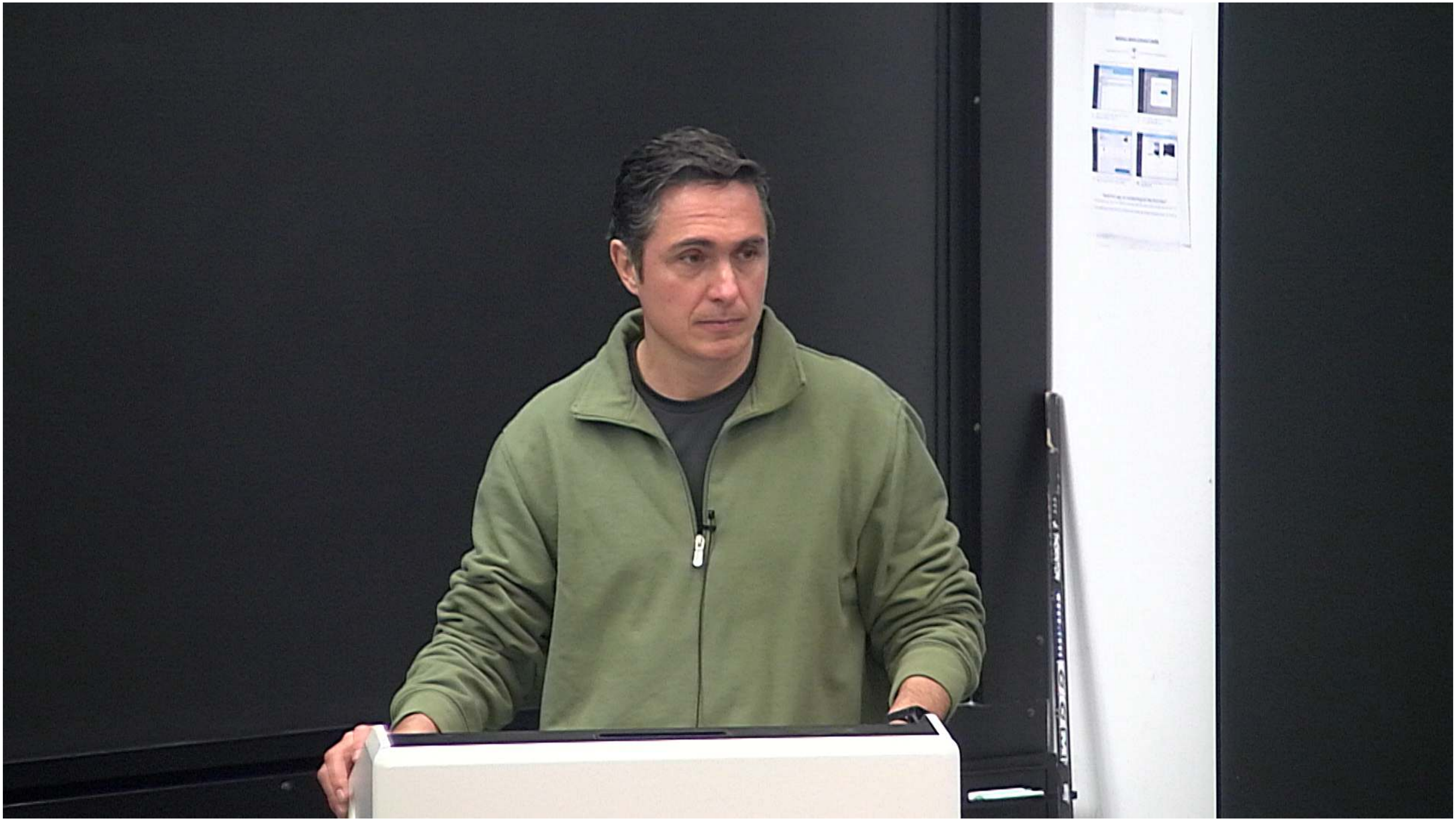
Title: PSI 2018/2019 - Strong Field Gravity - Lecture 6

Speakers: William East

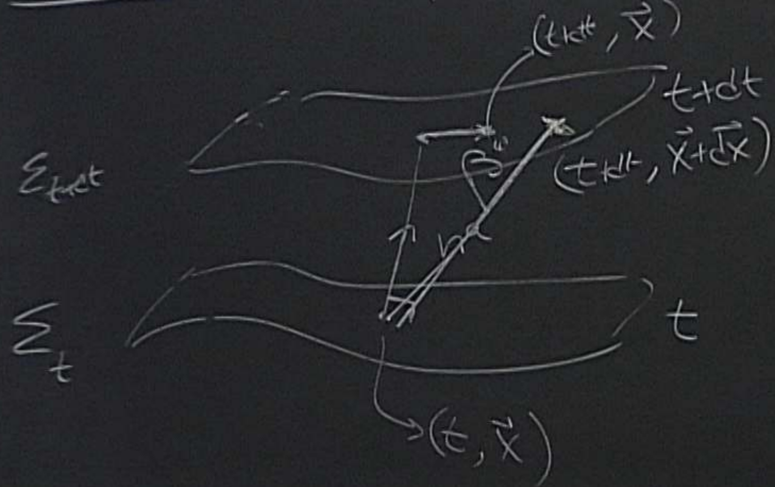
Collection: PSI 2018/2019 - Strong Field Gravity (East)

Date: March 11, 2019 - 9:00 AM

URL: http://pirsa.org/19030013



3+1 Decomposition : $(M, g_{ab}) \to G_{ab} = 8\pi T_{ab}$



each Σ is spacelike

$$\Delta\tau = dt\,\alpha$$

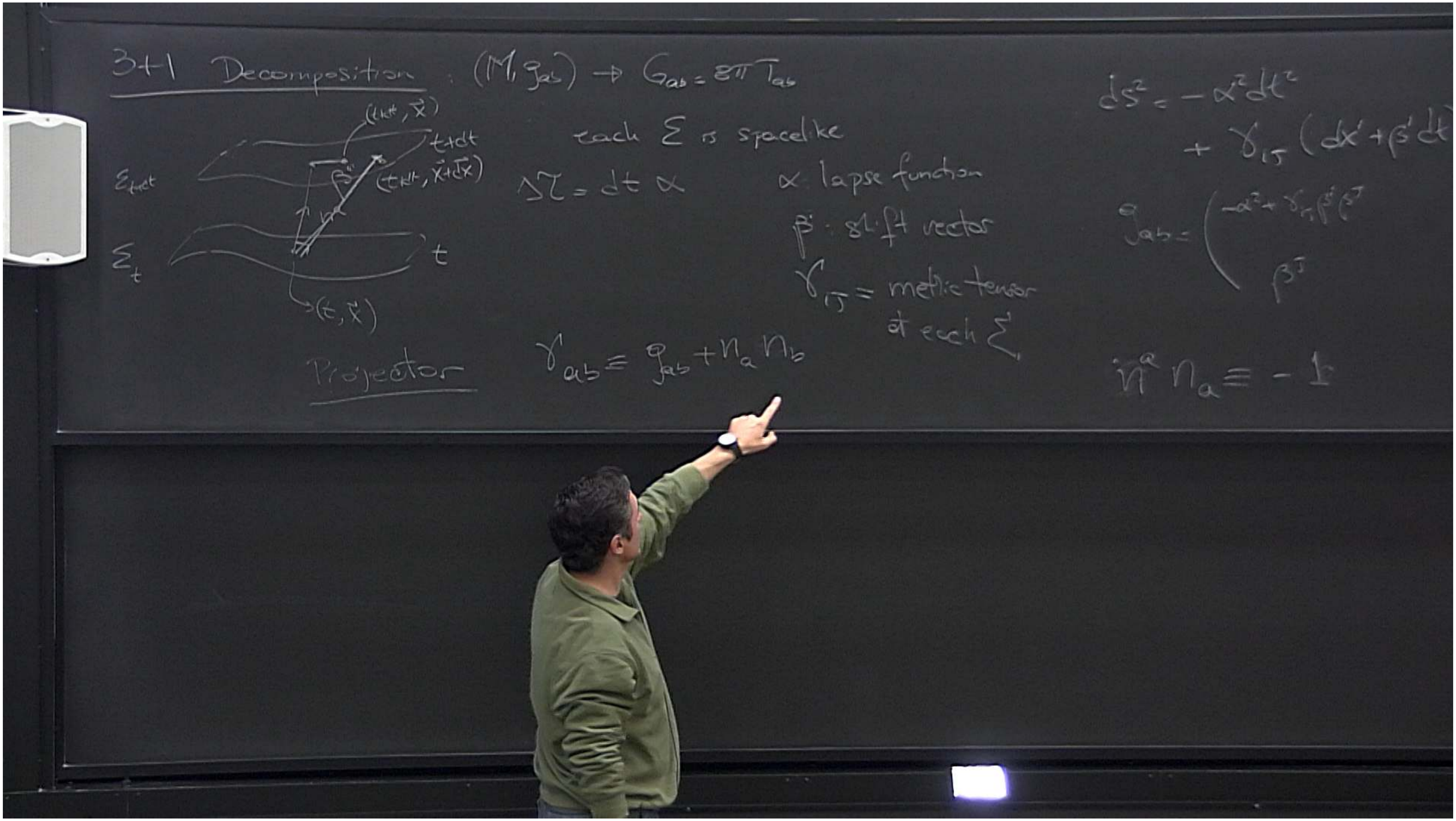
α: lapse function

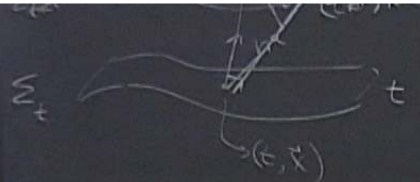
β^i: shift vector

γ_{ij} = metric tensor at each Σ

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \gamma_{ij}\beta^i\beta^j & \beta^i \\ \beta^j & \gamma_{ij} \end{pmatrix}; \quad g^{ab} = \begin{pmatrix} -\frac{1}{\alpha^2} & \beta^i/\alpha^2 \\ \beta^j/\alpha^2 & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}$$


3+1 Decomposition : $(M, g_{ab}) \to G_{ab} = 8\pi T_{ab}$
each Σ is spacelike
$\Delta\tau = dt\,\alpha$
α : lapse function
β^i : shift vector
γ_{ij} = metric tensor at each Σ
$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)$
Projector
$\gamma_{ab} \equiv g_{ab} + n_a n_b$
$n^a n_a \equiv -1$
Σ_{t+dt}
Σ_t
t+dt
t
$(t, \vec{x})$



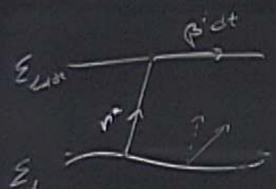
α: lapse function

β^i: shift vector

γ_{ij} = metric tensor at each Σ

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \gamma_{ij}\beta^i\beta^j & \beta^i \\ \beta^j & \gamma_{ij} \end{pmatrix}; \quad g^{ab} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}$$

Projector $\gamma_{ab} \equiv g_{ab} + n_a n_b$ $\qquad n^a n_a \equiv -1$



$$n_a = (-\alpha, 0)$$
$$n^a = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right) \Rightarrow \left(\frac{\partial}{\partial t}\right)^a = \alpha n^a + \beta^a$$

Extrinsic curvature $K_{ab} \equiv -\gamma^c{}_a \nabla_c n_b = -(g^c{}_a + n^c n_a)\nabla_c n_b = -\nabla_a n_b + n_a n^c \nabla_c n_b$

Extrinsic curvature $K_{ab} \equiv -\gamma^c{}_a \nabla_c n_b = -(g^c{}_a + n^c n_a)\nabla_c n_b = -(\nabla_a n_b + n_a \overbrace{n^c \nabla_c n_b})$

$$= -(\nabla_a n_b + n_a a_b)$$

$K_{ab} = K_{ba}$

$\gamma_{ab} = g_{ab} + n_a n_b$

$\nabla_a t = n_a = \xi_a$

n^a

$$\underline{K_{ab} = \gamma^c{}_a \nabla_c n_b = \gamma^c{}_a \nabla_c \xi_b = \gamma^c{}_a \nabla_b \xi_c = \gamma^c{}_a \nabla_b n_c = K_{ba}}$$

$\left(\frac{\partial}{\partial t}\right)^a$

$$\mathcal{L}_n \gamma_{ab} = n^c \nabla_c \gamma_{ab} + \gamma_{ac} \nabla_b n^c + \gamma_{bc} \nabla_a n^c$$

$$= n^c \nabla_c (\cancel{g_{ab}} + n_a n_b) + (g_{ac} + n_a n_c)\nabla_b n^c + (g_{bc} + n_b n_c)\nabla_a n^c$$

$$\to n^c \nabla_c n_a n_b + (n^c \nabla_c n_b) n_a + \underbrace{\nabla_b n_a + \nabla_a n_b}$$

$$\to (\gamma^c{}_b - g^c{}_b)\nabla_c n_a + (\gamma^c{}_a - g^c{}_a)\nabla_c n_b +$$

$$\gamma^c{}_b D_c n_a + \gamma^c{}_a D_c n_b - \cancel{D_b n_a} - \cancel{D_a n_b} + \cancel{D_b n_a} + \cancel{D_a n_b}$$

$$= -2K_{ab} \quad \Rightarrow \quad \boxed{\mathcal{L}_n \gamma_{ab} = -2K_{ab}}$$

$$\partial_t \gamma_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij}$$

Property

$$\mathcal{L}_{\bar n} \gamma_{ab} = \frac{1}{\phi} \mathcal{L}_{(\phi n)} \gamma_{ab}$$

$$K_{ab} = -\frac{1}{2\alpha} \mathcal{L}_{\alpha n} \gamma_{ab} = -\frac{1}{2\alpha}\left(\mathcal{L}_{\vec t} - \mathcal{L}_{\vec \beta}\right)\gamma_{ab}$$

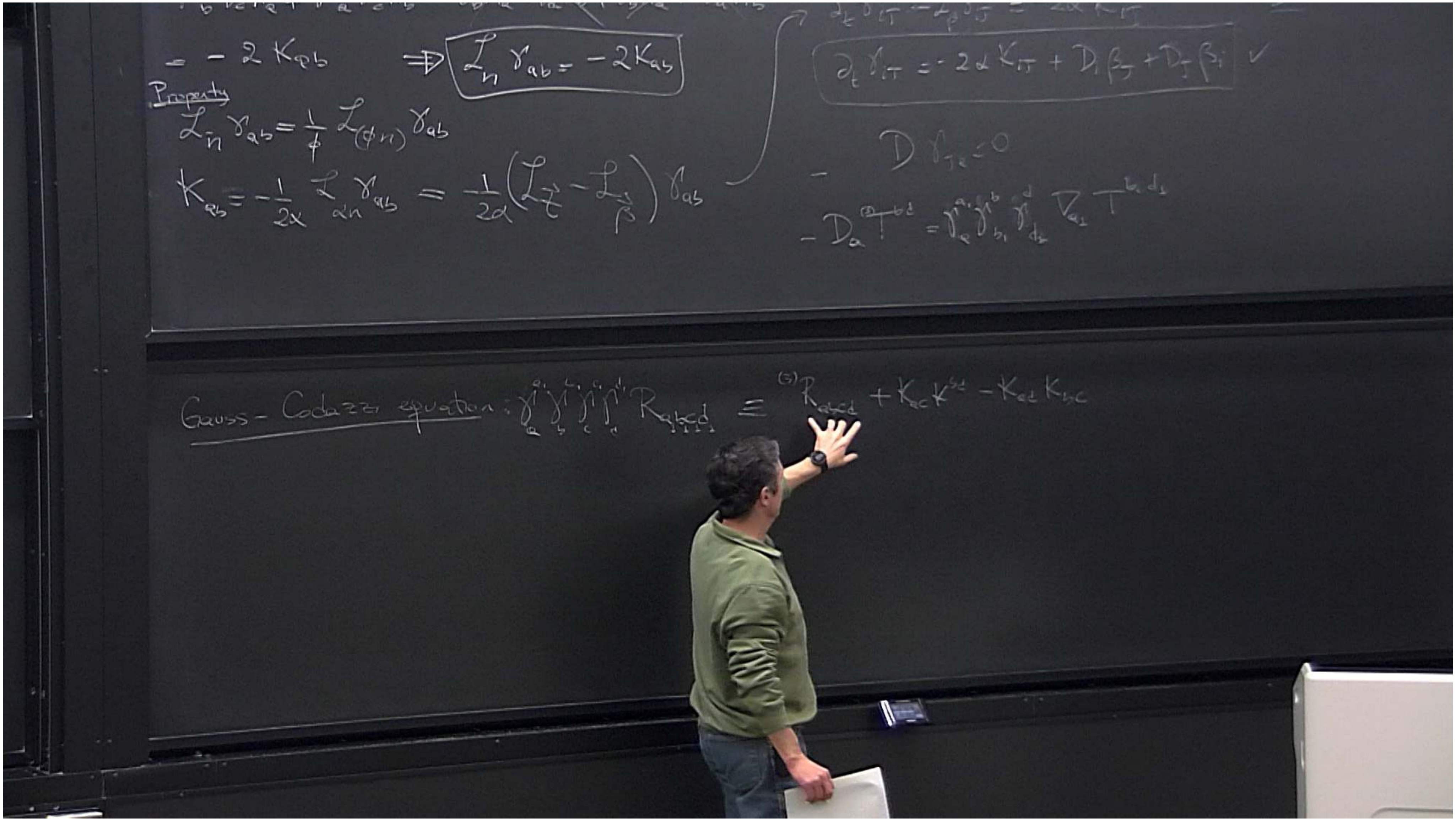
n_b

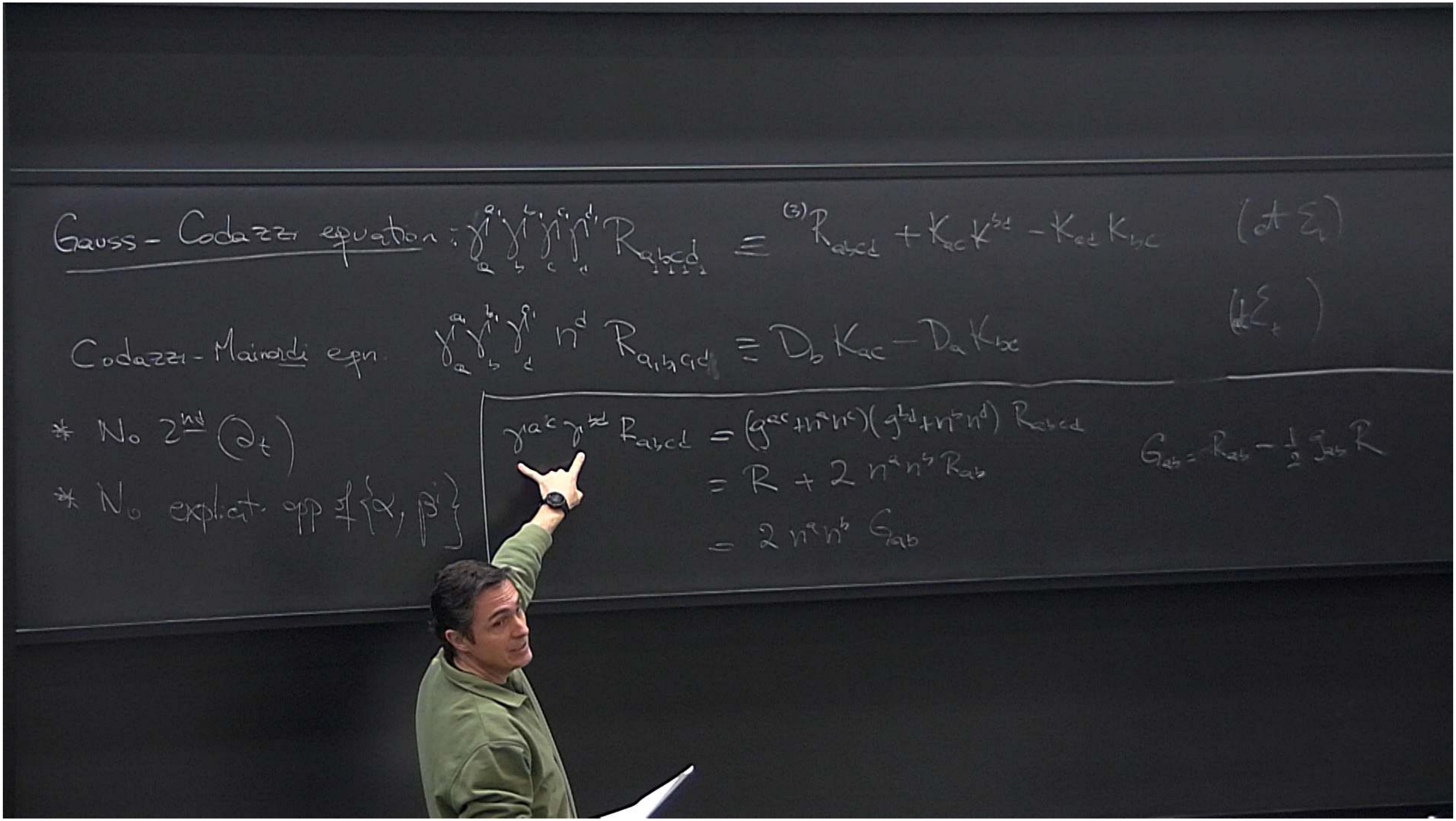
$$\partial_t \gamma_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij} \quad \checkmark$$

Ex

$$\boxed{\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i} \quad \checkmark$$

$$D_i \gamma_{jk} = 0$$


Property
Gauss - Codazzi equation



Gauss-Codazzi equation: $\gamma^{a}_{a'}\gamma^{b}_{b'}\gamma^{c}_{c'}\gamma^{d}_{d'} R_{abcd} \equiv {}^{(3)}R_{a'b'c'd'} + K_{ac}K^{bd} - K_{ad}K_{bc}$ (at Σ_t)
Codazzi-Mainardi eqn. $\gamma^{a}_{a'}\gamma^{b}_{b'}\gamma^{c}_{c'} n^{d} R_{abcd} \equiv D_b K_{ac} - D_a K_{bc}$ (at Σ_t)
* No 2^{nd} (∂_t)
* No explicit app of $\{\alpha, \beta^i\}$
$\gamma^{ac}\gamma^{bd} R_{abcd} = (g^{ac} + n^a n^c)(g^{bd} + n^b n^d) R_{abcd}$
$= R + 2 n^a n^b R_{ab}$
$= 2 n^a n^b G_{ab}$
$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

Gauss - Codazzi equation : $\gamma^{a_1}{}_{a}\gamma^{b_1}{}_{b}\gamma^{c_1}{}_{c}\gamma^{d_1}{}_{d} R_{a_1b_1c_1d_1} \equiv {}^{(3)}R_{abcd} + K_{ac}K^{bd} - K_{ad}K_{bc}$ (at Σ_t)

Codazzi - Mainardi eqn. $\gamma^{a_1}{}_{a}\gamma^{b_1}{}_{b}\gamma^{c_1}{}_{c}\, n^d R_{a_1b_1c_1d} \equiv D_b K_{ac} - D_a K_{bc}$ (at Σ_t)

* No 2^{nd} (∂_t)

* No explicit app of $\{\alpha, \beta^i\}$

$$\begin{aligned}\gamma^{ac}\gamma^{bd} R_{abcd} &= (g^{ac}+n^a n^c)(g^{bd}+n^b n^d) R_{abcd}\\ &= R + 2 n^a n^b R_{ab}\\ &= 2 n^a n^b G_{ab}\end{aligned}$$

$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

From G-C $\quad \gamma^{a_1c_1}\gamma^{b_1d_1} R_{a_1b_1c_1d_1} = {}^{(3)}R + K^2 - K_{ab}K^{ab}$

$$\Rightarrow \quad 2 n^a n^b G_{ab} \equiv {}^{(3)}R + K^2 - K_{ab}K^{ab}$$

for $G_{ab} = 8\pi T_{ab} \Rightarrow \boxed{{}^{(3)}R + K^2 - K_{ab}K^{ab} \equiv 16\pi\rho} \rightarrow$ Hamilt constraint

with $\rho = n^a n^b T_{ab}$

Notice $\gamma^{ab} n^d G_{bd} = n^d \gamma^{ab} R_{bd}$

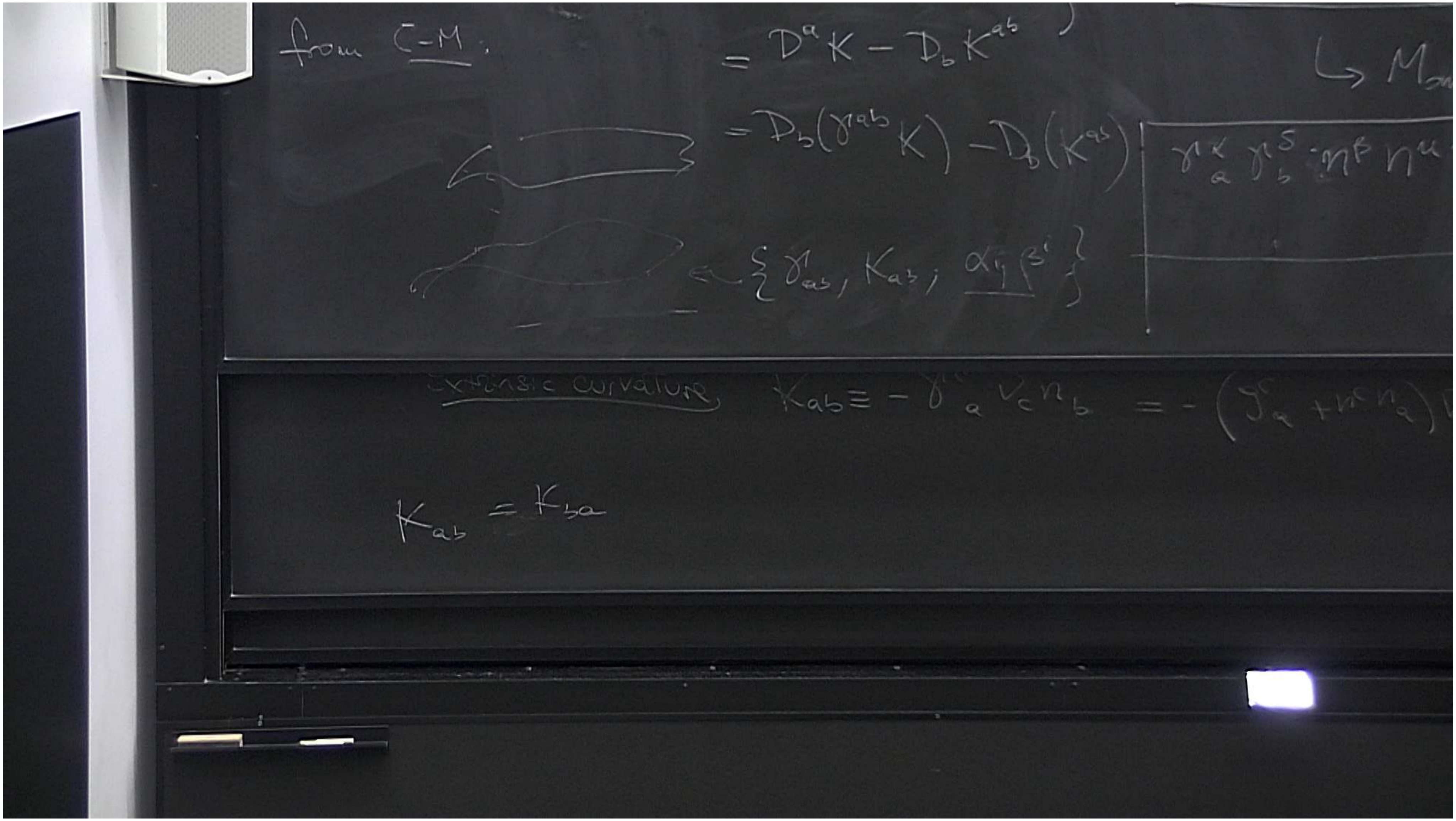
from C-M:

$$= D^a K - D_b K^{ab}$$

$$= D_b(\gamma^{ab} K) - D_b(K^{ab})$$

$$\Rightarrow \boxed{D_b\left(K^{ab} - \gamma^{ab} K\right) = 8\pi J^a}$$

where


from C-M:
$= D^a K - D_b K^{ab}$
$= D_b(\gamma^{ab} K) - D_b(K^{ab})$
$\{\gamma_{ab}, K_{ab}; \alpha, \beta^i\}$
extrinsic curvature
$K_{ab} \equiv -\gamma^c{}_a \nabla_c n_b$
$K_{ab} = K_{ba}$