

Title: PSI 2018/2019 - Strong Field Gravity - Lecture 1

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Collection: PSI 2018/2019 - Strong Field Gravity (East)

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Geometric units : $G = c = 1$

Index conventions: $a, b, c, \dots$ 4 index (spacetime)
 $i, j, k, \dots$ 3 index (spatial)

$$U^a = (U^+, U^i)$$

East coast metric signature: $(-, +++)$

Metric : $ds^2 = g_{ab} dx^a dx^b$

Christoffel Symbols:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bc} + \partial_b g_{ac} - \partial_d g_{ab})$$

$$\nabla_a T^{b_1 \dots b_n}_{c_1 \dots c_n} = \partial_a T^{b_1 \dots b_n}_{c_1 \dots c_n} + \Gamma^{b_1}_{ad} T^{d b_2 \dots b_n}_{c_1 \dots c_n} + \dots + \Gamma^{b_n}_{ad} T^{b_1 \dots b_{n-1} d}_{c_1 \dots c_n} - \Gamma^d_{ca} T^{b_1 \dots b_n}_{d c_2 \dots c_n} - \dots$$

Metric : $ds^2 = g_{ab} dx^a dx^b$

Christoffel Symbols:

$$\Gamma_{ab}^c = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$$

$$\nabla_a T^{b_1 \dots b_n} = \partial_a T^{b_1 \dots b_n} + \sum_{c_1 \dots c_n} \Gamma_{c_1 a}^{b_1} T^{b_2 \dots b_n} - \sum_{c_1 \dots c_n} \Gamma_{c_1 a}^{c_1} T^{b_1 \dots b_n} - \dots$$

Metric compatible: $\nabla_a g_{bc} = 0$

Riemann tensor: $(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R^c{}_{dab} V^d$

Ricci tensor: $R_{ab} = R^c{}_{acb}$

Ricci scalar: $R = R_{ab} g^{ab}$

Einstein tensor: $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$

Einstein Equations: $G_{ab} = 8\pi T_{ab}$

Perfect fluid $T_{ab} = (\rho + P)u^a u^b + P g_{ab}$

S-E conservation: $\nabla_a T^{ab} = 0$

Null energy condition: $T_{ab} k^a k^b \geq 0$

$$(\rho + \rho)(K^a u_a)^2 \geq 0$$

$$(\rho + p)(k^a u_a)^2 \geq 0$$

Kerr Black Holes

- Only possible stationary, vacuum BH solutions
- Perturbations decay rapidly (Week 3)
- Final state generic collapse : (M, J) + GWs

Metric in Boyer-Lindquist coordinates

$$ds^2 = - \left(\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) dt^2 - \frac{2a \sin^2 \theta (r^2 + \bar{a}^2 - \Delta)}{\Sigma} dt d\phi + \left(\frac{(r^2 + \bar{a}^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2 \\ + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2 \cos^2 \theta$$

$$\Delta = r^2 + a^2 - 2Mr$$

$$a = J/M, \quad \bar{a} = J/M^2$$

Fix $\bar{a}$ let $M \rightarrow 0$

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \right) dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2$$

$$x = (r^2 + a^2)^{1/2} \sin \theta \cos \phi$$

$$y = (r^2 + a^2)^{1/2} \sin \theta \sin \phi$$

$$z = r \cos \theta$$

Symmetries of Kerr

dt^2

$$\text{Killing: } \mathcal{L}_K g_{ab} = 0$$

$$K^c \partial_c g_{ab} + (\partial_a K^c) g_{cb} + (\partial_b K^c) g_{ac} = 0$$

$$\partial \rightarrow \nabla$$

$$\mathcal{L}_K g_{ab} = 0 + \nabla_a K_b + \nabla_b K_a = \frac{1}{2} \nabla_K K_{ab}$$

Killing tensor

$$\nabla_{(a} K_{bc)} = 0$$

$$\text{Kerr: } K_{ab} = r^2 g_{ab} + 2\Sigma \ell_{(a} n_{b)}$$

Horizons

$$\Sigma = 0, \Delta = 0$$

- Coordinates breakdown

$$R_{abcd} R^{abcd}$$

$$\Sigma = 0 \Rightarrow$$

blows up $\Sigma = 0$

$$r^2 + a^2 \cos^2 \theta = 0$$

$r = \text{constant}$ null

$$g^{ab} \partial_a r \partial_b r = 0$$

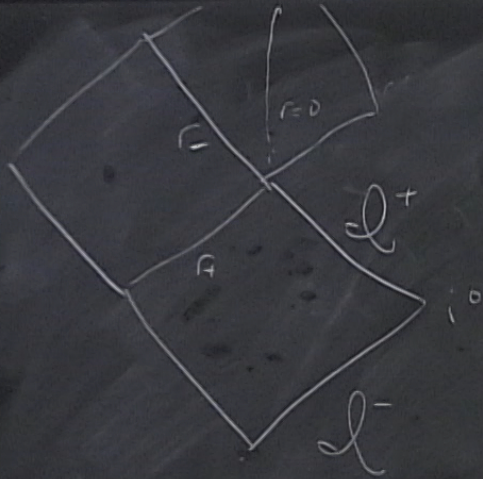
$$\Rightarrow g^{rr} = 0 = \frac{\Delta}{\Sigma}$$

$$\Delta = (r - r_+)(r - r_-)$$

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}$$

Outer / inner horizon

$$a > M$$

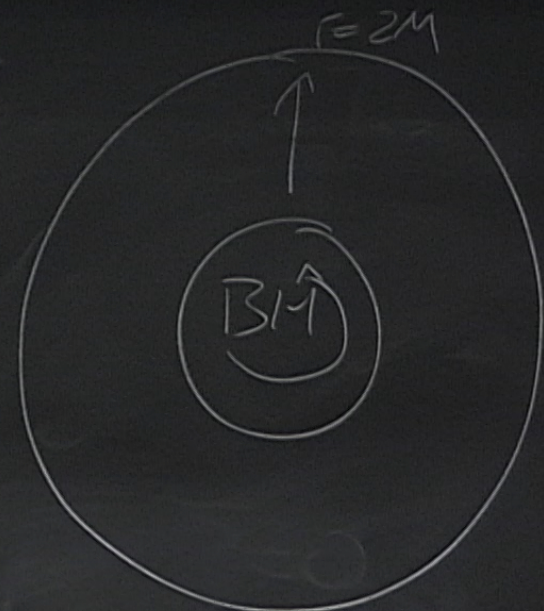
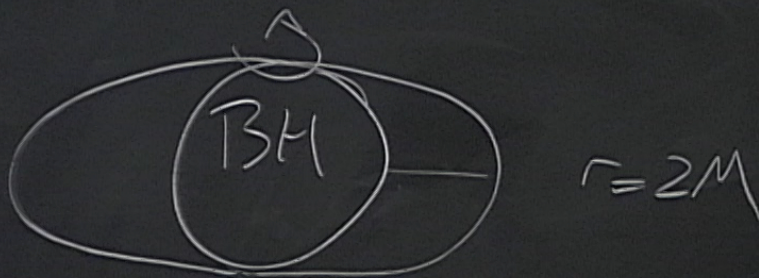


Ergosphere

$$\uparrow^a \uparrow_a = g_{tt} = \frac{a^2 \sin^2 \theta}{\Sigma} \geq 0 \quad \text{at } r=r_+$$

null, $\uparrow^a \uparrow_a = 0 \Rightarrow g_{tt} = 0 \Rightarrow \Delta = a^2 \sin^2 \theta$

$$r^2 + a^2 \cos^2 \theta - 2Mr = 0$$



u^a timelike

$$-1 = g_{ab} u^a u^b = \underbrace{g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + g_{\phi\phi}(u^\phi)^2}_{>0} + \underbrace{2g_{t\phi} u^t u^\phi}_{<0}$$

$$g^{t\phi} < 0 \quad u^t = \frac{dt}{d\tau} > 0 \quad \Rightarrow \quad u^\phi = \frac{d\phi}{d\tau} > 0$$

$$\chi^a = \alpha \left(\dot{\phi}^a + \Omega \hat{\phi}^a \right)$$

$$\text{ZAMO} : \hat{\phi}_a \chi^a = 0 \Rightarrow g_{\phi t} + g_{\phi\phi} \Omega = 0$$

$$\Omega = \frac{d\phi}{dt} = - \frac{g_{\phi t}}{g_{\phi\phi}} \rightarrow \frac{\bar{a}}{(2g)} = \Omega_H$$

$$\chi^a = \frac{\uparrow^a}{r} + \Omega_H \frac{\uparrow^a}{\phi}$$

$$\chi^a \chi_a \rightarrow 0 \quad r \rightarrow r_+$$

For $\bar{a} \ll 1$

$$r_+ \approx \left(2 - \frac{1}{2}\bar{a}^2\right)M$$

$$\Omega_H \approx \frac{\bar{a}}{4M}$$

For $\bar{a} = 1 - \epsilon$

$\epsilon \gg 1$

$$r_+ = (1 + \sqrt{2\epsilon})M$$

$$\Omega_H \approx \frac{1 - \sqrt{2\epsilon}}{2M}$$