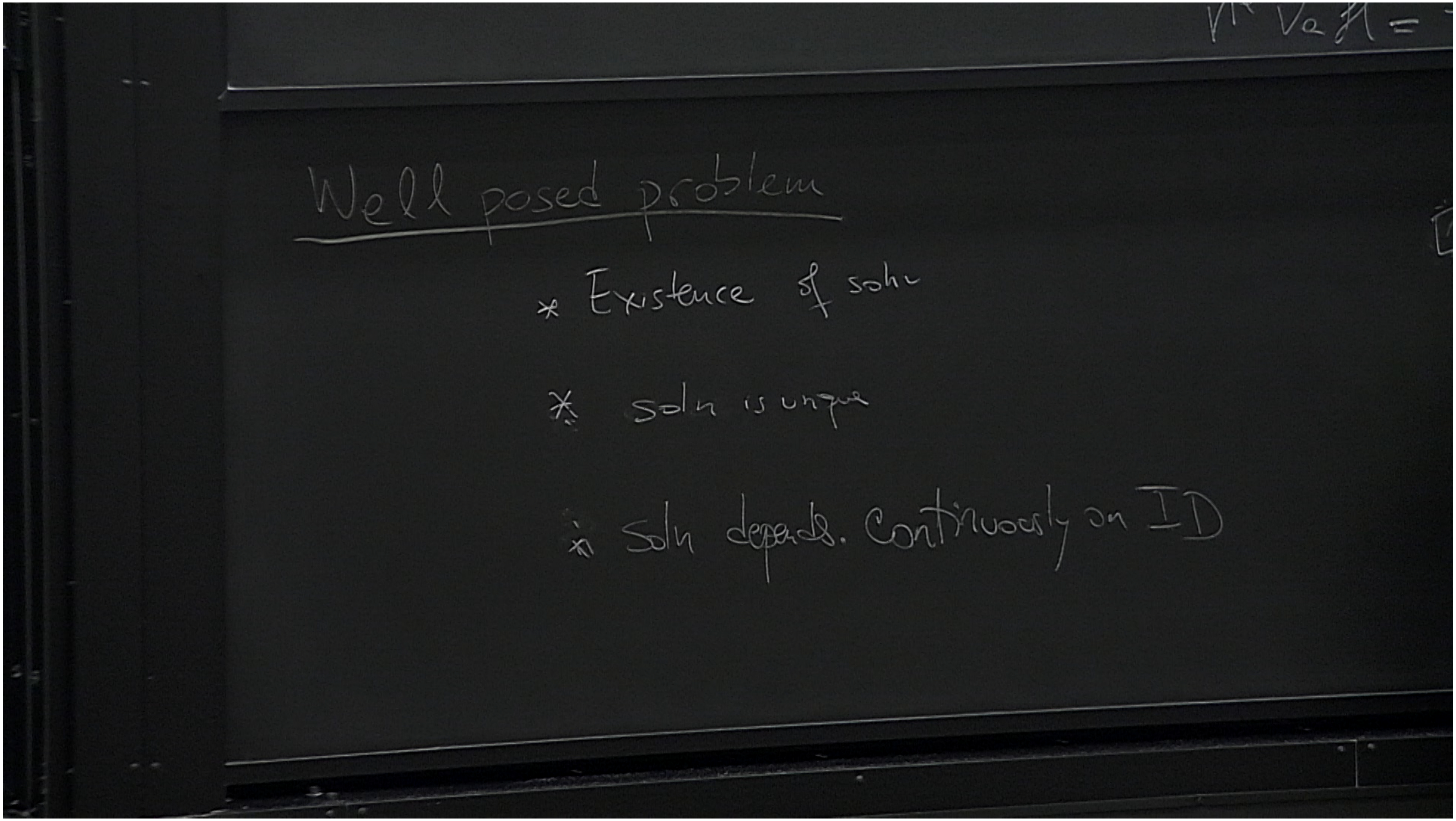
$$h^b{}_c = g^b{}_c + n^b n_c$$

$$\gamma^b{}_c \nabla^a(\ ) = \nabla^a[\gamma^b{}_c[\ ]) + [\mathcal{E}_{ab}]\ \nabla^a(\gamma^b{}_c)$$

$$\rightarrow n^a D_a M_b = -D^c \mathcal{E}_{bc} + [\mathcal{E}\ ](Dn)$$

$$n^a D_a \mathcal{H} = -D_c M^c + [\mathcal{E}\ ](Dn)$$


$\Box^2\phi = 0$
$\phi = f(x)$
$\partial_t \phi$


Well posed problem
* Existence of soln
* soln is unique
* soln depends continuously on ID

$$\|u(T)\| \leq K e^{\beta t} \|u(0)\|$$

K, β are universal
indep on ID & BD

$$u_{,t} = u_{,x} + v_{,x}$$
$$v_{,t} = v_{,x} + u$$

$$\rightarrow U_{,t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} U_{,x} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$U(x,t) \equiv e^{st} e^{i\omega x} \hat{U}(\omega, 0) \rightarrow \left(\mathbb{1} + i\omega \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t \right) e^{i\omega(t+x)}$$

$$\|u(T)\| \leq K e^{\beta t} \|u(0)\|$$

K, β are universal
indep on ID & BD

$$u_{,t} = u_{,x} + v_{,x}$$
$$v_{,t} = v_{,x} + u$$

$$\rightarrow U_{,t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} U_{,x} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$U(x,t) \equiv e^{st} e^{i\omega x} \hat{U}(\omega,0) \Rightarrow \left(\mathbb{1} + i\omega \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t\right) e^{i\omega(t+x)} \qquad e^{\omega t}$$

$$u_{,t} = \sum_{j=1}^{D} A_j D_j u .$$

Assume A_j cons coeff

Consider solns of the form $u(x,t) \equiv e^{i\vec{\omega}\cdot\vec{x}} \, \hat{u}(\omega,t)$

$$\hat{u}_t = i|\omega| \underbrace{\sum_{j=1}^{D} A_j \omega_j'}\, \hat{u} \quad \text{with} \quad \omega' = \vec{\omega}/|\omega|$$

$$= i|\omega| \, P(\omega') \, \hat{u}$$

if eigenvalues of $P(\omega^i)$ are real

linearly indep.

& there is a complete set of eigenvectors

$\Rightarrow$ strongly hyperbolic