

Title: PSI 2018/2019 - Quantum Field Theory III - Lecture 15

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Chiral Anomaly from Functional Integral

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Outline: $\mathcal{D}\psi' \mathcal{D}\bar{\psi}'$

$$\partial_\mu J_5^\mu = ?$$

exact result
instantons

Massless QED in $D=4$ Euclidean

↑
not essential

↑
 $i\not{D}$ is hermitian

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$$

$$\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix} \quad i=1,2,3$$

Hermitian

$$\gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$$

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S[\psi, \bar{\psi}, A]}$$

$$\psi \rightarrow \psi' = e^{i\theta \gamma^5} \psi$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} e^{i\theta \gamma^5}$$

$$A_\mu \rightarrow A'_\mu = A_\mu$$

$$S \rightarrow S' = S + \int d^4x \theta \partial_\mu J_5^\mu$$

$$0 = \left. \frac{\delta Z[A]}{\delta \theta} \right|_{\theta=0} = \frac{\delta}{\delta \theta(x)} \left[\int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' e^{-S - \int d^4x \theta \partial_n J_5^n} \right] \Big|_{\theta=0}$$

$$i \not{D} \phi_n = \lambda_n \phi_n \quad \begin{array}{l} \nwarrow \text{orthonormal set of eigenfunctions} \\ \uparrow \text{real eigenvalue} \end{array}$$

$$\int d^4x \phi_n^\dagger(x) \phi_m(x) = \delta_{nm} \quad \text{orthonormal}$$

$$\sum_n \phi_{n\alpha}^\dagger(x) \phi_{n\beta}(y) = \delta(x-y) \delta_{\alpha\beta} \quad \text{complete}$$

$$\Psi(x) = \sum_n a_n \phi_n(x)$$

$a_n, \bar{b}_n$ Grassman variables

$$\bar{\Psi}(x) = \sum_n \bar{b}_n \phi_n^\dagger(x)$$

$$\mathcal{D}\Psi \mathcal{D}\bar{\Psi} = \prod_n da_n \prod_m d\bar{b}_m$$

$$\Psi'(x) = \sum_n a'_n \phi_n(x)$$

$$a_n = \langle \phi_n | \Psi' \rangle$$

$$= \langle \phi_n | e^{i\theta \gamma^5} | \Psi \rangle$$

$$= \sum_m a_m \underbrace{\langle \phi_n | e^{i\theta \gamma^5} | \phi_m \rangle}_{C_{nm}}$$

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$$

5 real variables

$$\sum_n a_n \underbrace{\langle \phi_n | e^{i\Theta \gamma^5} | \phi_m \rangle}_{C_{nm}}$$

$$\begin{aligned} \Pi d a_n &= (\det C)^{-1} \Pi d a_n \\ &= e^{-\text{Tr} \log C} \Pi d a_n \\ &= e^{-i \text{Tr} \Theta \gamma^5} \Pi d a_n \end{aligned}$$

$$\text{Tr} \Theta \gamma^5 = \int d^4 x \phi_{m\alpha}^\dagger(x) \Theta(x) \gamma_{\alpha\beta}^5 \phi_{m\beta}(x)$$

↑
both matrix + functional trace

tr = matrix trace only

$$S \rightarrow S' = S + \int d^4x \theta \partial_\mu J_5^\mu$$

$$\sum_n \phi_{n\alpha}^\dagger(x) \phi_{n\beta}(y) = \delta(x-y) \delta_{\alpha\beta}$$

$$0 = \frac{\delta Z[A]}{\delta \theta(x)} \Big|_{\theta=0} = \frac{\delta}{\delta \theta(x)} \left(\int D\psi D\bar{\psi} e^{-S - \theta \partial_\mu J_5^\mu - 2i \text{Tr} \theta \gamma^5} \right) \Big|_{\theta=0}$$

$\uparrow$ same from $d\alpha \mp d\bar{\alpha}$ $\uparrow$ anomaly

$$\begin{aligned} \text{Tr} \theta \gamma^5 &= \int d^4x \underbrace{f^{(4)}(x-x)}_{=\infty} \theta(x) \underbrace{\text{tr} \gamma^5}_{=0} \\ &= \infty \cdot 0 \end{aligned}$$

$\uparrow$ UV divergence

$$\text{Hermitian} \rightarrow \gamma^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$$

regulator: inserting $f(k^2)$ into integrand

$$f(0) = 1$$

$$f(\pm\infty) = f'(\pm\infty) = \dots = 0 \quad \text{eg } f(k^2) = e^{-k^2} \text{ or } \frac{1}{1+k^2}$$

$$\text{Tr } \Theta \gamma^5 = \lim_{\lambda \rightarrow \infty} \int d^4x \sum_n \text{tr} \Theta \phi_n^+ \gamma^5 f\left(-\frac{\lambda_n^2}{\Lambda^2}\right) \phi_n$$

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$$= \lim_{\Lambda \rightarrow \infty} \int d^4x \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \sum_n \Theta e^{-ik'y} \tilde{\phi}_n^+(k') \gamma^5 f\left(\frac{|\vec{0}|^2}{\Lambda^2}\right) \tilde{\phi}_n(k) e^{ik'y}$$

$$\text{Hermitian} \rightarrow \gamma^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$$

regulator inserting $f(k^2)$ into integrand

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$$f(\pm\infty) = f'(\pm\infty) = \dots = 0$$

$$\text{e.g. } f(k^2) = e^{-k^2} \text{ or } \frac{1}{1+k^2}$$

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$$= \lim_{\Lambda \rightarrow \infty} \int d^4x \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \sum_n \text{tr} \Theta e^{-ik'x} \tilde{\phi}_n^+(k') \gamma^5 \underbrace{e^{ikx} f\left(-\frac{(\not{D}-K)^2}{\Lambda^2}\right) \tilde{\phi}_n(k)}_{f\left(\frac{(\not{D})^2}{\Lambda^2}\right) \tilde{\phi}_n(k) e^{ikx}}$$

$$\text{Hermitian} \rightarrow \gamma^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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$$= \lim_{\Lambda \rightarrow \infty} \int d^4x \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \sum_n \Theta e^{-ik'x} \bar{\phi}_n(k') \gamma^5 f\left(\frac{(\not{D})^2}{\Lambda^2}\right) \phi_n(k) e^{ikx}$$

$$= \infty - 0$$

UV divergence

$$\sum \tilde{\varphi}_n^\dagger(k') \tilde{\varphi}_n(k) = (2\pi)^4 \delta(k-k') \underset{\substack{\uparrow \\ \text{spinor}}}{1} \quad \text{FT of completeness}$$

$$\text{Tr } \Theta \gamma^5 = \lim_{\Lambda \rightarrow \infty} \int d^4x \frac{d^4k}{(2\pi)^4} \text{Tr } \gamma^5 \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{-i \not{D}}{\Lambda} \right)^{2N} f^{(N)}\left(\frac{k^2}{\Lambda^2}\right)$$

$$= \infty - 0$$

W divergence

$$\sum \tilde{\varphi}_n^+(k') \tilde{\varphi}_n(k) = (2\pi)^4 \delta(k-k') \underset{\text{spinor}}{1} \quad \text{FT of completeness}$$

$$\text{Tr } \Theta \gamma^5 = \lim_{\Lambda \rightarrow \infty} \int d^4x \frac{d^4k}{(2\pi)^4} \Theta \text{tr } \gamma^5 \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{-i \not{D}}{\Lambda} \right)^{2N} f^{(N)}\left(\frac{k^2}{\Lambda^2}\right)$$

only $N=2$ contributes

$$\text{tr } \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4 \epsilon^{\mu\nu\rho\sigma} \quad \text{tr } \gamma^5 = \text{tr } \gamma^5 \gamma^\mu \gamma^\mu = 0 \quad N=0 + N=1 \text{ vanish}$$

$N=3$ have extra factors of $\frac{1}{\Lambda^2} \rightarrow$ vanish as $\Lambda \rightarrow \infty$

$$\epsilon D D D D \rightarrow \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + \text{integrate by parts to relate } f^{(2)} \text{ to } f(0)$$

$$\text{Tr } \Theta \gamma^5 = \int d^4x \frac{e^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

↑ this anomaly

non-abelian add tr (integrand is instanton density)

$$\boxed{\partial_\mu J_5^\mu = \frac{e^2}{6\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}}$$

in Nakagaki

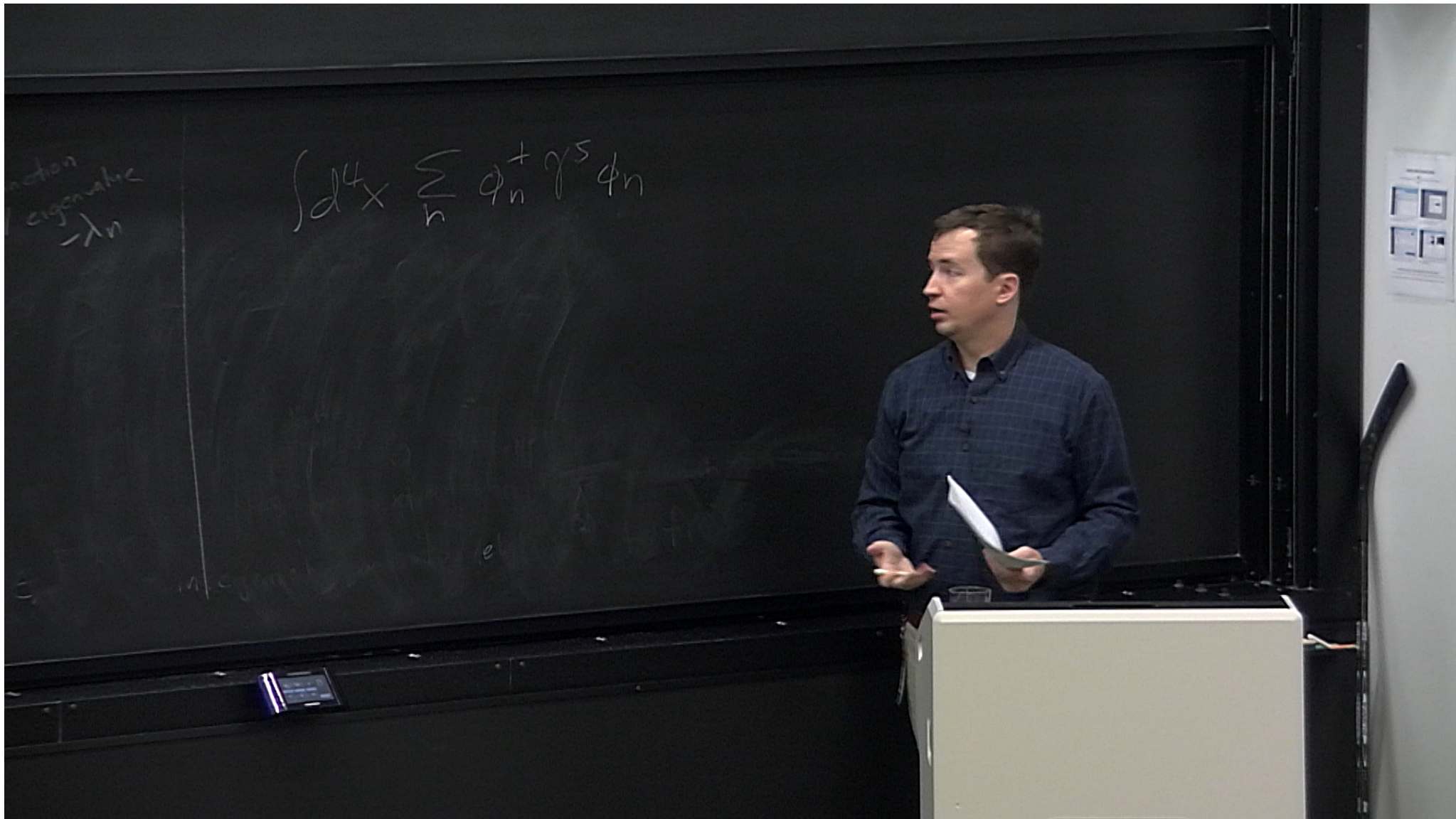
Zero Modes

For $\lambda_n \neq 0$

$$(i\cancel{D})(\gamma^5 \phi_n) = -i\gamma^5 \cancel{D} \phi_n = -\lambda_n (\gamma^5 \phi_n)$$

eigenfunction
w/ eigenvalue
 $-\lambda_n$

ϕ_n and $\gamma^5 \phi_n$ orthogonal if $\phi_n \neq 0$



$$\int d^4x \sum_n \phi_n^\dagger \gamma^5 \phi_n = n_+ - n_- \equiv \text{Index}(iD)$$

only zero
modes
contribute

$n_{\pm}$ = # of zero modes
with eigenvalue ± 1
of γ^5

$$\int d^4x \sum_n \phi_n^\dagger \gamma^5 \phi_n = n_+ - n_- \equiv \text{Index}(iD)$$

only zero
modes
contribute

$n_{\pm}$ = # of zero modes
with eigenvalue ± 1
of γ^5

anomaly is integer $\rightarrow$ no corrections

Instantons

$$\langle n+v | n \rangle = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[-\frac{1}{2g^2} \int F^2 - i \int \bar{\psi} \not{D} \psi \right]$$

$$= \left(\int \mathcal{D}A \underbrace{\det i \not{D}} \exp \left[-\frac{1}{2g^2} \int F^2 \right] \right)$$

= 0 due zero eigenvalues

$$\text{Tr } \theta \gamma^5 = \int d^4x \frac{\theta e^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

this anomaly

non-abelian add tr (Integrand is instanton density)

$$\partial_\mu J_S^\mu = \frac{e^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \quad \text{in Minkowski}$$

Instantons

$$\langle n+1 | n \rangle = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[\frac{i}{4} \int d^4x F^2 - i \bar{\psi} \not{D} \psi \right]$$

$$= \int \mathcal{D}A \underbrace{\det i \not{D}} \exp \left[\frac{i}{4} \int d^4x F^2 \right]$$

= 0 due zero eigenvalues

chiral transformation can eliminate θ parameter