Title: PSI 2018/2019 - Strong Field Gravity - Lecture 2

Speakers: William East

Collection: PSI 2018/2019 - Strong Field Gravity (East)

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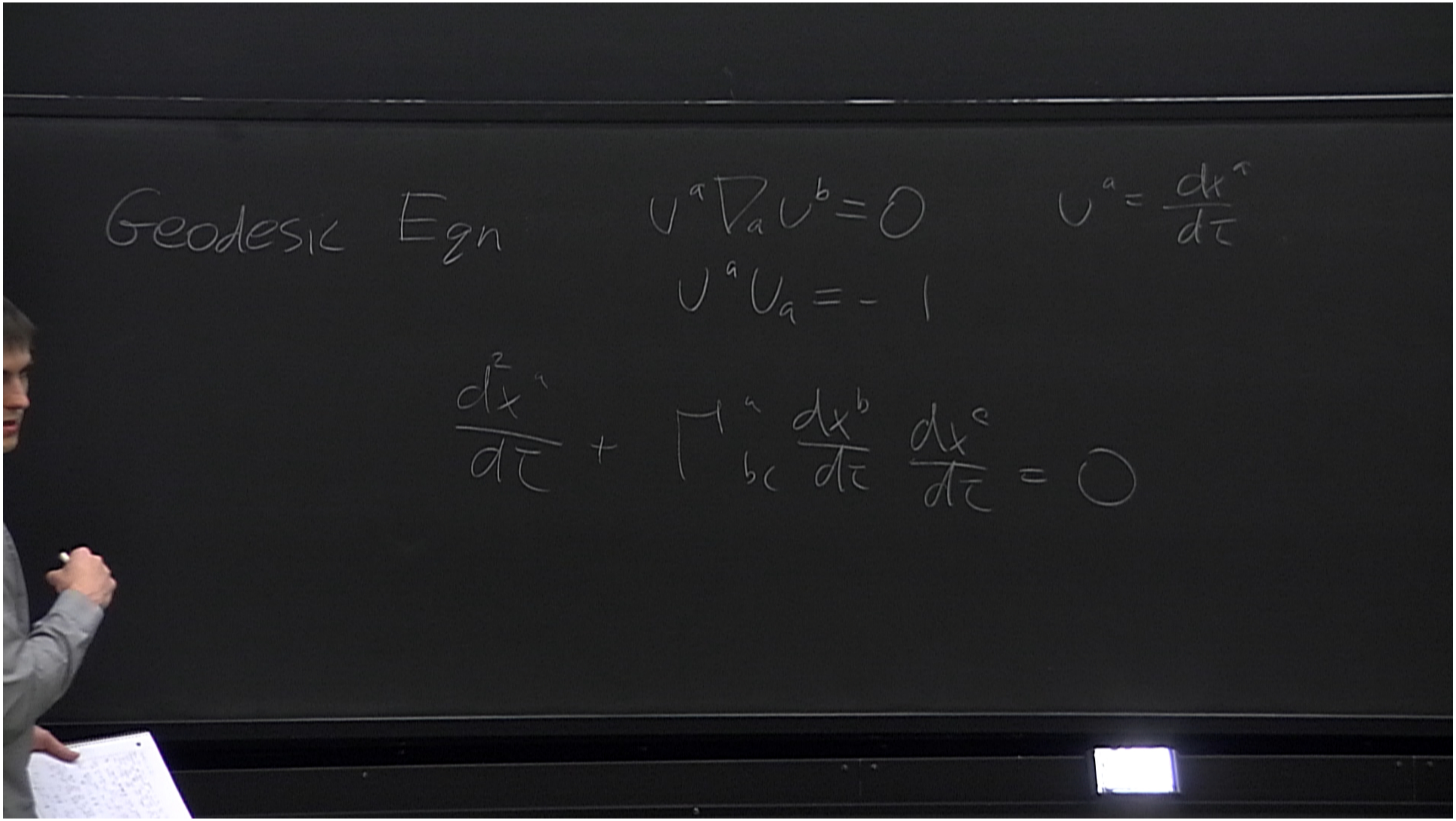
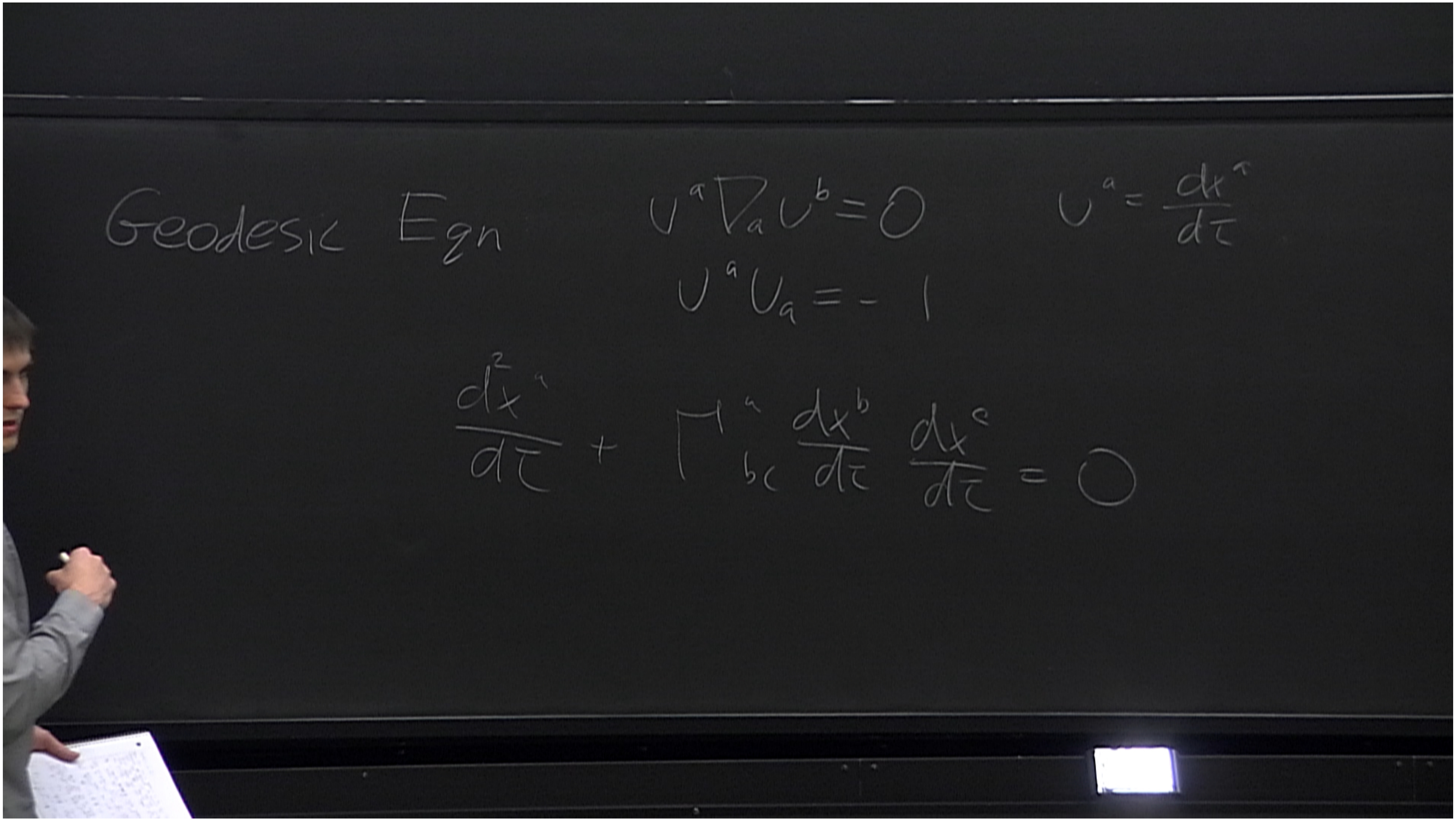
URL: http://pirsa.org/19030009


Geodesic Eqn
$u^a \nabla_a u^b = 0$
$u^a = \frac{dx^a}{d\tau}$
$u^a u_a = -1$
$\frac{d^2x^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$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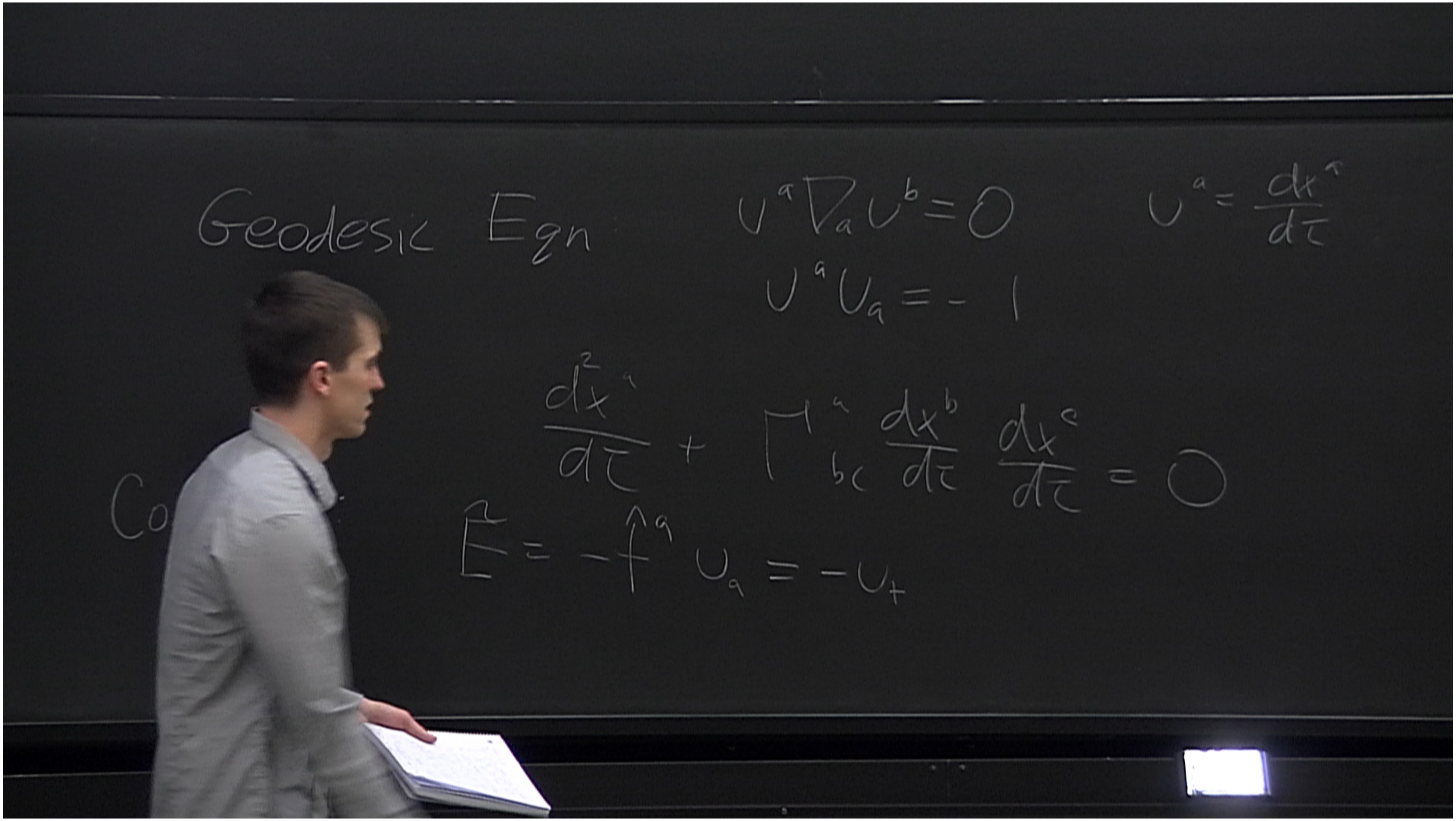
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$E = -\hat{t}^a u_a = -u_t$

Geodesic Eqn $\quad u^a \nabla_a u^b = 0 \qquad u^a = \frac{dx^a}{d\tau}$

$$u^a u_a = -1$$

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

Constants:

$$\tilde{E} = -\hat{t}^a u_a = -u_t = -(g_{tt} u^t + g_{t\phi} u^\phi) =$$

$$\tilde{J} = \hat{\phi}^a u_a = u_\phi$$

$$\hat{t}^a = (1, 0, 0, 0)$$

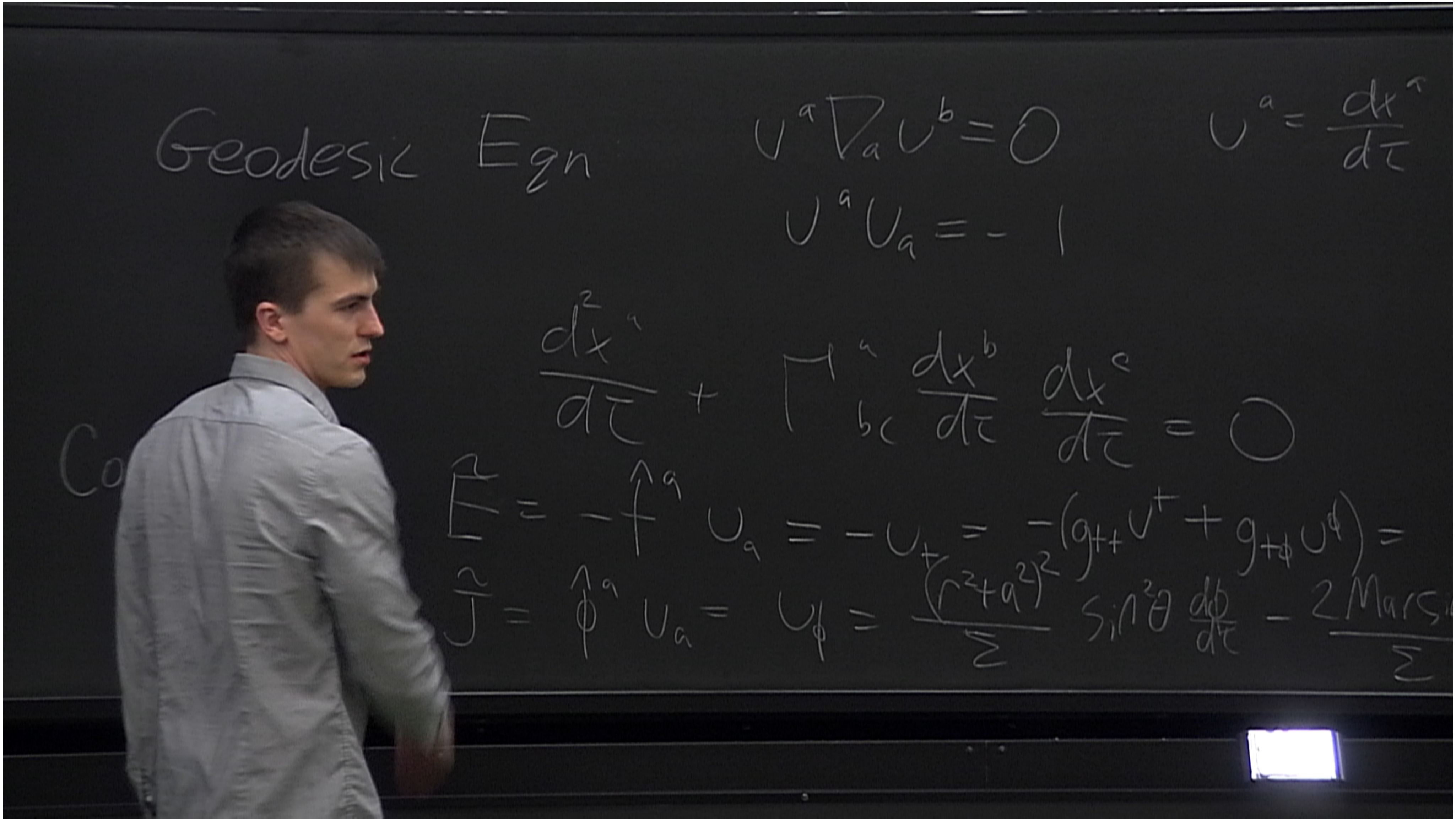
$$\hat{\phi}^a = (0, 0, 0, 1)$$

$$\Delta = r^2 + a^2 - 2Mr$$

$$\Sigma = r^2 + a^2 \cos^2\theta$$

$$\left(1 - \frac{2Mr}{\Sigma}\right)\frac{dt}{d\tau} + \frac{2Mar\sin^2\theta}{\Sigma}\frac{d\phi}{d\tau}$$

$$\ldots n^2\theta \quad \frac{dt}{d\tau}$$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Co
$\tilde{E} = -\hat{t}^a u_a = -u_t = -(g_{tt} u^t + g_{t\phi} u^\phi) =$
$\tilde{J} = \hat{\phi}^a u_a = u_\phi = \frac{(r^2+a^2)^2}{\Sigma} \sin^2\theta \frac{d\phi}{d\tau} - \frac{2Mars}{\Sigma}$

$$C = K_{ab} u^a u^b$$

$$\theta = \frac{\pi}{2} \quad , u^\theta = 0 = u_\theta$$

$$-1 = g^{ab} u_a u_b = g^{tt}(\tilde{E})^2 + g^{rr}(u_r)^2 + g^{\phi\phi}(\tilde{J})^2 - 2 g^{t\phi} \tilde{E} \tilde{J}$$

$$C = K_{ab} u^a u^b$$

$$\theta = \frac{\pi}{2}, \quad u^\theta = 0 = u_\theta$$

$$-1 = g^{ab} u_a u_b = g^{tt}(\tilde{E})^2 + g^{rr}(u_r)^2 + g^{\phi\phi}(\tilde{J})^2 - 2g^{t\phi}\tilde{E}\tilde{J}$$

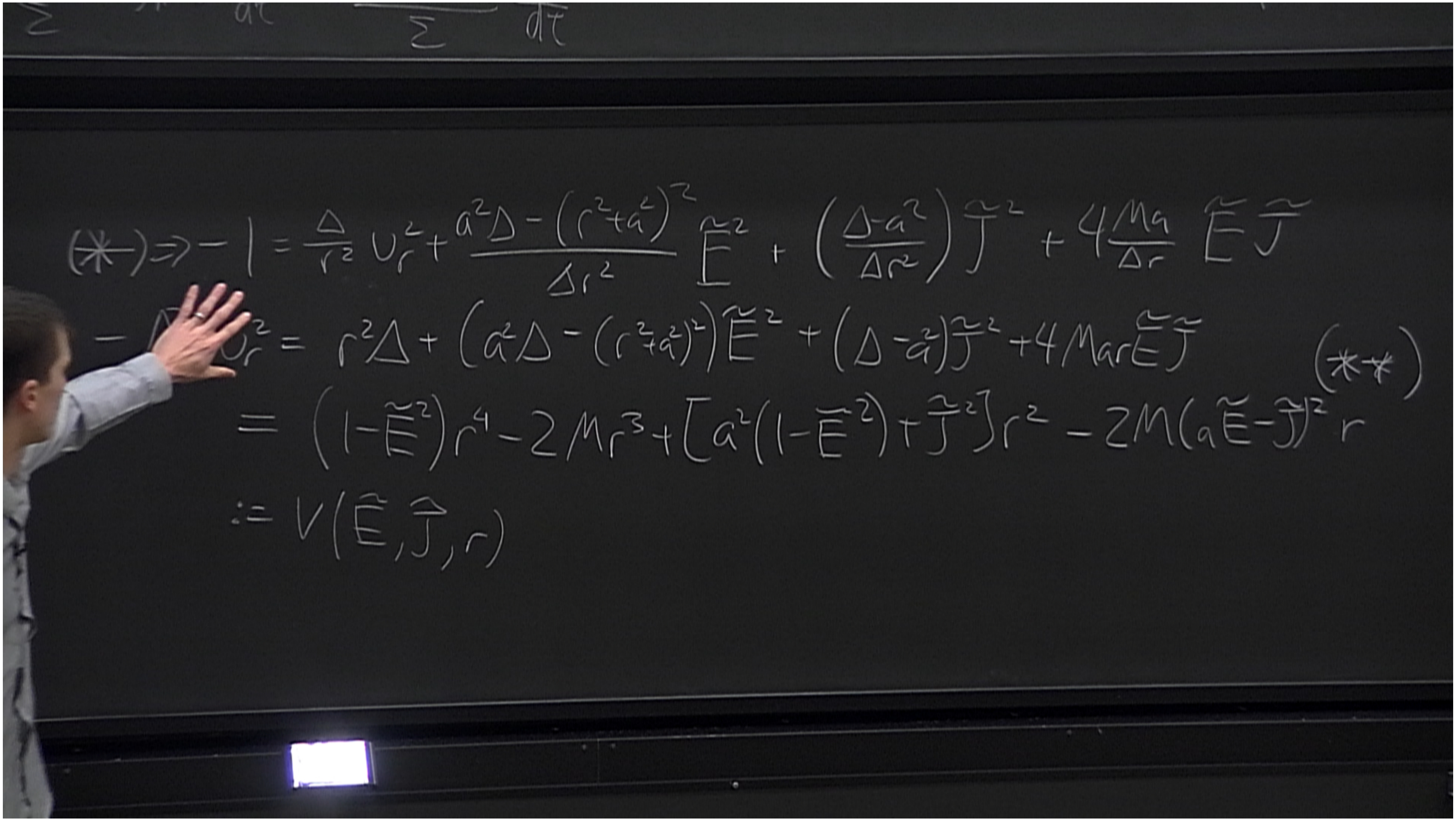
$$u_r = g_{rr} u^r = \frac{r^2}{\Delta}\frac{dr}{d\tau}$$

$$g^{tt} = \frac{-\left((r^2+a^2)^2 - a^2\Delta\right)}{\Delta r^2}$$

$$g^{rr} = \frac{\Delta}{r^2}\ , \quad g^{rr} = \frac{\Delta}{r^2}\ , \quad g^{\phi\phi} = \frac{\Delta - a^2}{\Delta r^2}$$

$$g^{t\phi} = \frac{-2Ma}{\Delta r}$$

$$(*) \Rightarrow -1 = \frac{\Delta}{r^2} v_r^2 + \frac{a^2\Delta - (r^2+a^2)^2}{\Delta r^2} \tilde{E}^2 + \left(\frac{\Delta - a^2}{\Delta r^2}\right) \tilde{J}^2 + 4\frac{Ma}{\Delta r} \tilde{E} \tilde{J}$$


$(*) \Rightarrow -1 = \frac{\Delta}{r^2} v_r^2 + \frac{a^2\Delta - (r^2+a^2)^2}{\Delta r^2} \tilde{E}^2 + \left(\frac{\Delta - a^2}{\Delta r^2}\right) \tilde{J}^2 + 4\frac{Ma}{\Delta r} \tilde{E}\tilde{J}$
$-\Delta^2 v_r^2 = r^2\Delta + (a^2\Delta - (r^2+a^2)^2)\tilde{E}^2 + (\Delta - a^2)\tilde{J}^2 + 4Mar\tilde{E}\tilde{J}$
$(**)$
$= (1-\tilde{E}^2)r^4 - 2Mr^3 + [a^2(1-\tilde{E}^2) + \tilde{J}^2]r^2 - 2M(a\tilde{E} - \tilde{J})^2 r$
$:= V(\tilde{E}, \tilde{J}, r)$

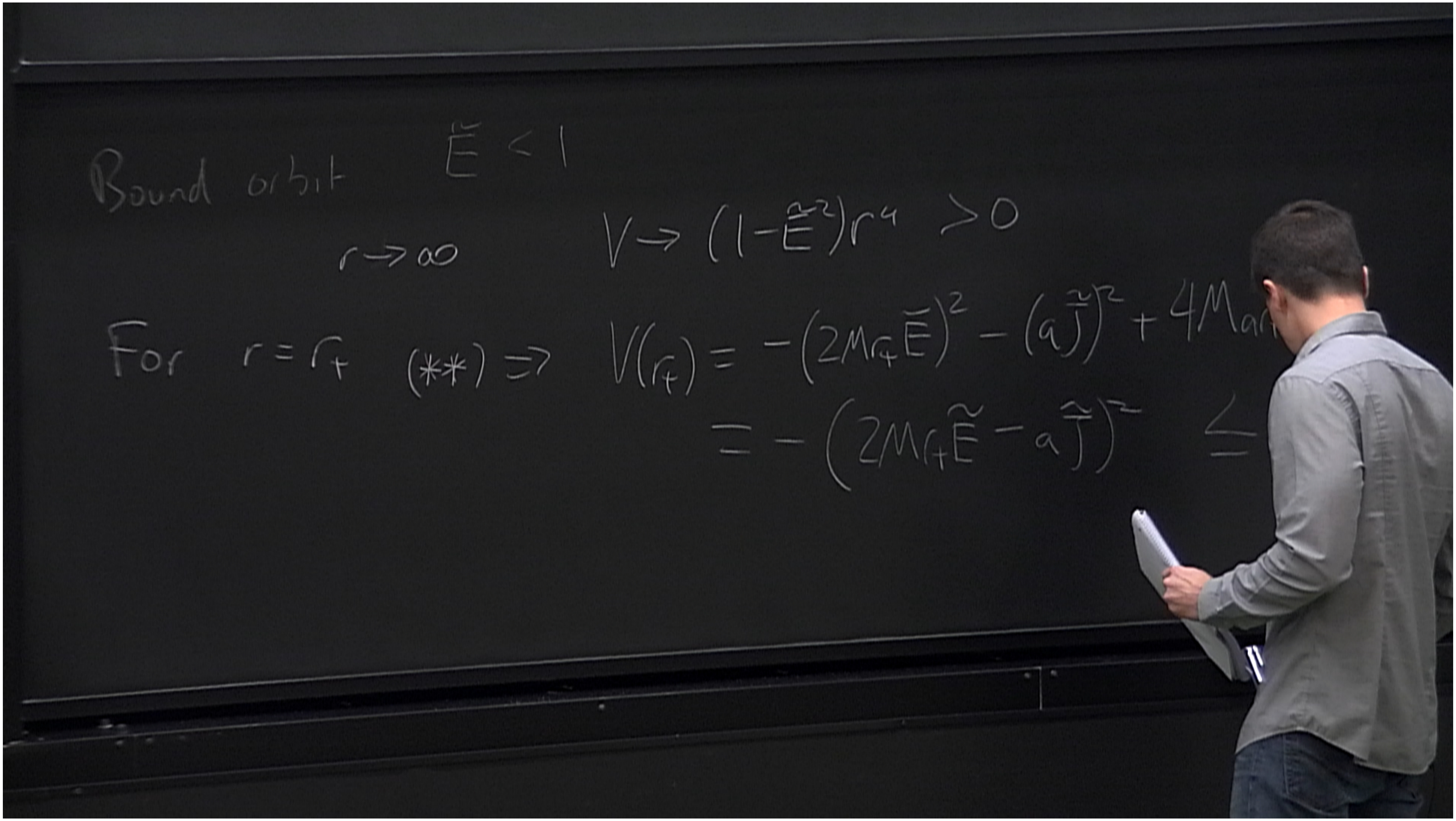
$$(*) \Rightarrow -1 = \frac{\Delta}{r^2} u_r^2 + \frac{a^2\Delta - (r^2+a^2)^2}{\Delta r^2}\tilde{E}^2 + \left(\frac{\Delta - a^2}{\Delta r^2}\right)\tilde{J}^2 + 4\frac{Ma}{\Delta r}\tilde{E}\tilde{J}$$

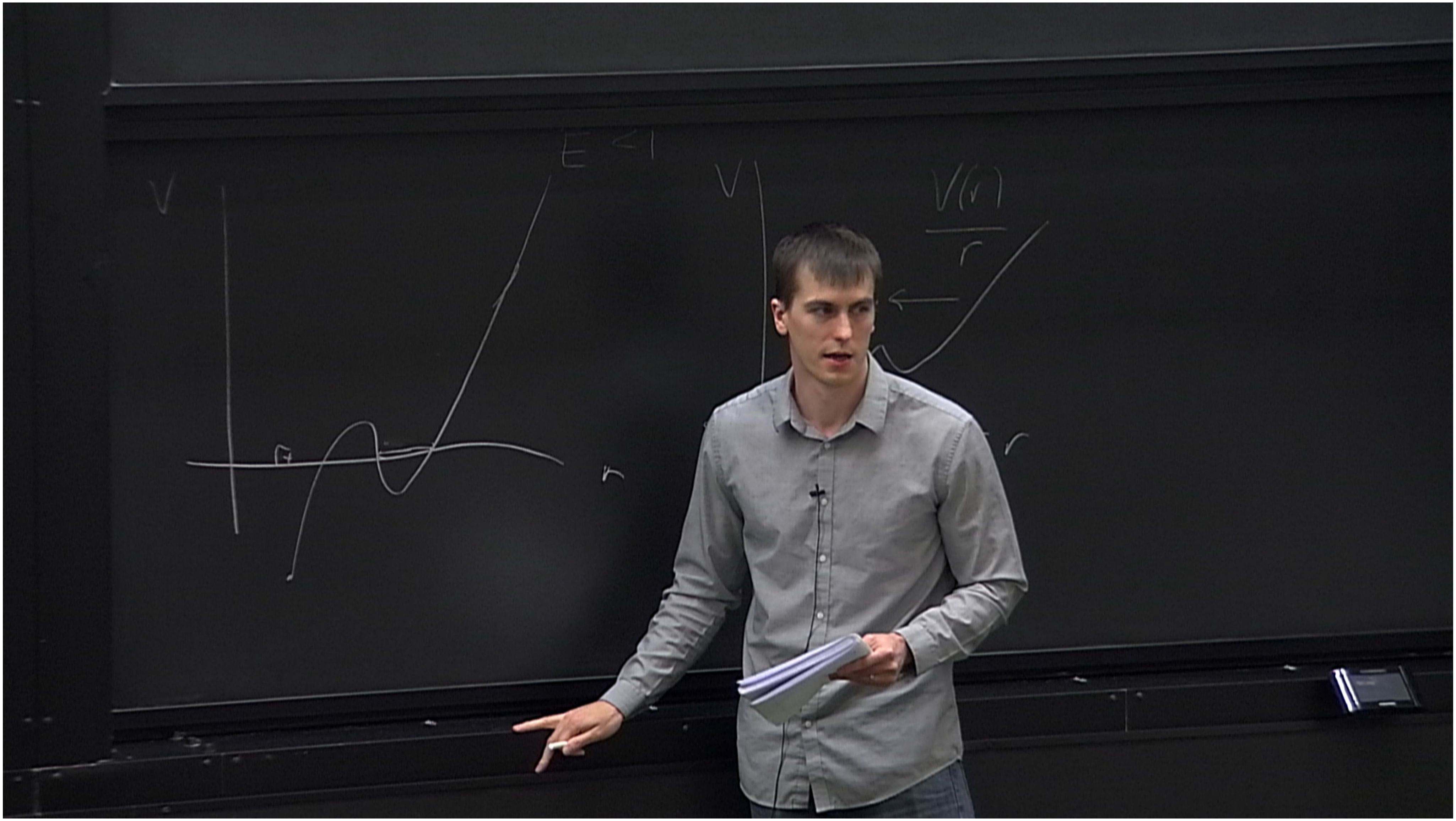
$$-\Delta^2 u_r^2 = r^2\Delta + \left(a^2\Delta - (r^2+a^2)^2\right)\tilde{E}^2 + (\Delta - a^2)\tilde{J}^2 + 4Mar\tilde{E}\tilde{J} \qquad (**)$$

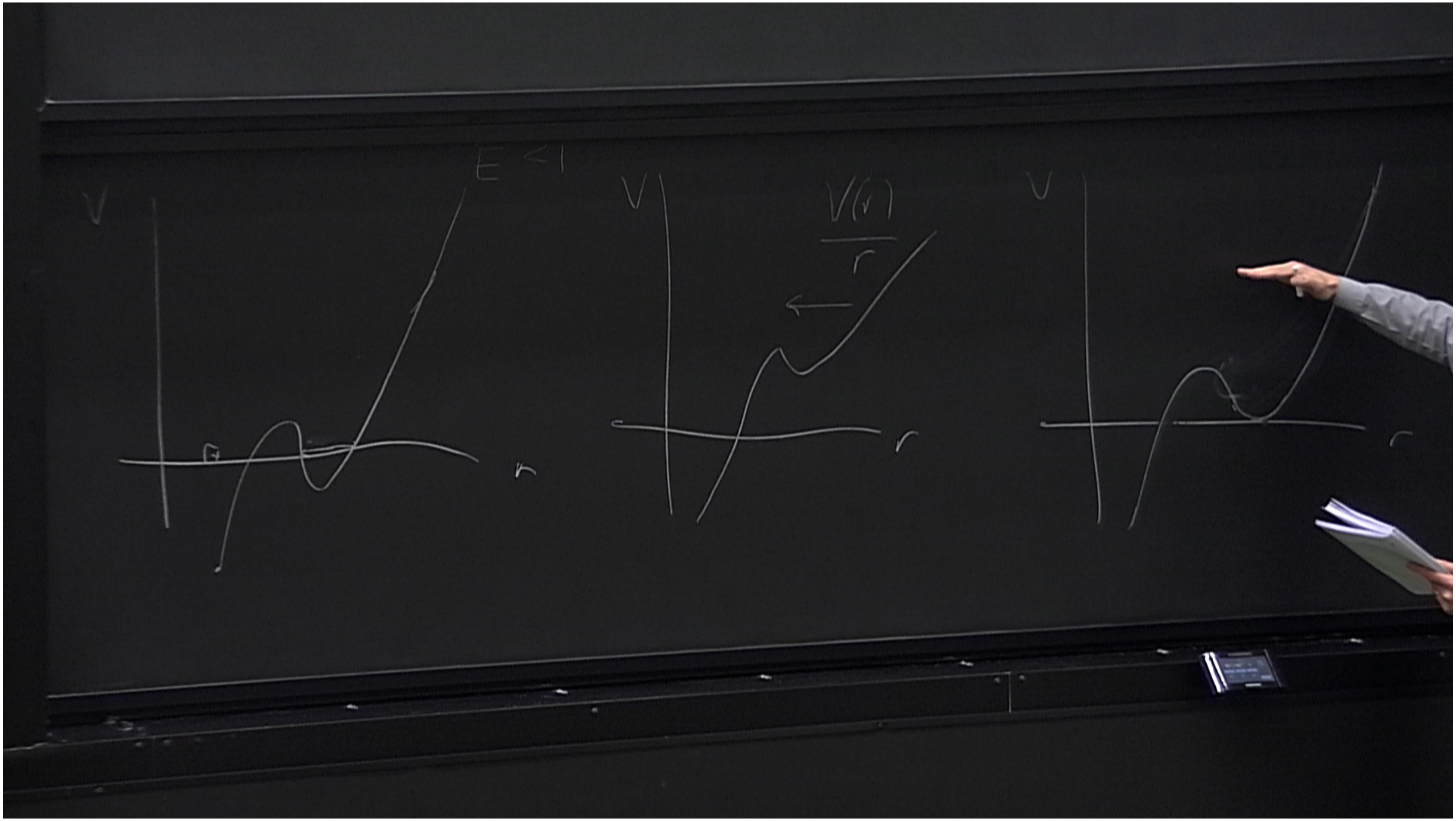
$$= (1-\tilde{E}^2) r^4 - 2Mr^3 + \left[a^2(1-\tilde{E}^2) + \tilde{J}^2\right] r^2 - 2M(a\tilde{E} - \tilde{J})^2 r$$

$$:= V(\tilde{E}, \tilde{J}, r)$$

Turing points $\left(\frac{dr}{d\tau} = 0\right)$ when $V = 0$


Bound orbit $\tilde{E} < 1$
$r \to \infty$ $V \to (1-\tilde{E}^2)r^4 > 0$
For $r = r_+$ $(**) \Rightarrow$ $V(r_+) = -(2Mr_+\tilde{E})^2 - (a\tilde{J})^2 + 4Ma$
$= -(2Mr_+\tilde{E} - a\tilde{J})^2 \leq$


E < 1
V
r
V
V(r)/r
r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V
r
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r
V
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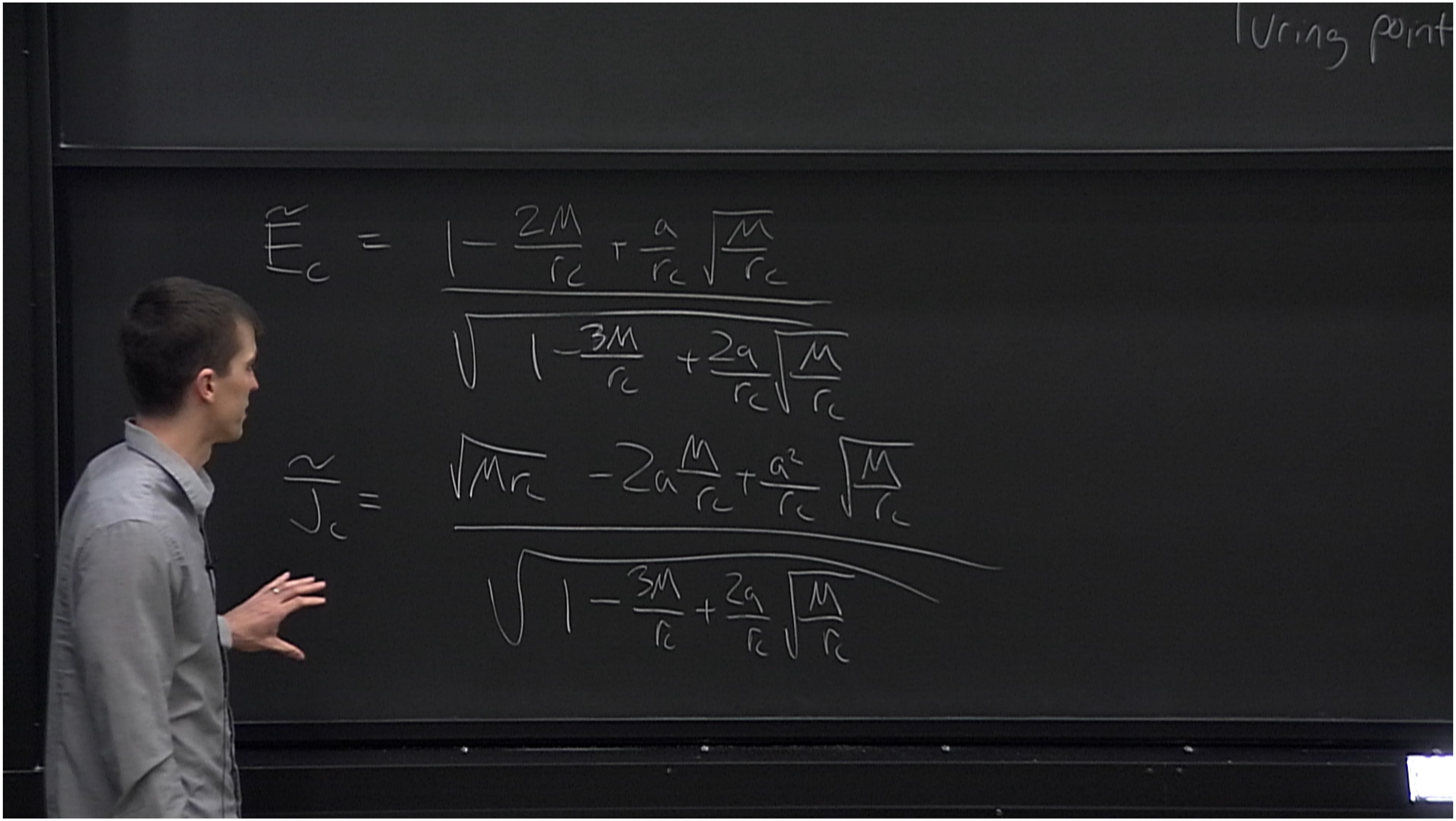
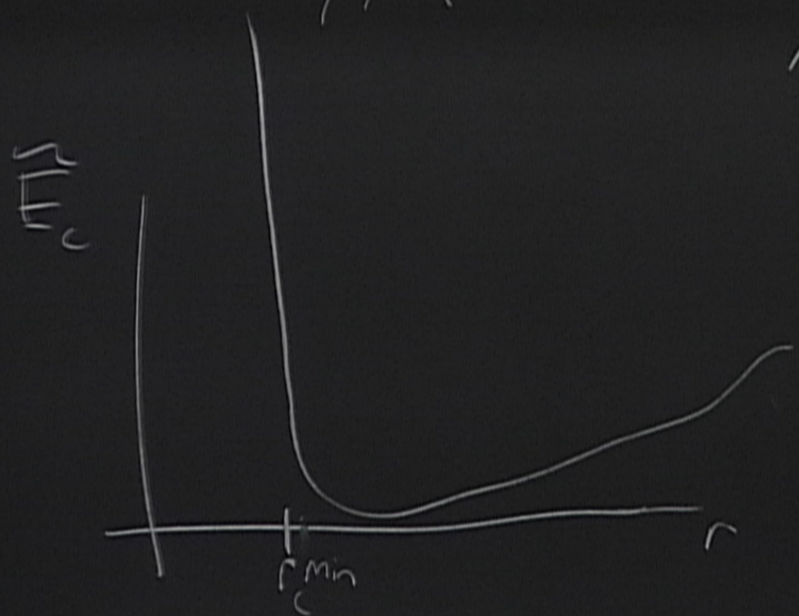

Turning point
$\tilde{E}_c = \dfrac{1 - \frac{2M}{r_c} + \frac{a}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$
$\tilde{J}_c = \dfrac{\sqrt{M r_c} - 2a\frac{M}{r_c} + \frac{a^2}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$

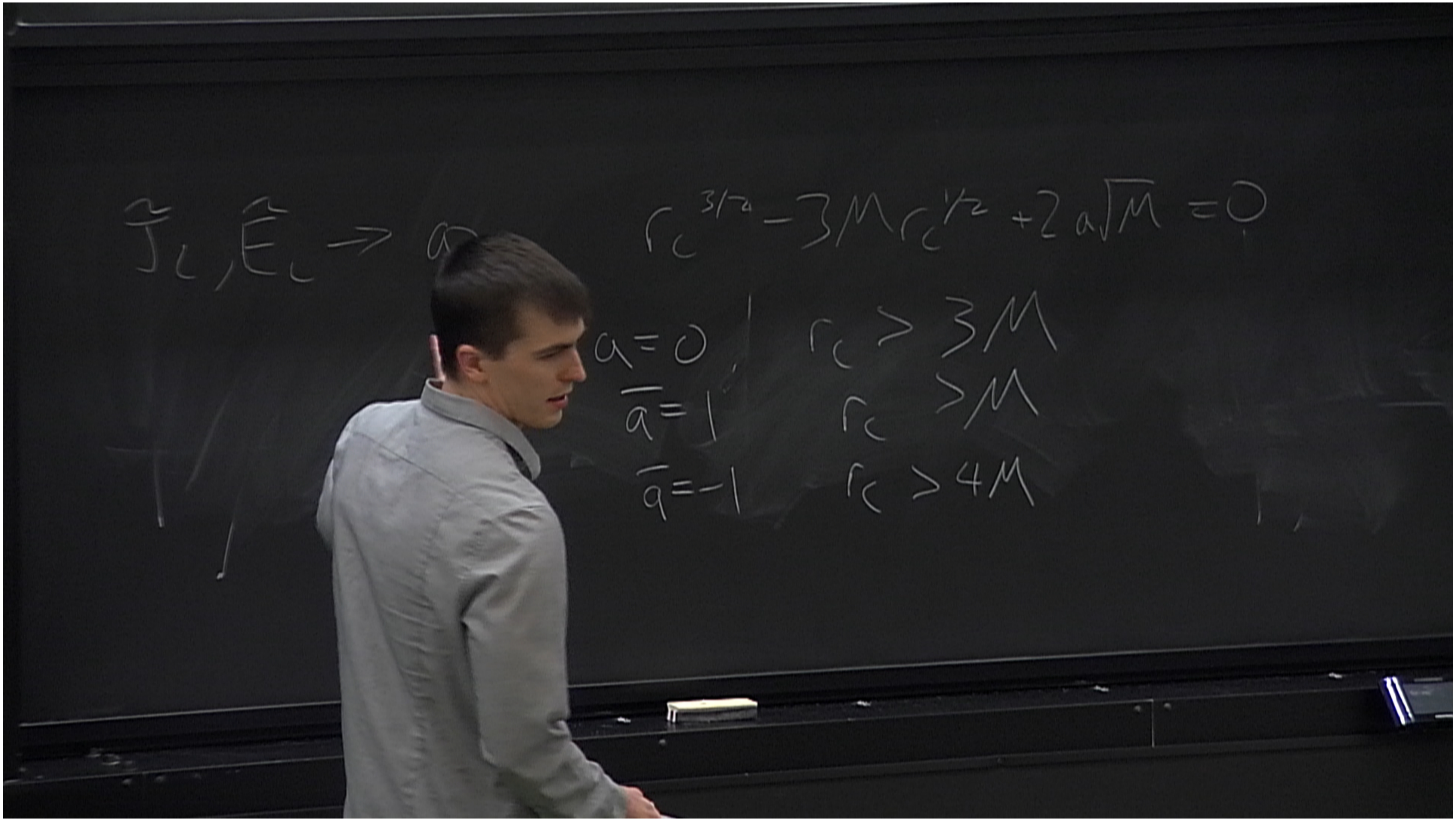
For $r_c/M >> 1$, $\tilde{E}_c = 1 - \frac{M}{2r_c}$

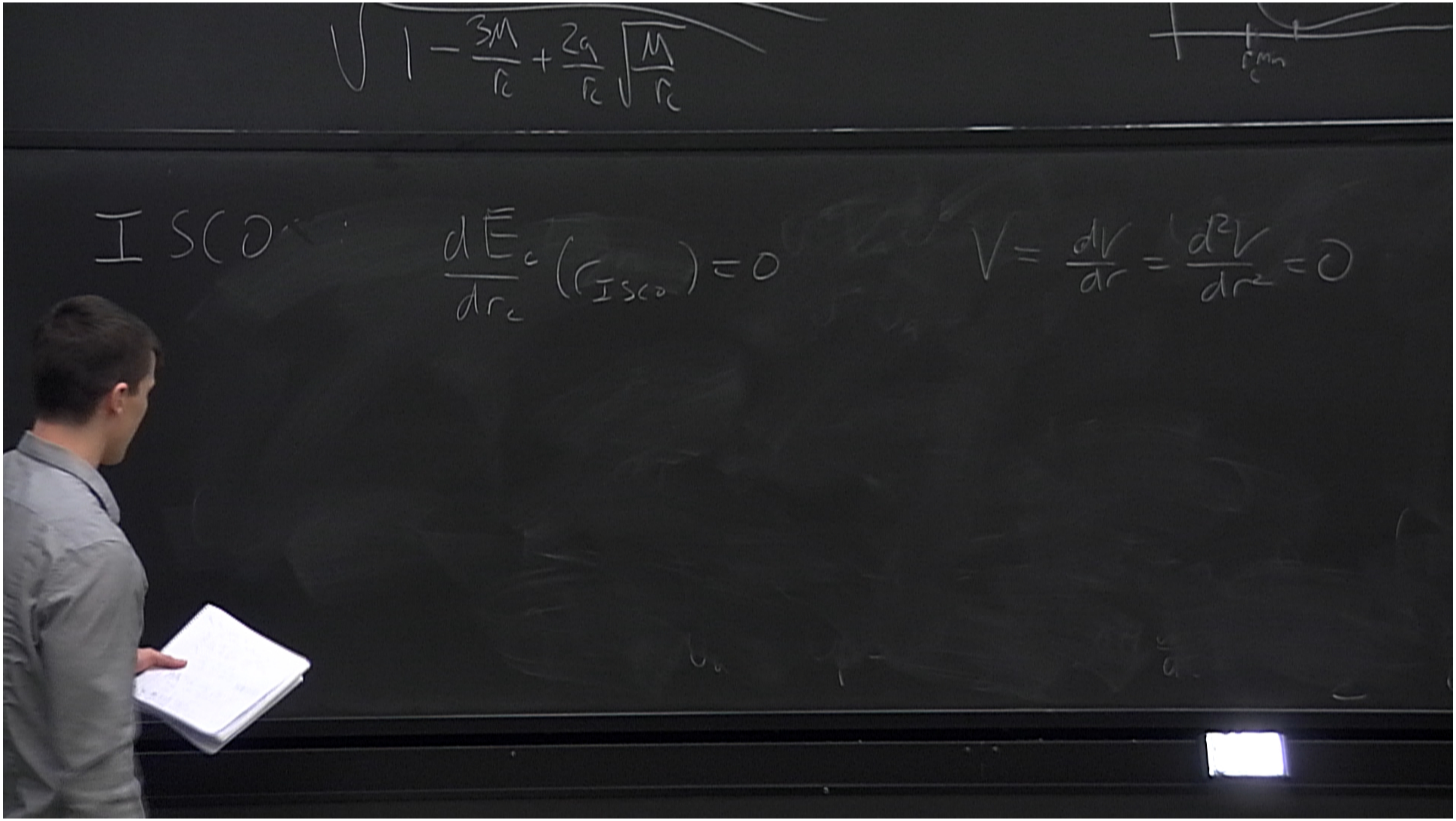
$\tilde{J}_c = \sqrt{M r_c}$

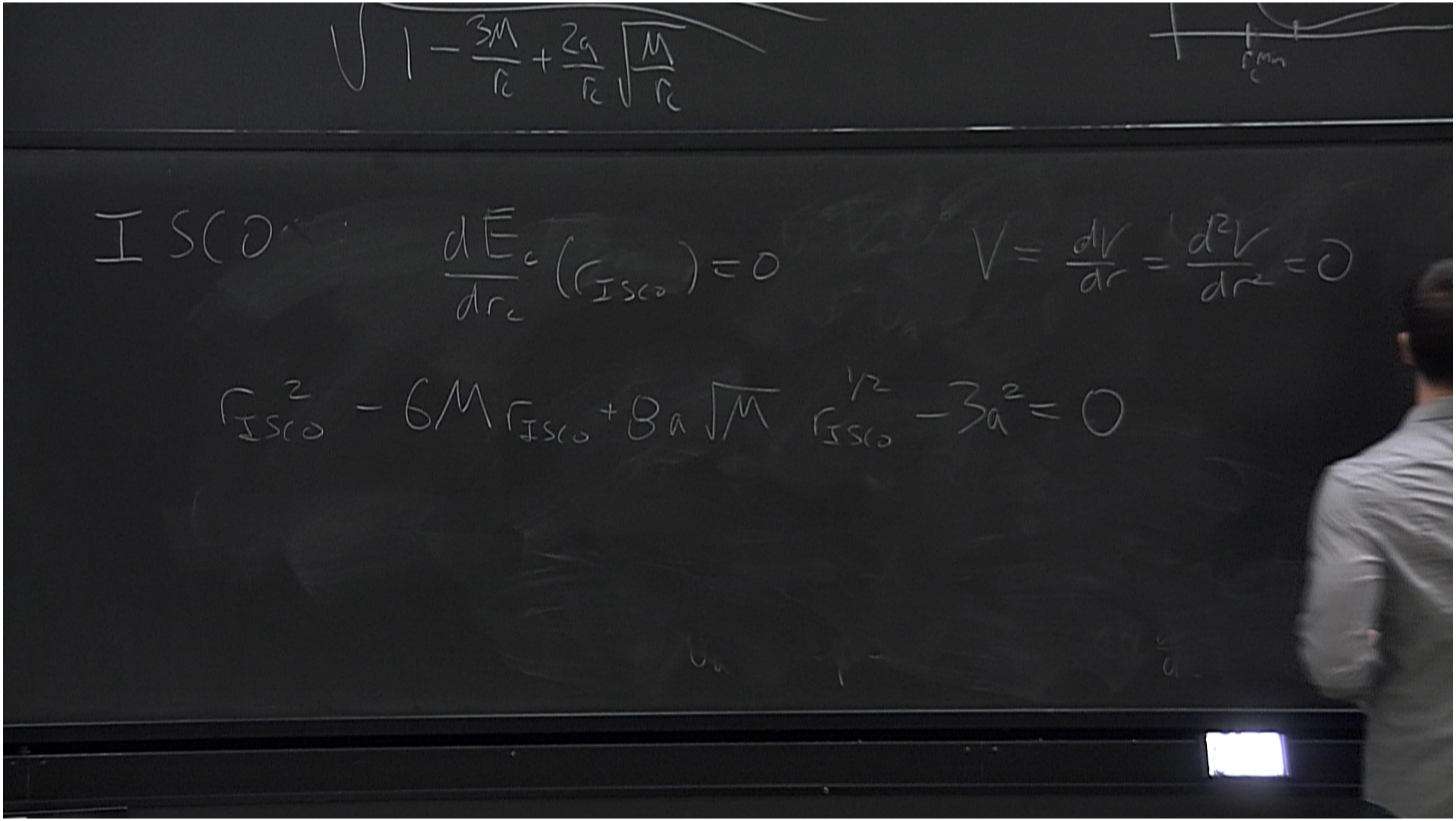
For $r_c/M \gg 1$, $\tilde{E}_c = 1 - \frac{M}{2r_c}$

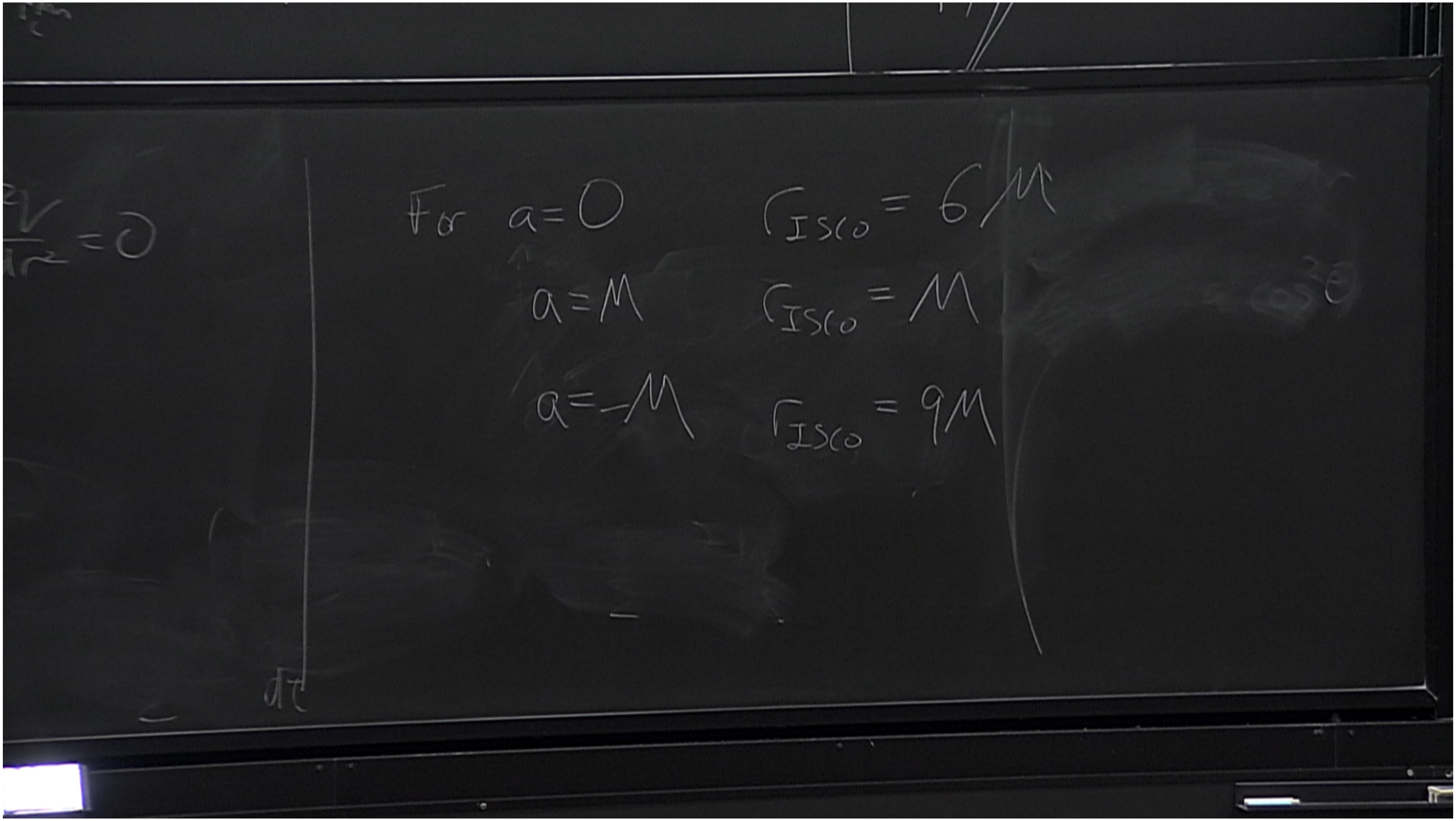
$\tilde{J}_c = \sqrt{M r_c}$

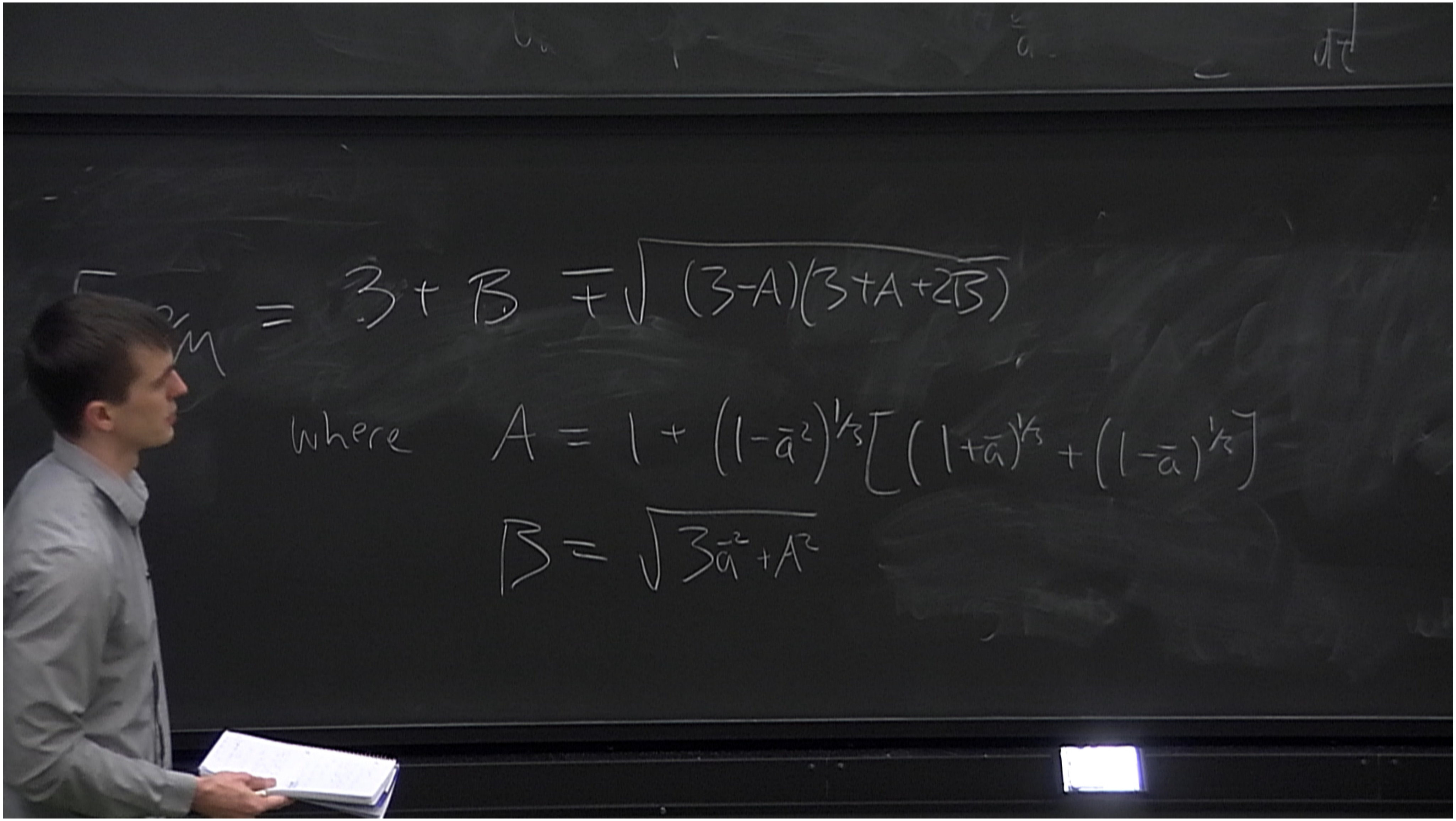


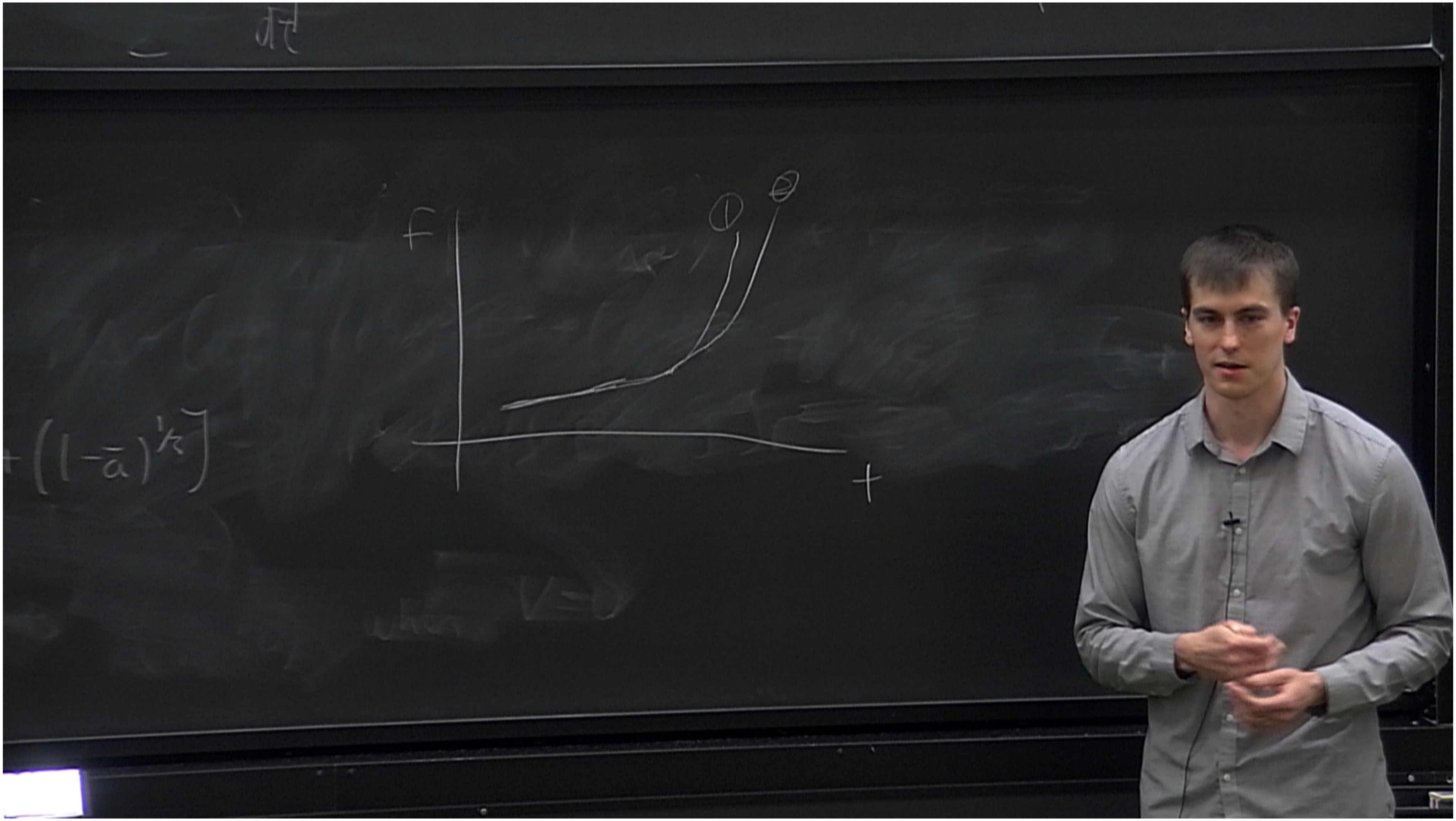

$\tilde{J}_c, \tilde{E}_c \to a$
$r_c^{3/2} - 3M r_c^{1/2} + 2a\sqrt{M} = 0$
$a = 0$ $r_c > 3M$
$\bar{a} = 1$ $r_c > M$
$\bar{a} = -1$ $r_c > 4M$


$\sqrt{1-\frac{3M}{r_c}+\frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}$
r_c^{min}
ISCO:
$\frac{dE_c}{dr_c}(r_{ISCO})=0$
$V=\frac{dV}{dr}=\frac{d^2V}{dr^2}=0$


$\sqrt{1-\frac{3M}{r_c}+\frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}$
ISCO:
$\frac{dE_c}{dr_c}(r_{ISCO})=0$
$V=\frac{dV}{dr}=\frac{d^2V}{dr^2}=0$
$r_{ISCO}^2-6Mr_{ISCO}+8a\sqrt{M}\,r_{ISCO}^{1/2}-3a^2=0$


For $a=0$ $r_{ISCO} = 6M$
$a=M$ $r_{ISCO} = M$
$a=-M$ $r_{ISCO} = 9M$


$= 3 + B \mp \sqrt{(3-A)(3+A+2B)}$
where $A = 1 + (1-\bar{a}^2)^{1/3}\left[(1+\bar{a})^{1/3} + (1-\bar{a})^{1/3}\right]$
$B = \sqrt{3\bar{a}^2 + A^2}$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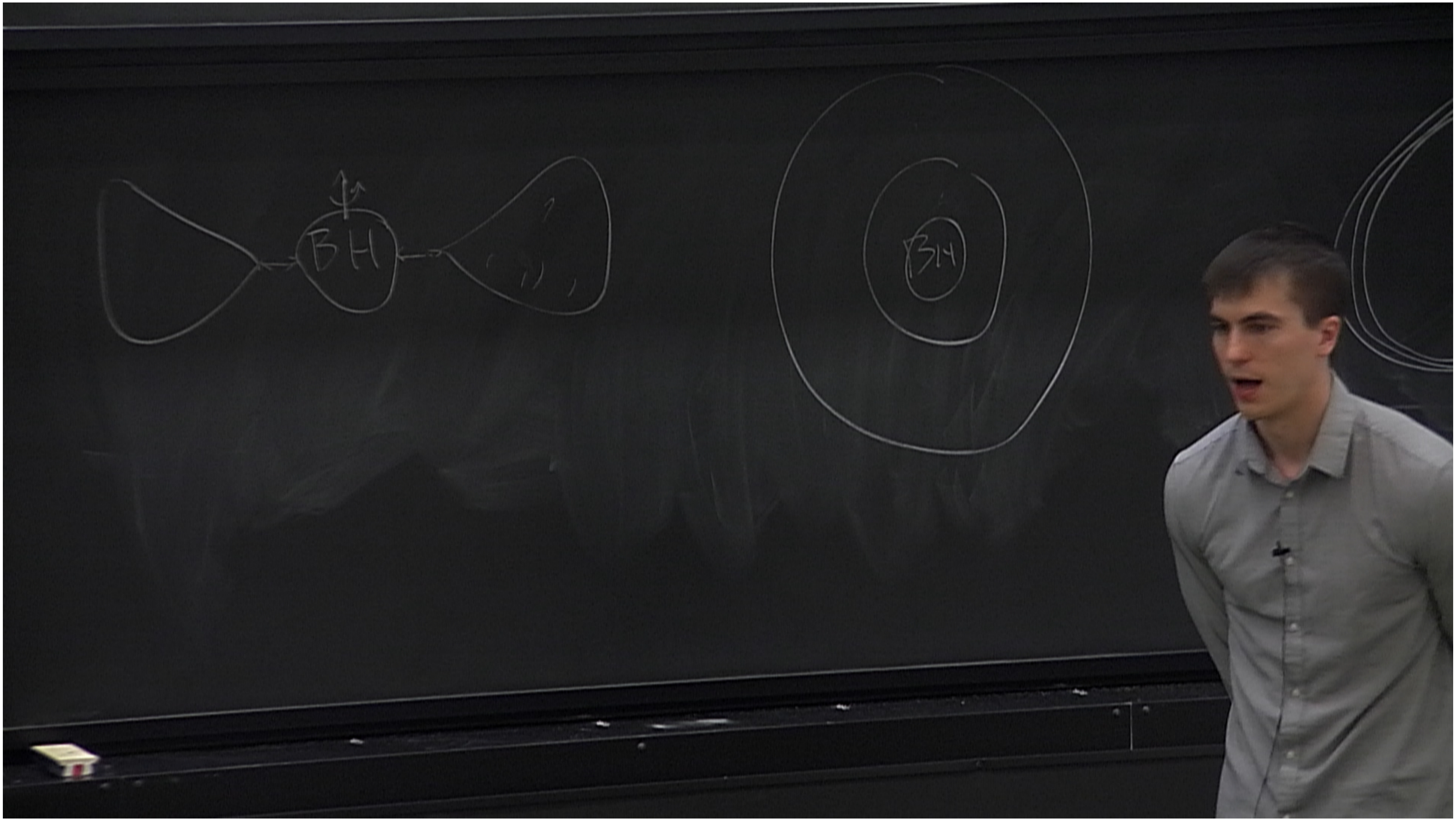
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