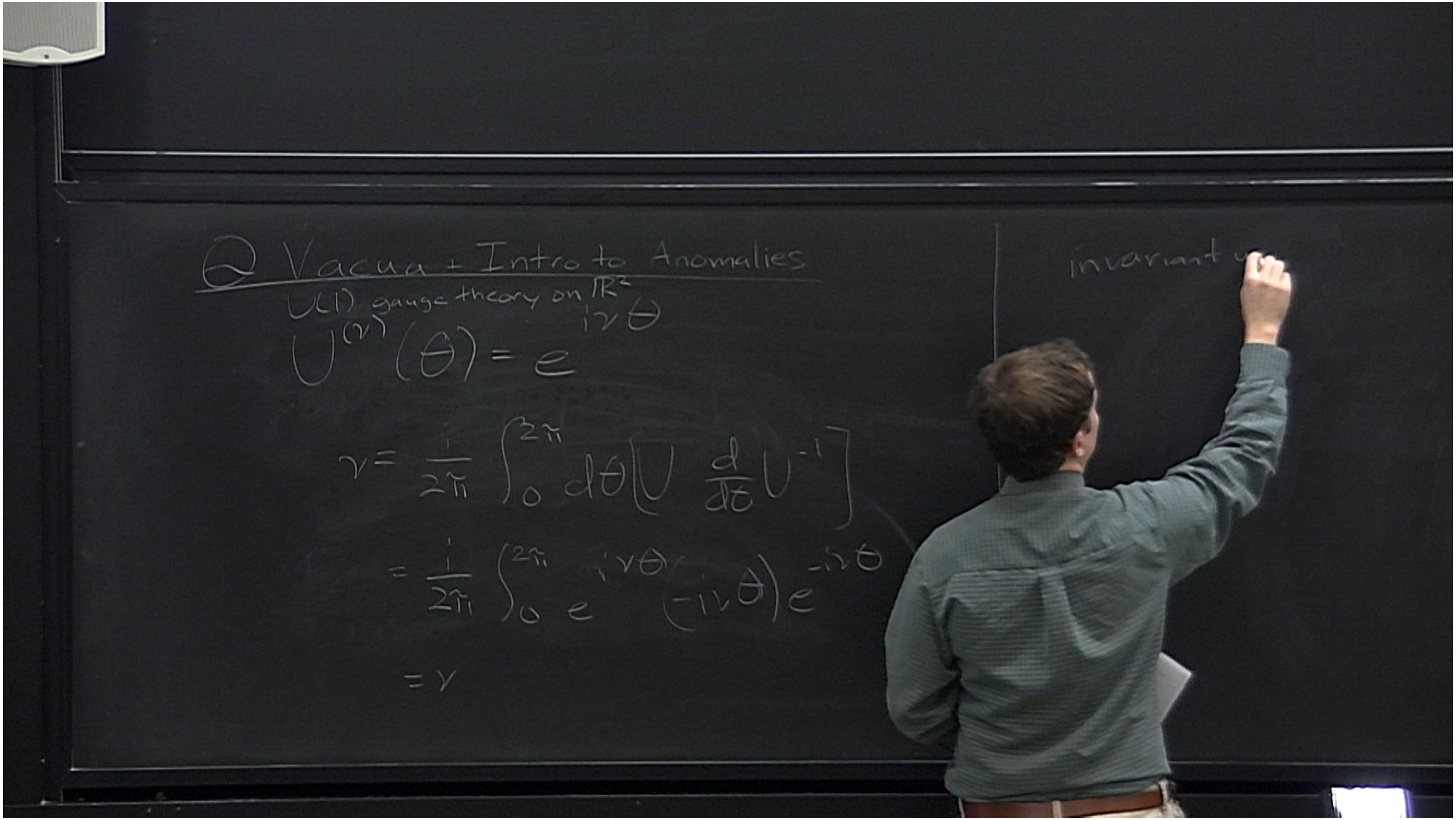
Title: PSI 2018/2019 - Quantum Field Theory III - Lecture 13

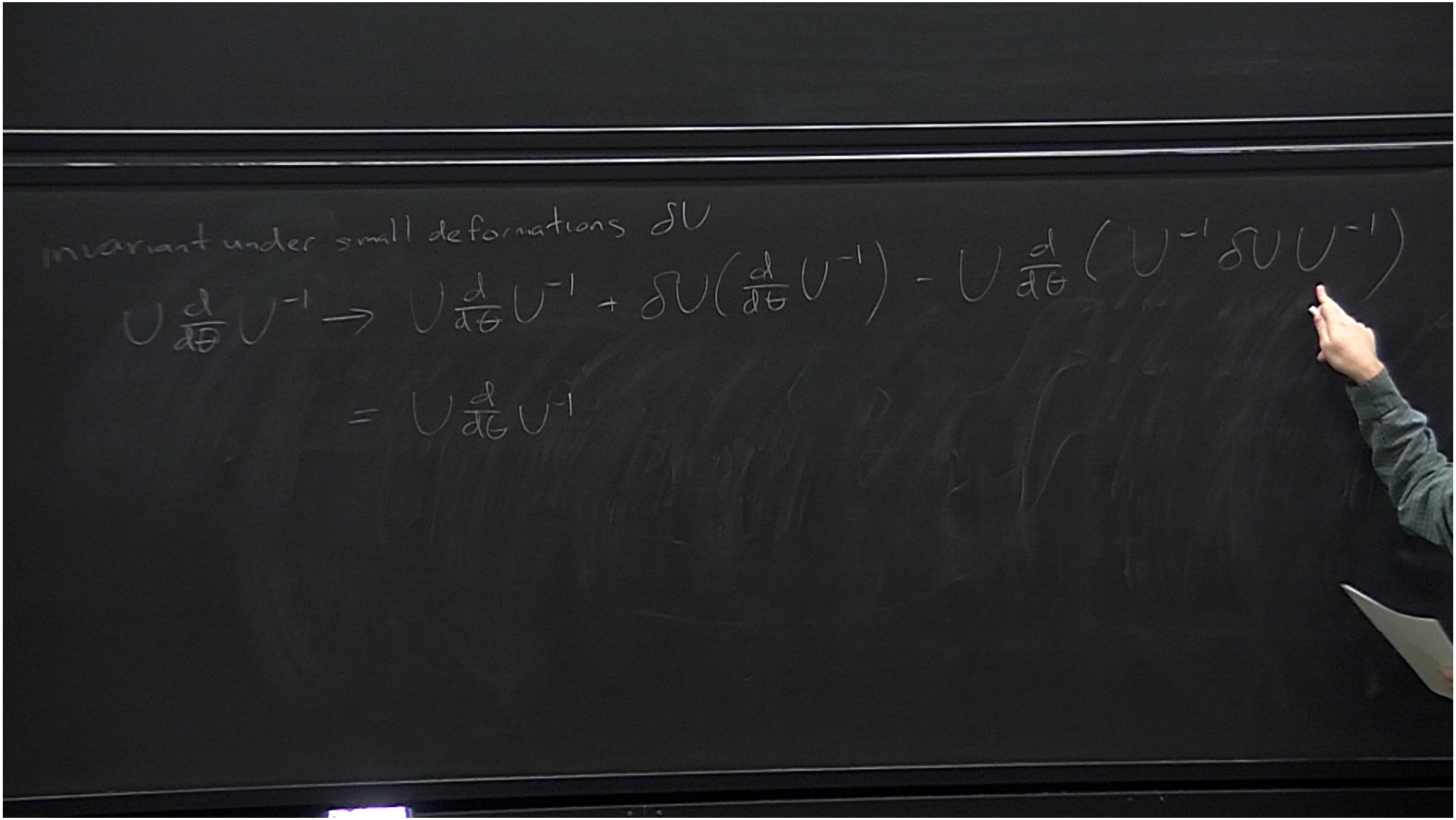
Speakers: Dan Wohns

Collection: PSI 2018/2019 - Quantum Field Theory III (Gomis, Wohns and Ali)

Date: March 14, 2019 - 2:00 PM

URL: http://pirsa.org/19030005


2 Vacua + Intro to Anomalies
U(1) gauge theory on $\mathbb{R}^2$
$U^{(\nu)}(\theta) = e^{i\nu\theta}$
$\nu = \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[U \frac{d}{d\theta} U^{-1}\right]$
$= \frac{1}{2\pi}\int_0^{2\pi} e^{i\nu\theta}(-i\nu)e^{-i\nu\theta}$
$= \nu$
invariant


invariant under small deformations δU
$U\frac{d}{d\theta}U^{-1} \to U\frac{d}{d\theta}U^{-1} + \delta U\left(\frac{d}{d\theta}U^{-1}\right) - U\frac{d}{d\theta}\left(U^{-1}\delta U U^{-1}\right)$
$= U\frac{d}{d\theta}U^{-1}$

invariant under small deformations δU

$$U\frac{d}{d\theta}U^{-1} \to U\frac{d}{d\theta}U^{-1} + \delta U\left(\frac{d}{d\theta}U^{-1}\right) - U\frac{d}{d\theta}\left(U^{-1}\delta U U^{-1}\right)$$

$$= U\frac{d}{d\theta}U^{-1} - U\left(\frac{d}{d\theta}U^{-1}\right)\delta U U^{-1} - \left(\frac{d}{d\theta}\delta U\right)U^{-1}$$

$$= U\frac{d}{d\theta}U^{-1} - \frac{d}{d\theta}\left(U^{-1}\delta U\right)$$

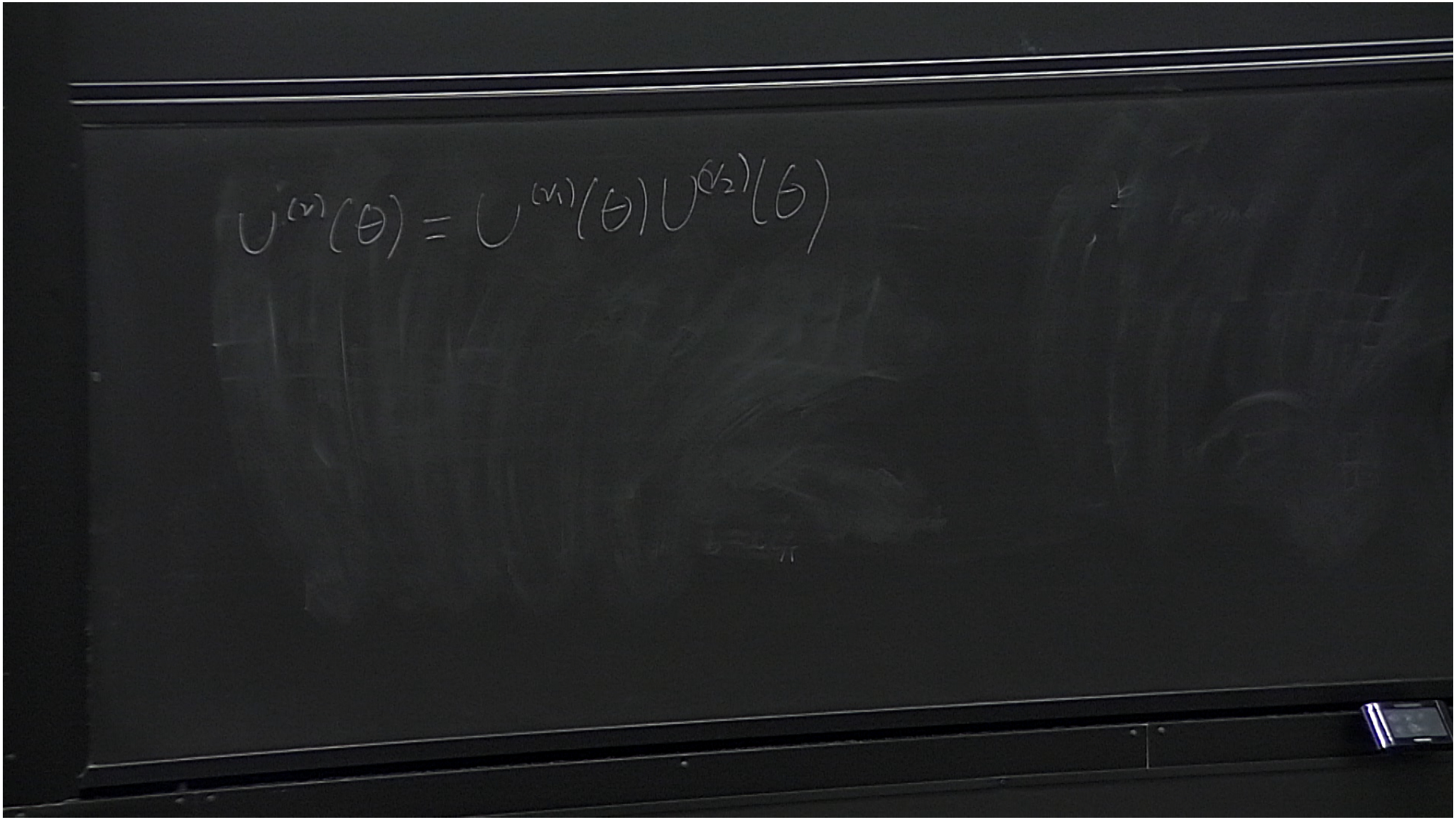
invariant under small deformations δU

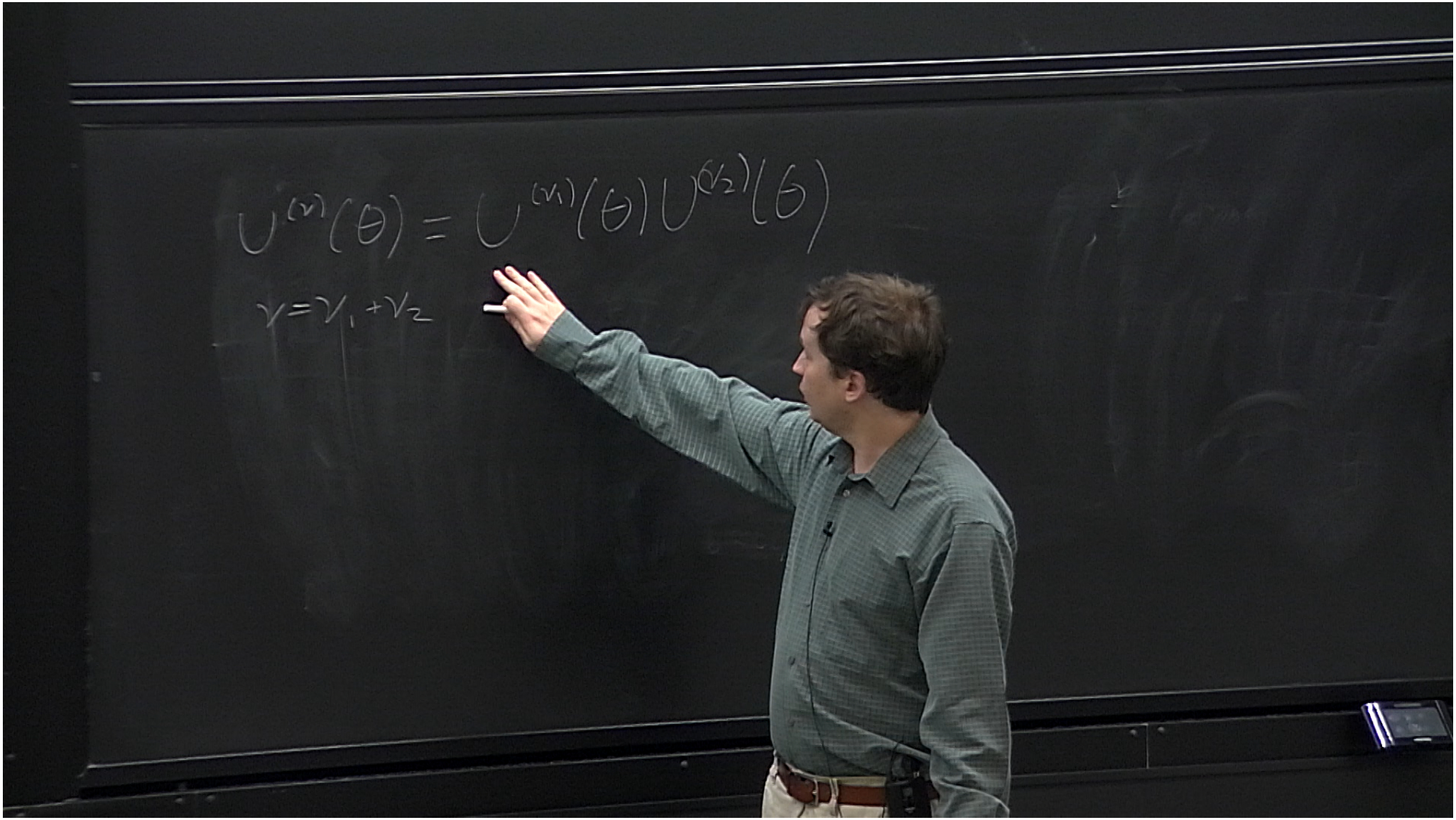
$$U\frac{d}{d\theta}U^{-1} \to U\frac{d}{d\theta}U^{-1} + \delta U\left(\frac{d}{d\theta}U^{-1}\right) - U\frac{d}{d\theta}\left(U^{-1}\delta U U^{-1}\right)$$

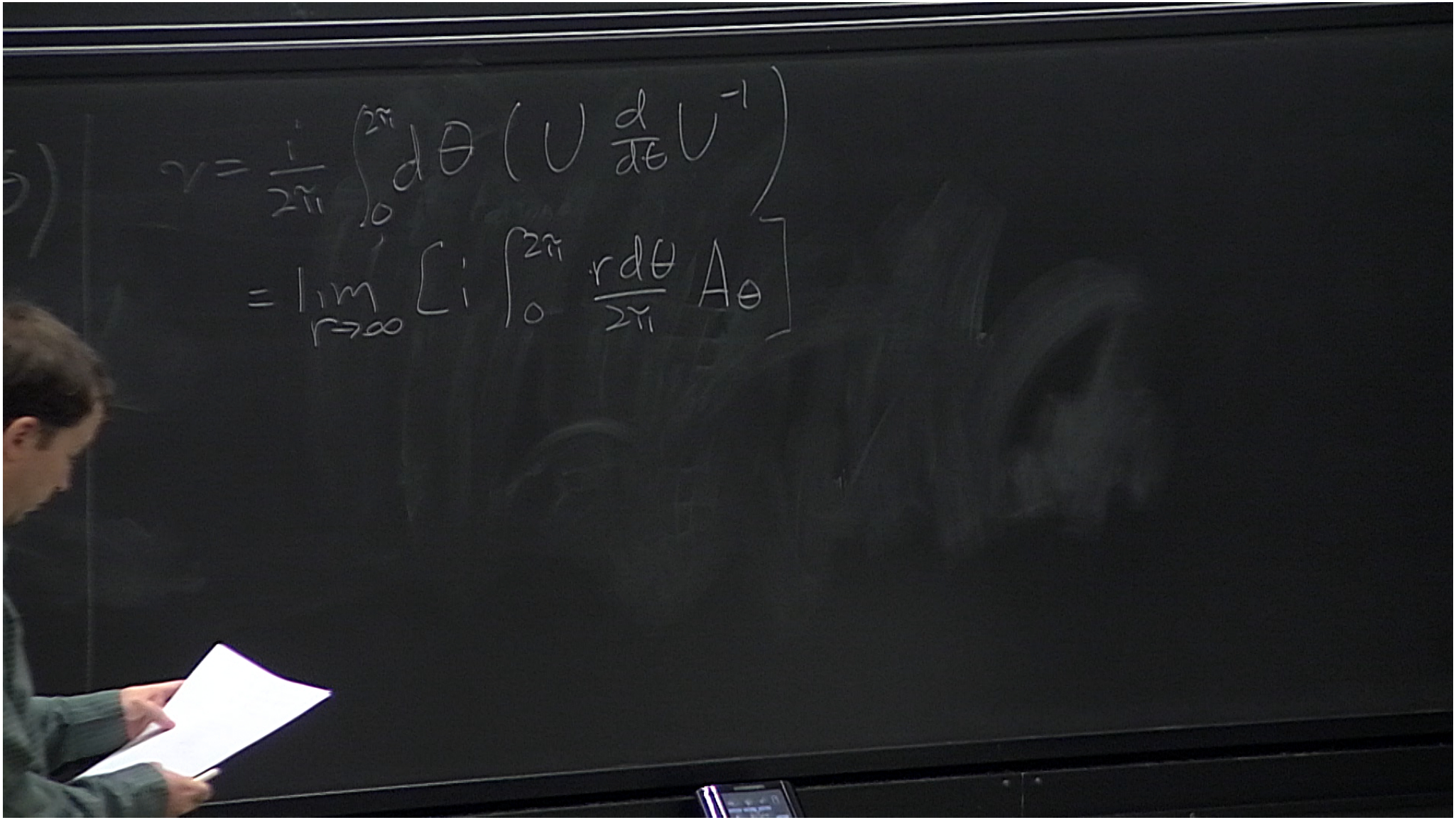
$$= U\frac{d}{d\theta}U^{-1} - \cancel{U\left(\frac{d}{d\theta}U^{-1}\right)\cdot\delta U U^{-1}} - \left(\frac{d}{d\theta}\delta U\right)U^{-1}$$

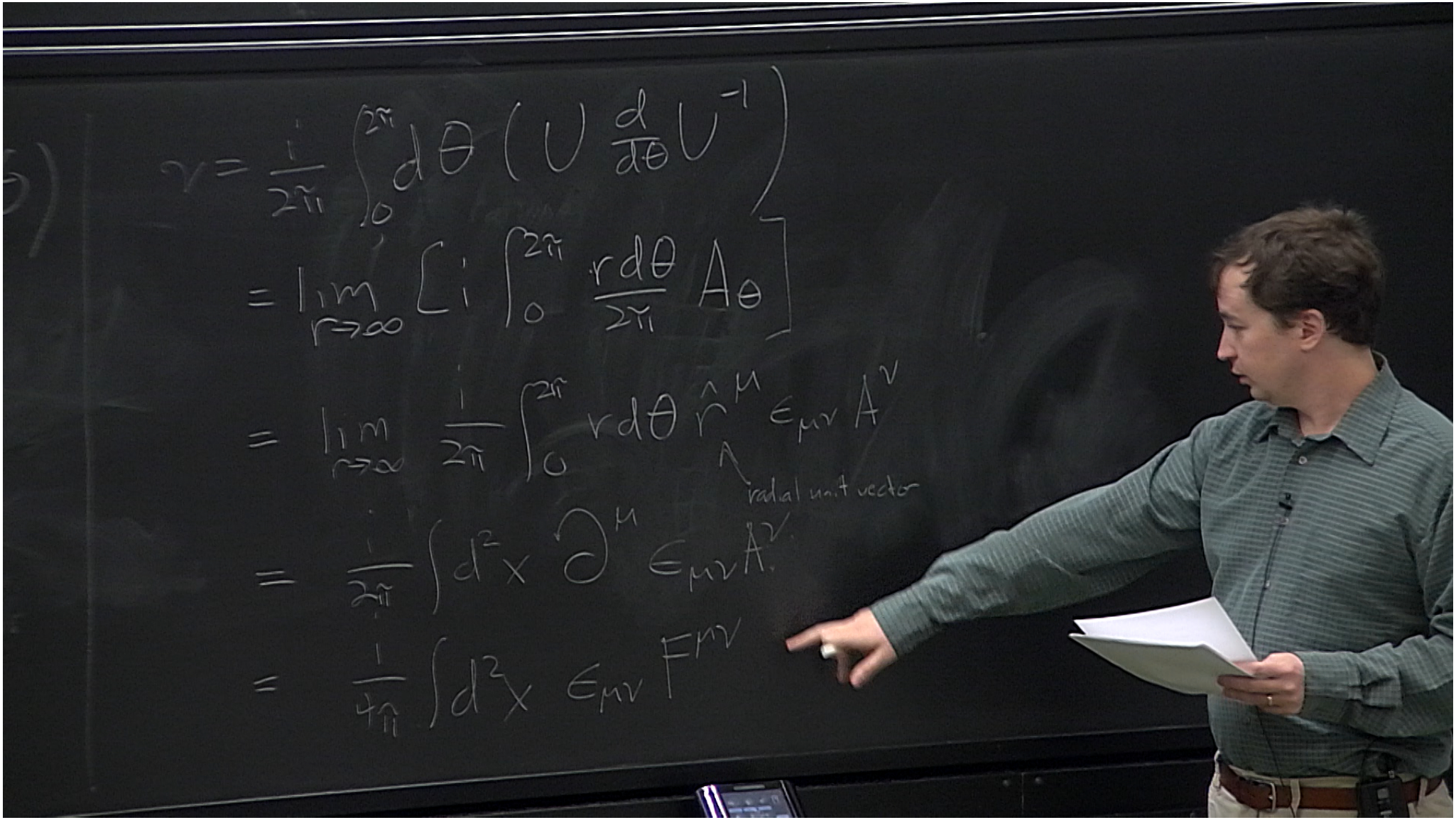
$$= U\frac{d}{d\theta}U^{-1} - \underbrace{\frac{d}{d\theta}\left(U^{-1}\delta U\right)}$$

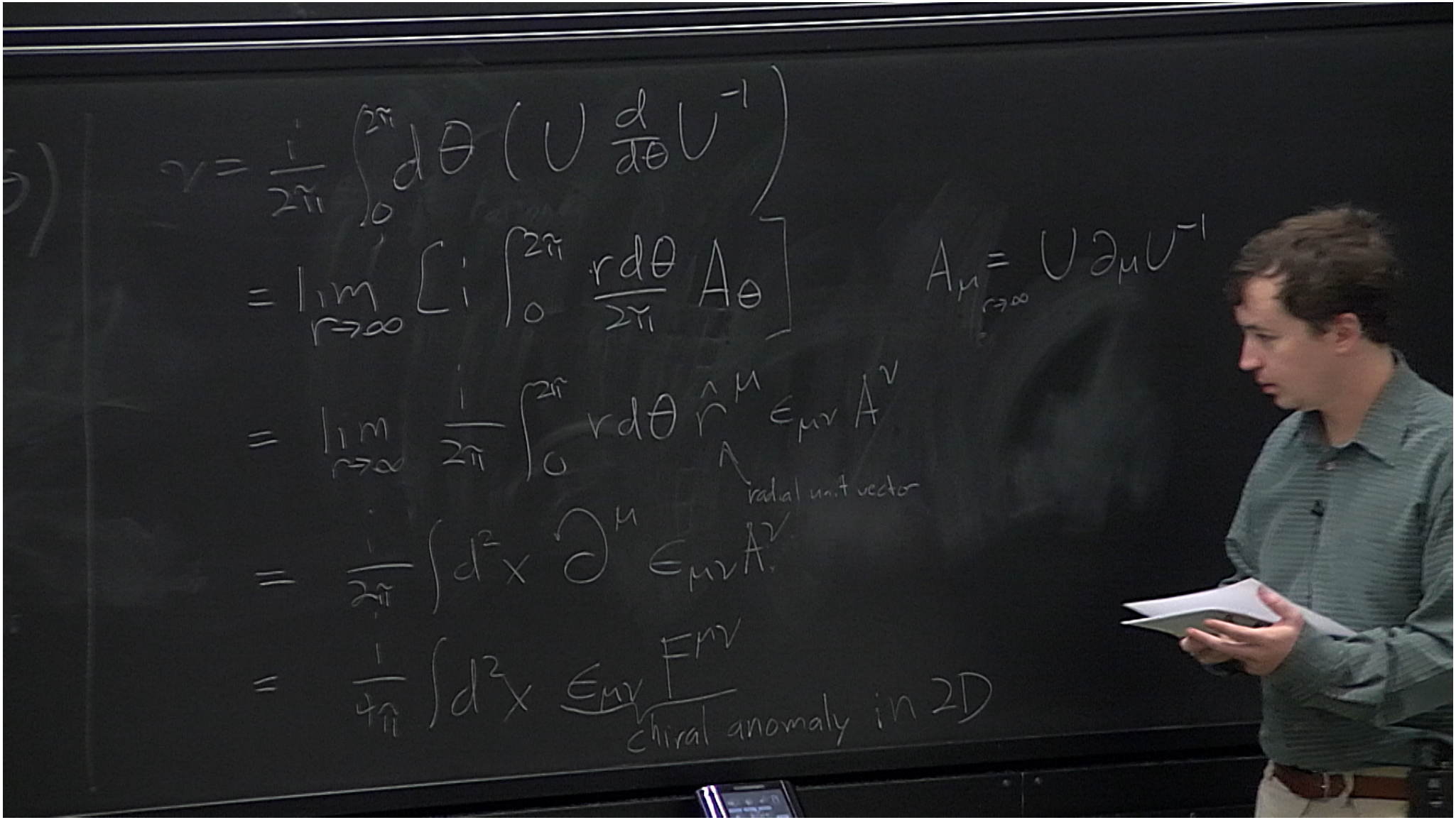
vanishes upon integration


$U^{(n)}(\theta) = U^{(n_1)}(\theta) U^{(n_2)}(\theta)$


$U^{(\gamma)}(\theta) = U^{(\gamma_1)}(\theta) U^{(\gamma_2)}(\theta)$
$\gamma = \gamma_1 + \gamma_2$


$\nu = \frac{i}{2\pi} \int_0^{2\pi} d\theta \left(U \frac{d}{d\theta} U^{-1} \right)$
$= \lim_{r \to \infty} \left[i \int_0^{2\pi} \frac{r\, d\theta}{2\pi} A_\theta \right]$


$\nu = \frac{1}{2\pi i}\int_0^{2\pi} d\theta \left(U \frac{d}{d\theta} U^{-1}\right)$
$= \lim_{r\to\infty}\left[i\int_0^{2\pi} \frac{r\,d\theta}{2\pi} A_\theta\right]$
$= \lim_{r\to\infty} \frac{1}{2\pi}\int_0^{2\pi} r\,d\theta\, \hat{r}^\mu \epsilon_{\mu\nu} A^\nu$
radial unit vector
$= \frac{1}{2\pi}\int d^2x\, \partial^\mu \epsilon_{\mu\nu} A^\nu$
$= \frac{1}{4\pi}\int d^2x\, \epsilon_{\mu\nu} F^{\mu\nu}$


$\nu = \frac{1}{2\pi i}\int_0^{2\pi} d\theta \left(U \frac{d}{d\theta} U^{-1}\right)$
$= \lim_{r\to\infty}\left[i\int_0^{2\pi} \frac{r\,d\theta}{2\pi} A_\theta\right]$
$A_\mu \underset{r\to\infty}{=} U\partial_\mu U^{-1}$
$= \lim_{r\to\infty} \frac{i}{2\pi}\int_0^{2\pi} r\,d\theta\, \hat{r}^\mu \epsilon_{\mu\nu} A^\nu$
radial unit vector
$= \frac{i}{2\pi}\int d^2x\, \partial^\mu \epsilon_{\mu\nu} A^\nu$
$= \frac{1}{4\pi}\int d^2x\, \epsilon_{\mu\nu} F^{\mu\nu}$
chiral anomaly in 2D

Similar for non-abelian gauge theory in $\mathbb{R}^4$

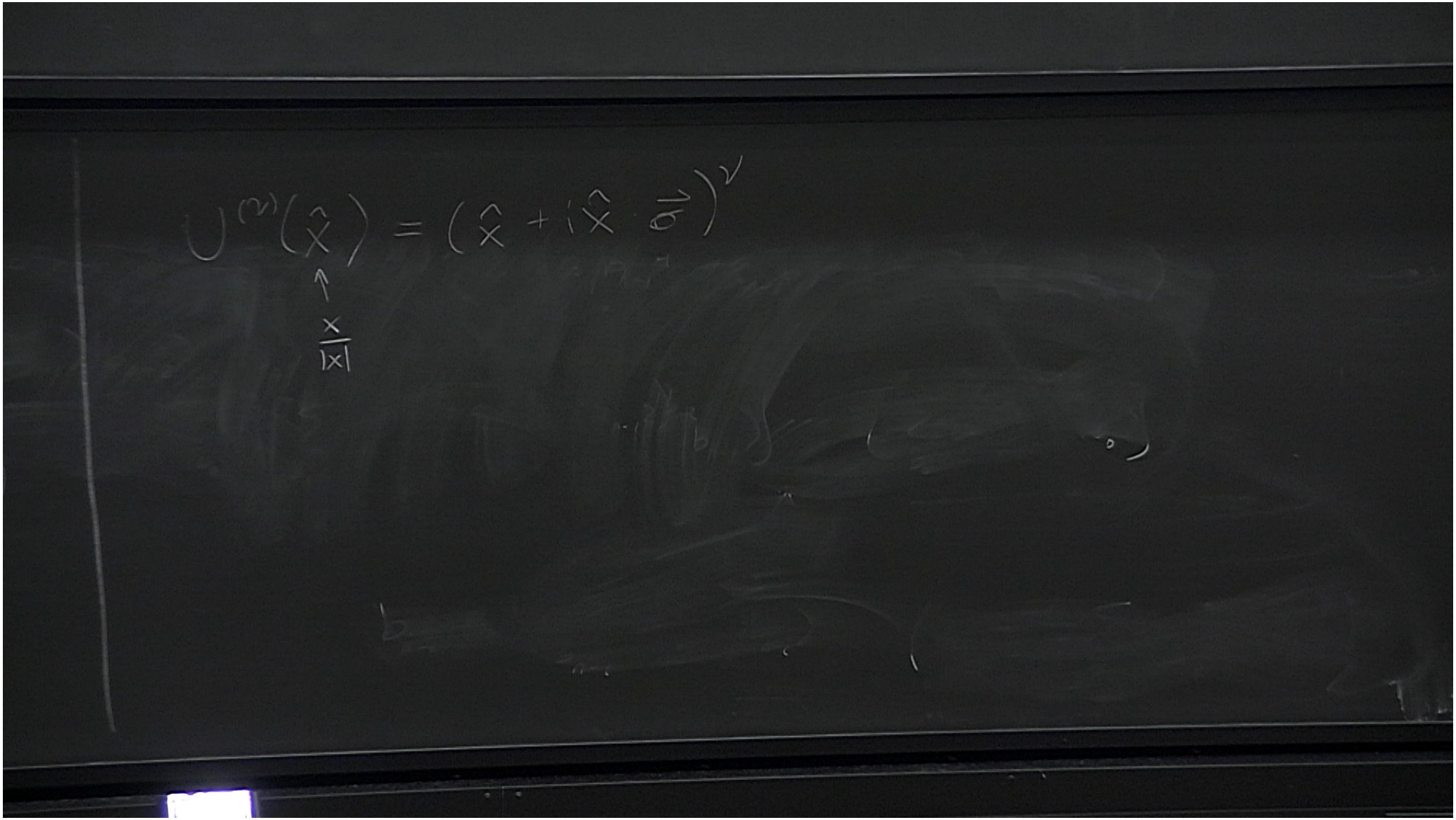
For $SU(2)$

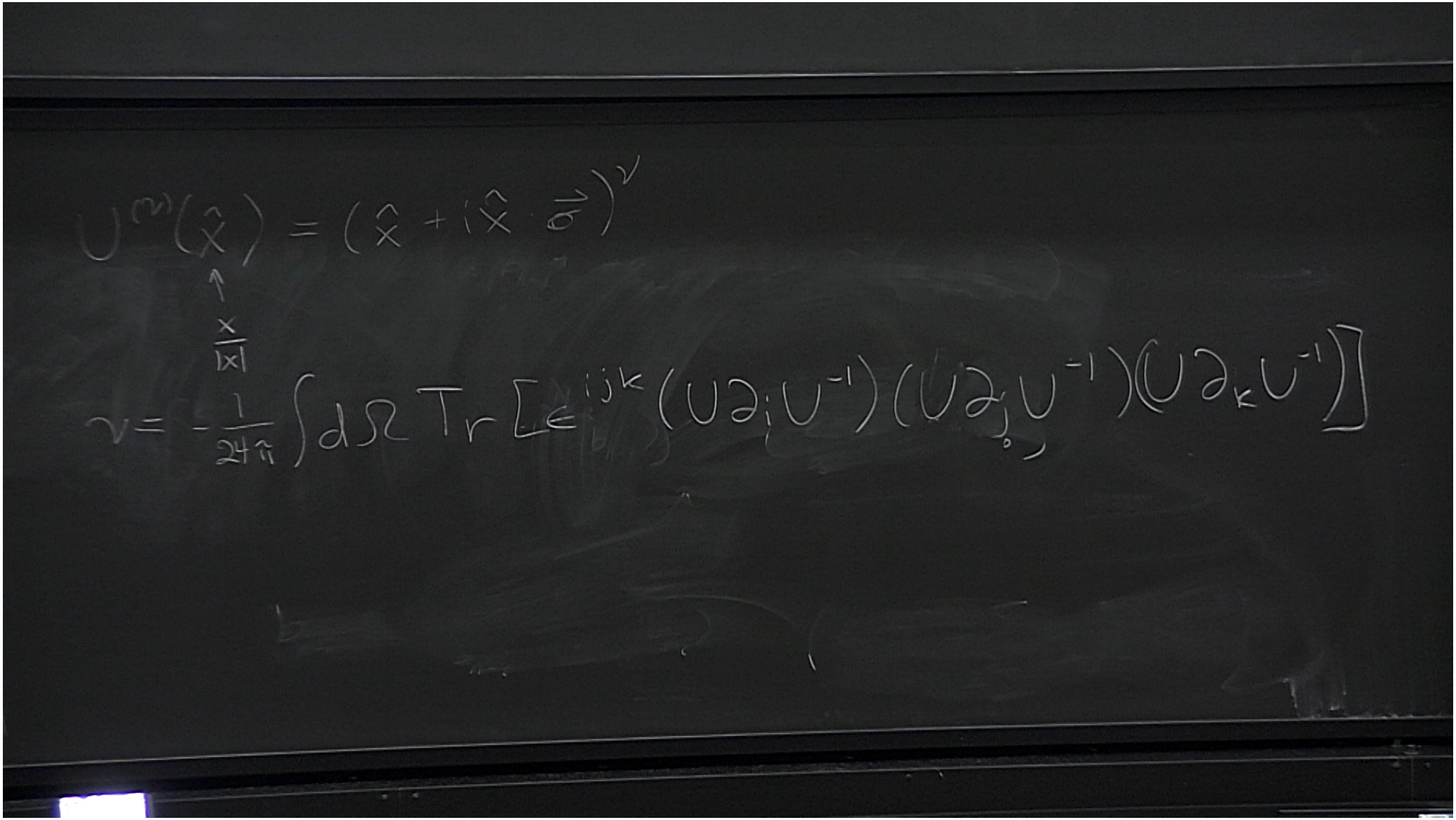
$$U = e^{i\lambda^i \frac{\sigma^i}{2}} = a\,\mathbb{1} + i b_i \sigma^i \qquad a, b_i \in \mathbb{R}$$

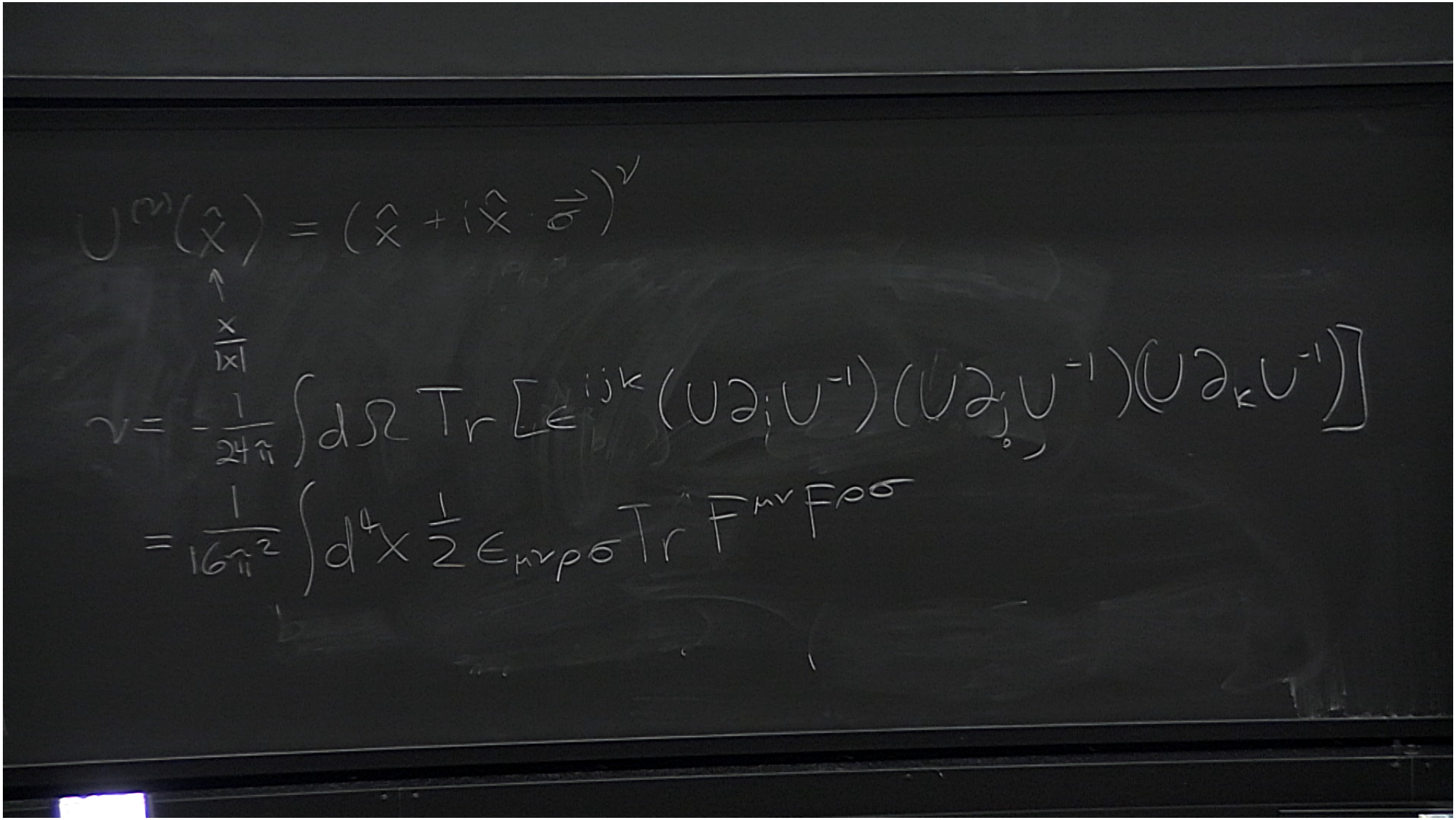
$$U^+ U = \mathbb{1} \Rightarrow a^2 + b_i b^i = 1 \qquad (3\text{-sphere})$$

homotopy classes of maps from $S^3 \to S^3$

(S^3 ↑ $\mathbb{R}^4$ at ∞; S^3 ↑ $SU(2)$)


$U^{(2)}(\hat{x}) = (\hat{x} + i\hat{x}\cdot\vec{\sigma})^2$
$\hat{x}$
$\frac{x}{|x|}$


$U^{(\nu)}(\hat{x}) = (\hat{x} + i\hat{x}\cdot\vec{\sigma})^{\nu}$
$\hat{x} = \frac{x}{|x|}$
$\nu = -\frac{1}{24\pi^2}\int d\Omega\, \mathrm{Tr}\left[\epsilon^{ijk}(U\partial_i U^{-1})(U\partial_j U^{-1})(U\partial_k U^{-1})\right]$


$U^{(\nu)}(\hat{x}) = (\hat{x} + i\hat{x}\cdot\vec{\sigma})^{\nu}$
$\hat{x} = \frac{x}{|x|}$
$\nu = -\frac{1}{24\pi^2}\int d\Omega\, \mathrm{Tr}\left[\epsilon^{ijk}(U\partial_i U^{-1})(U\partial_j U^{-1})(U\partial_k U^{-1})\right]$
$= \frac{1}{16\pi^2}\int d^4x \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\, F^{\mu\nu}F^{\rho\sigma}$

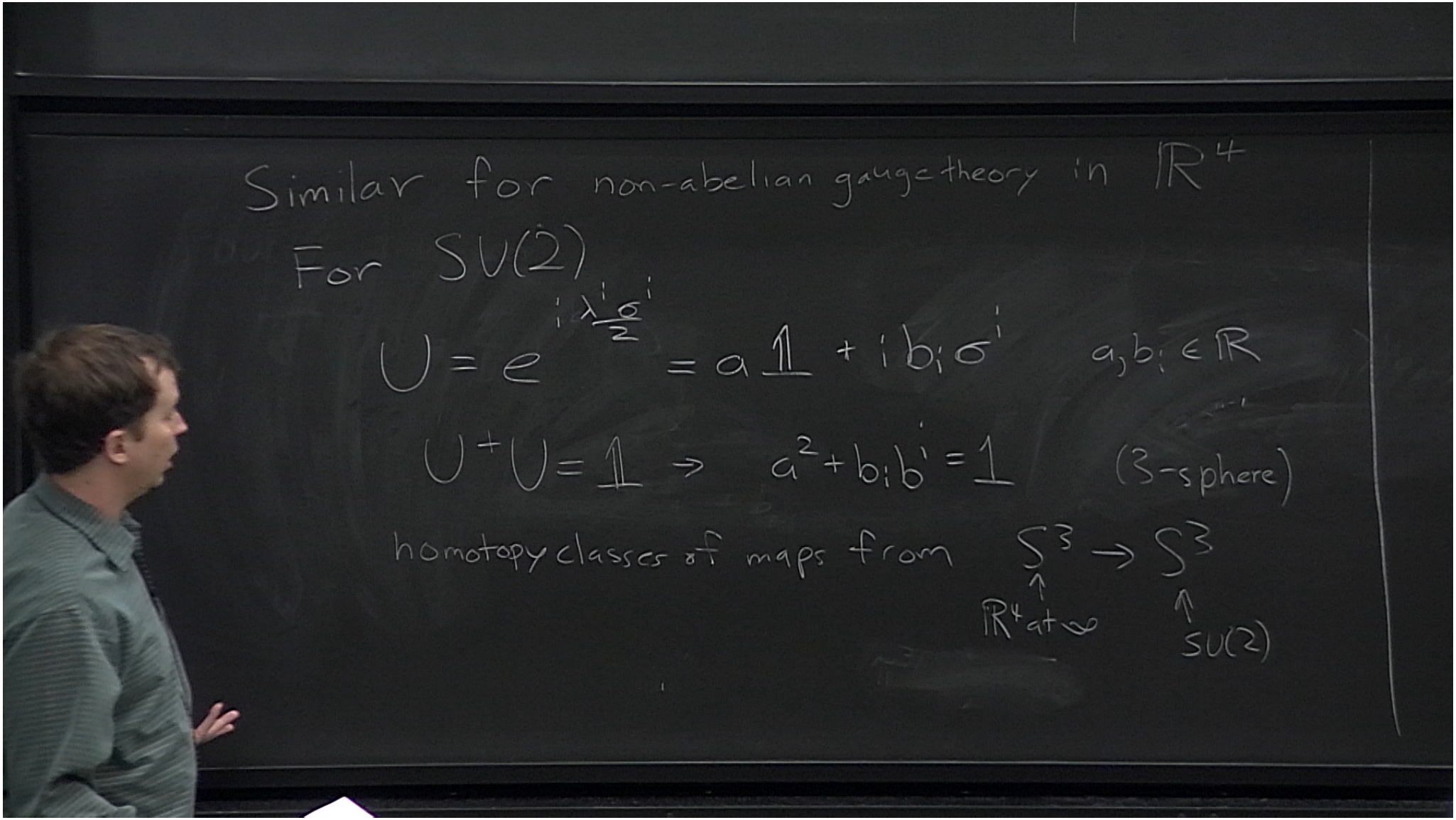
$$U^{(n)}(\hat{x}) = (\hat{x} + i\hat{x}\cdot\vec{\sigma})^n$$

$$\uparrow \frac{x}{|x|}$$

$$\nu = -\frac{1}{24\pi^2}\int d\Omega\, \mathrm{Tr}\left[\epsilon^{ijk}(U\partial_i U^{-1})(U\partial_j U^{-1})(U\partial_k U^{-1})\right]$$

$$= \frac{1}{16\pi^2}\int d^4x\, \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\, F^{\mu\nu}F^{\rho\sigma}$$

chiral anomaly in $D=4$


Similar for non-abelian gauge theory in $\mathbb{R}^4$
For $SU(2)$
$U = e^{i\lambda^i \frac{\sigma^i}{2}} = a\mathbb{1} + ib_i\sigma^i \quad a, b_i \in \mathbb{R}$
$U^+U = \mathbb{1} \Rightarrow a^2 + b_i b^i = 1$ (3-sphere)
homotopy classes of maps from $S^3 \to S^3$
$\mathbb{R}^4$ at ∞
$SU(2)$

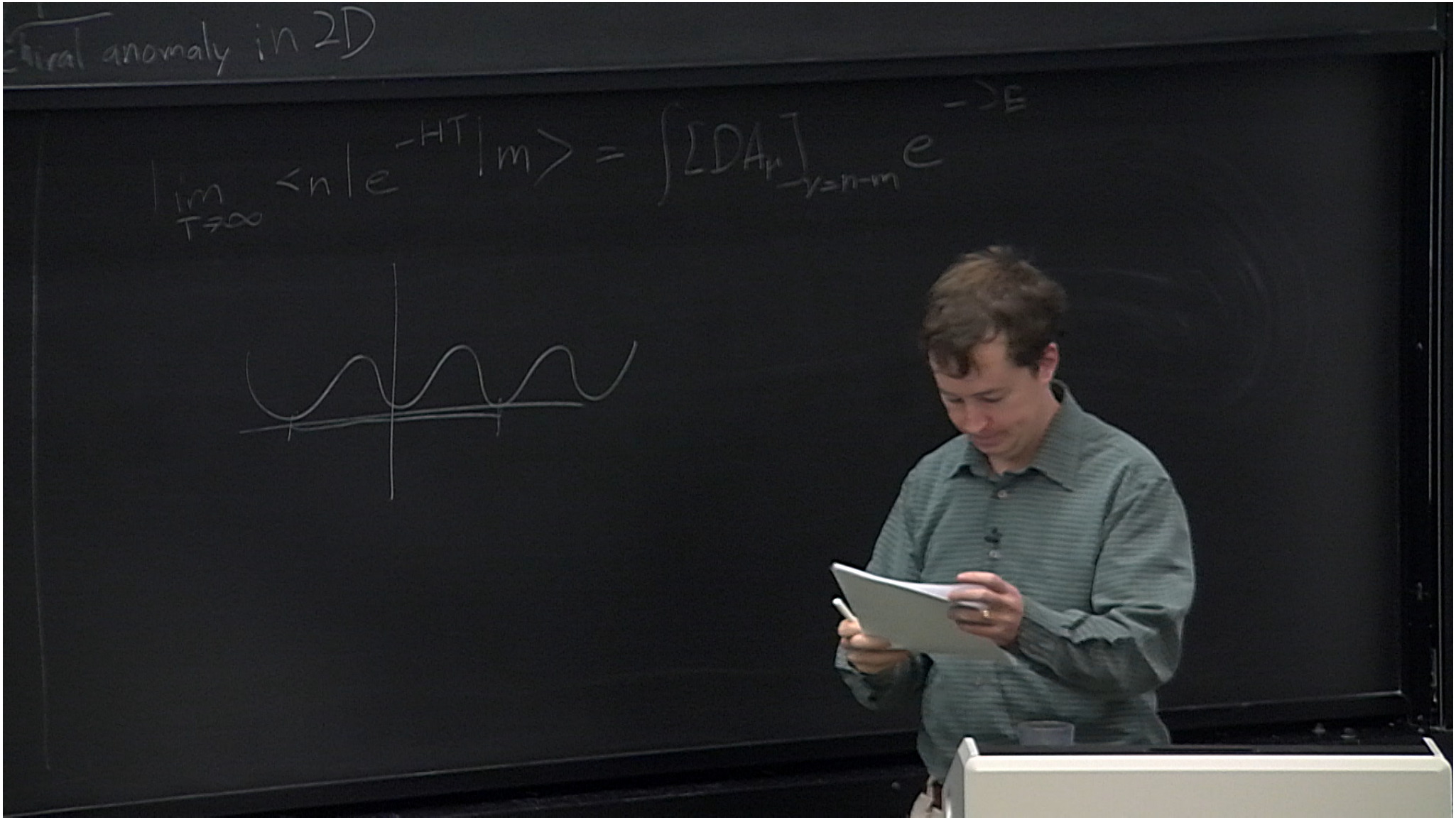
chiral anomaly

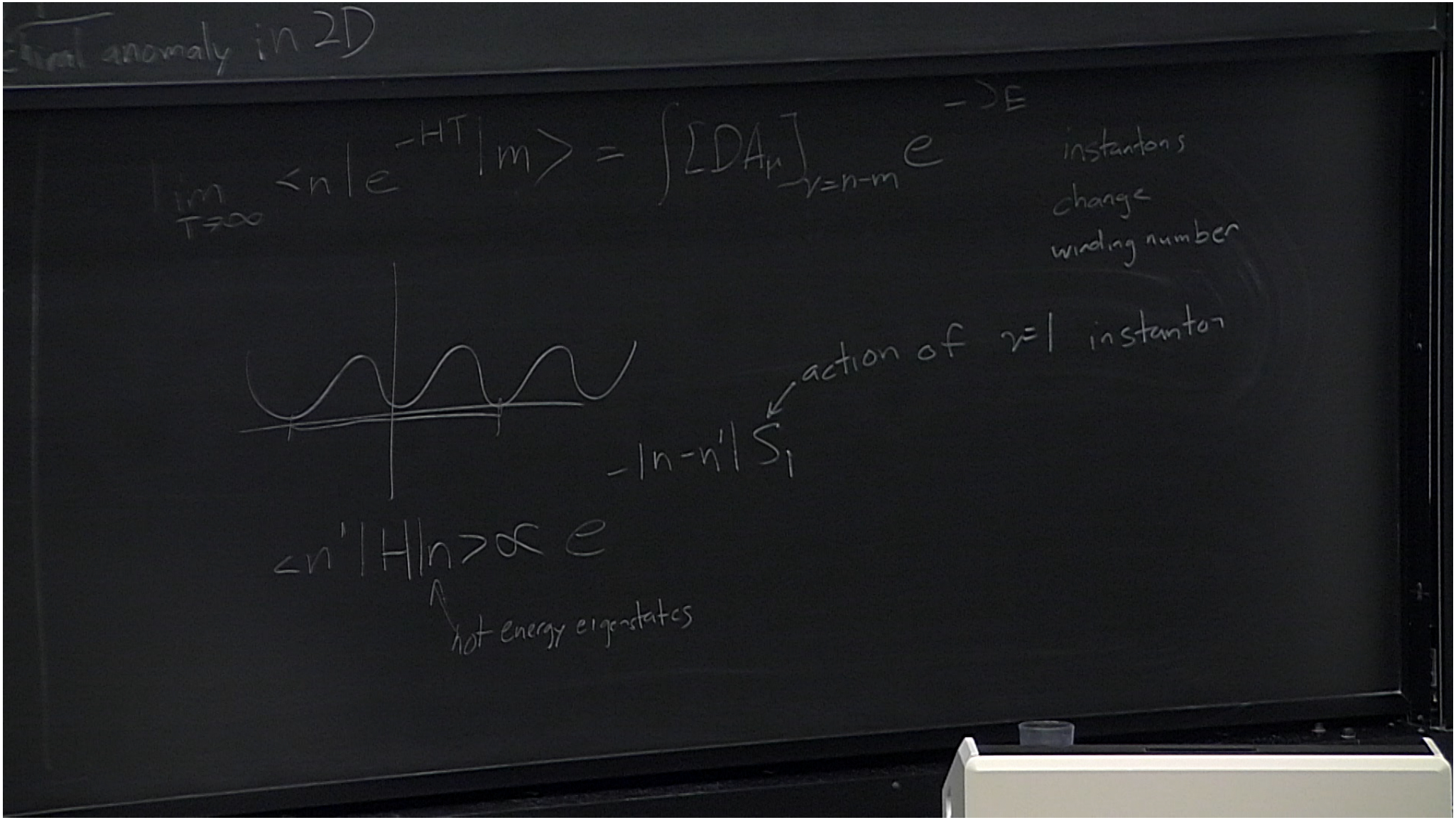
No analog of ν for $U(1)$ in $D=4$

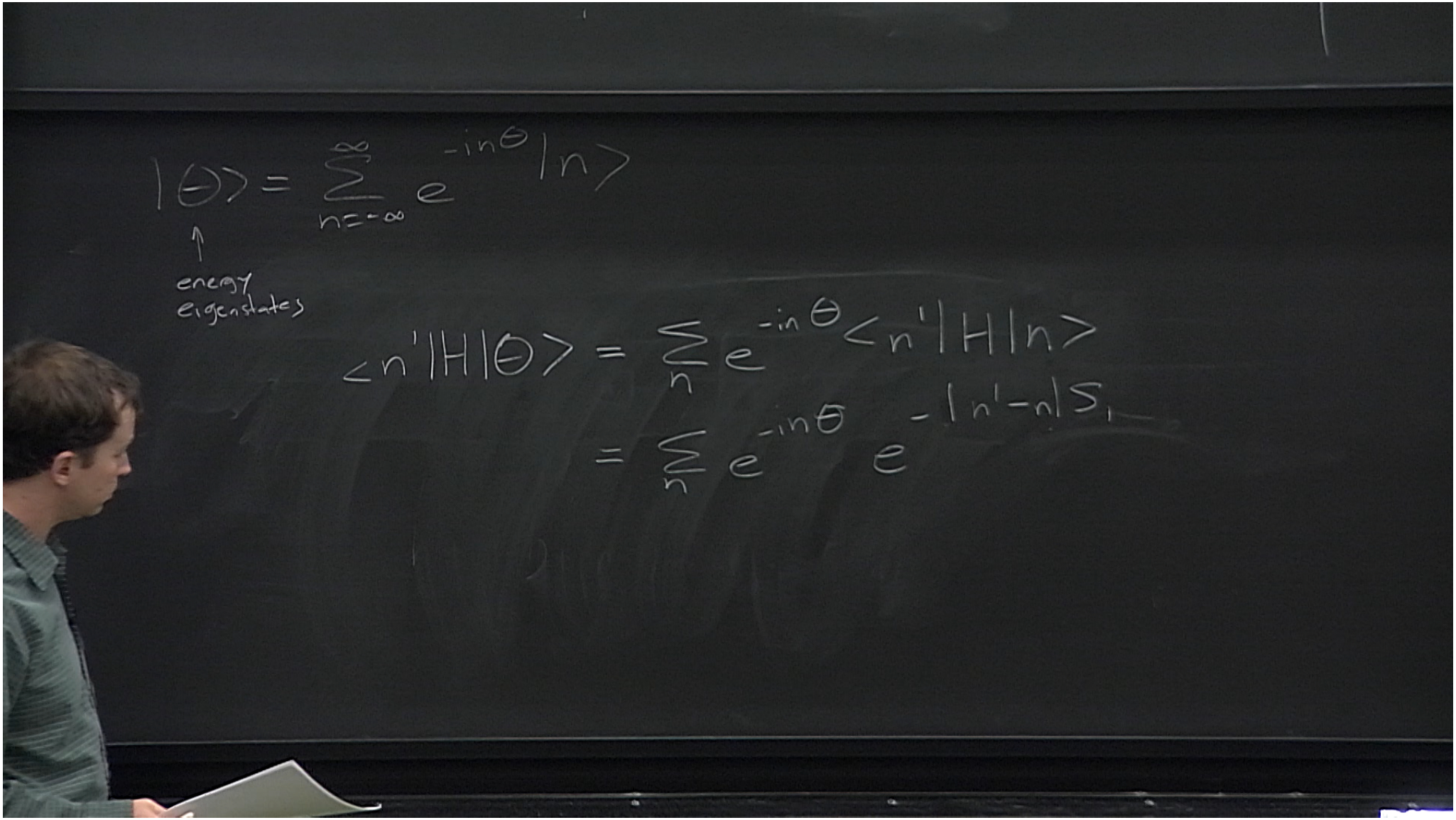
Theorem: any map from $S^3 \to G$ can be continuously

(G: simple non-abelian Lie group)

deformed to a map onto a $SU(2)$ subgroup of G


iral anomaly in 2D
lim T→∞ <n|e^{-HT}|m> = ∫[DA_μ]_{γ=n-m} e^{-S}


anomaly in 2D
$\lim_{T\to\infty} \langle n|e^{-HT}|m\rangle = \int [DA_\mu]_{\nu=n-m} e^{-\to E}$
instantons change winding number
$-|n-n'|S_1$
action of $\nu=1$ instanton
$\langle n'|H|n\rangle \propto e$
not energy eigenstates


$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta} |n\rangle$
energy eigenstates
$\langle n'|H|\theta\rangle = \sum_n e^{-in\theta} \langle n'|H|n\rangle$
$= \sum_n e^{-in\theta} e^{-|n'-n|s_1}$

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta} |n\rangle$$

↑ energy eigenstates

$$\langle n'|H|\theta\rangle = \sum_n e^{-in\theta} \langle n'|H|n\rangle$$

$$= \sum_n e^{-in\theta} e^{-|n'-n|s_1}$$

$$= \sum_m e^{-i(m+n')\theta} e^{-|m|s_1}$$

$$= \underbrace{e^{-in'\theta}}_{\langle n'|\theta\rangle} \underbrace{\sum_m e^{-im\theta} e^{-|m|s_1}}_{E_\theta}$$

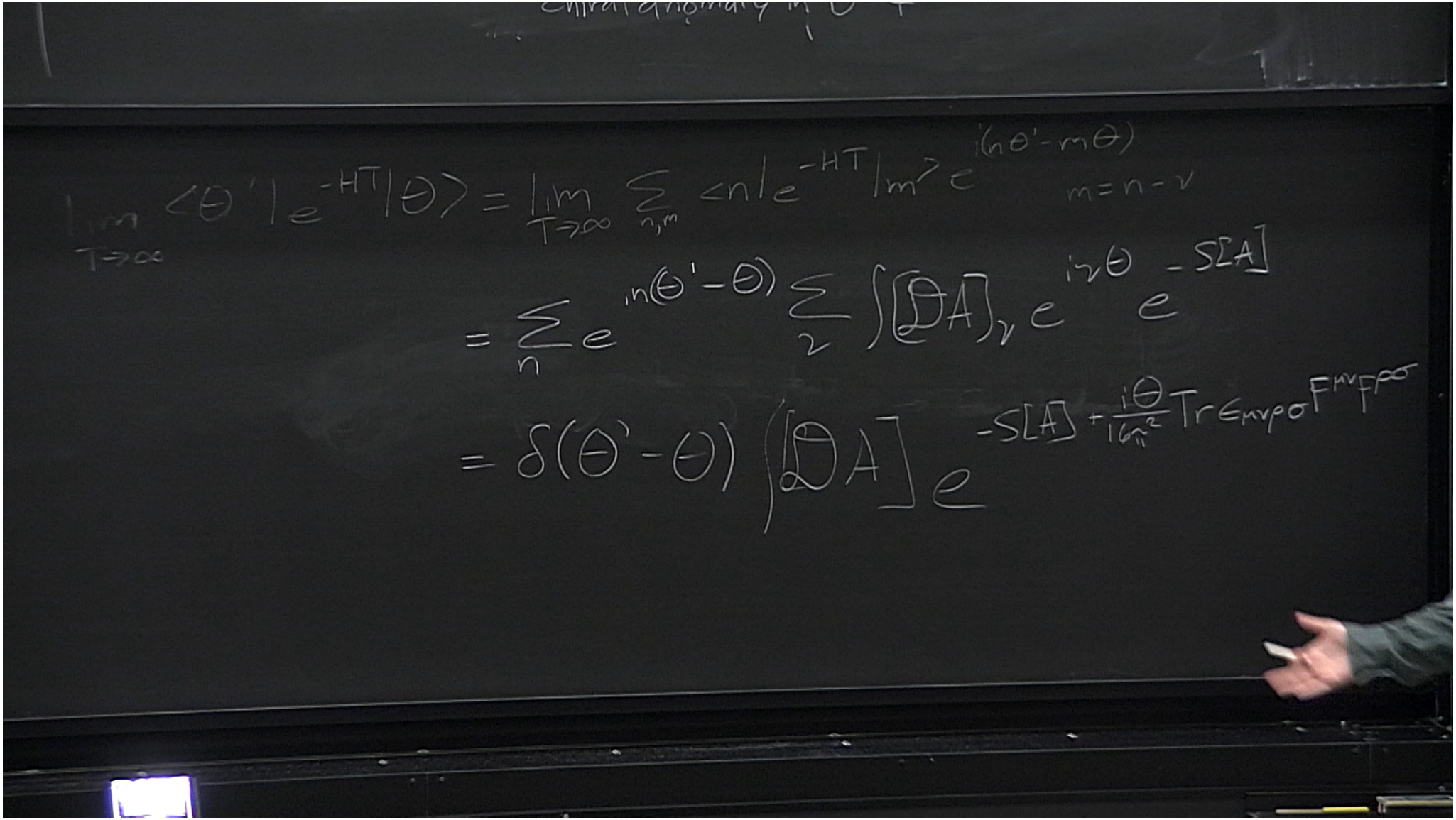
$$\lim_{T\to\infty} \langle\theta'|e^{-HT}|\theta\rangle = \lim_{T\to\infty} \sum_{n,m} \langle n|e^{-HT}|m^2\rangle e^{n\theta' - m\theta}$$

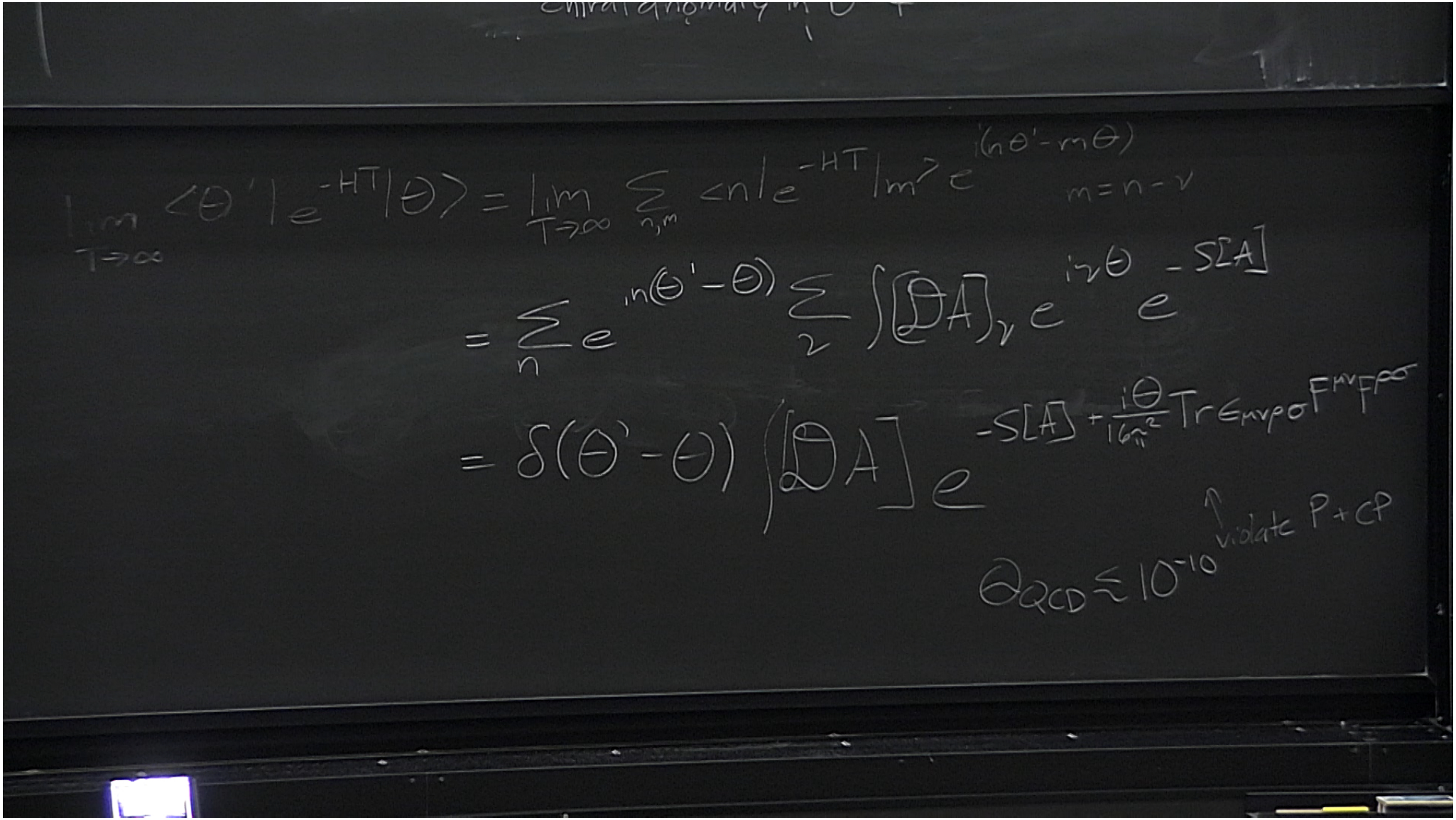
$$= \sum_n e^{in(\theta'-\theta)}$$

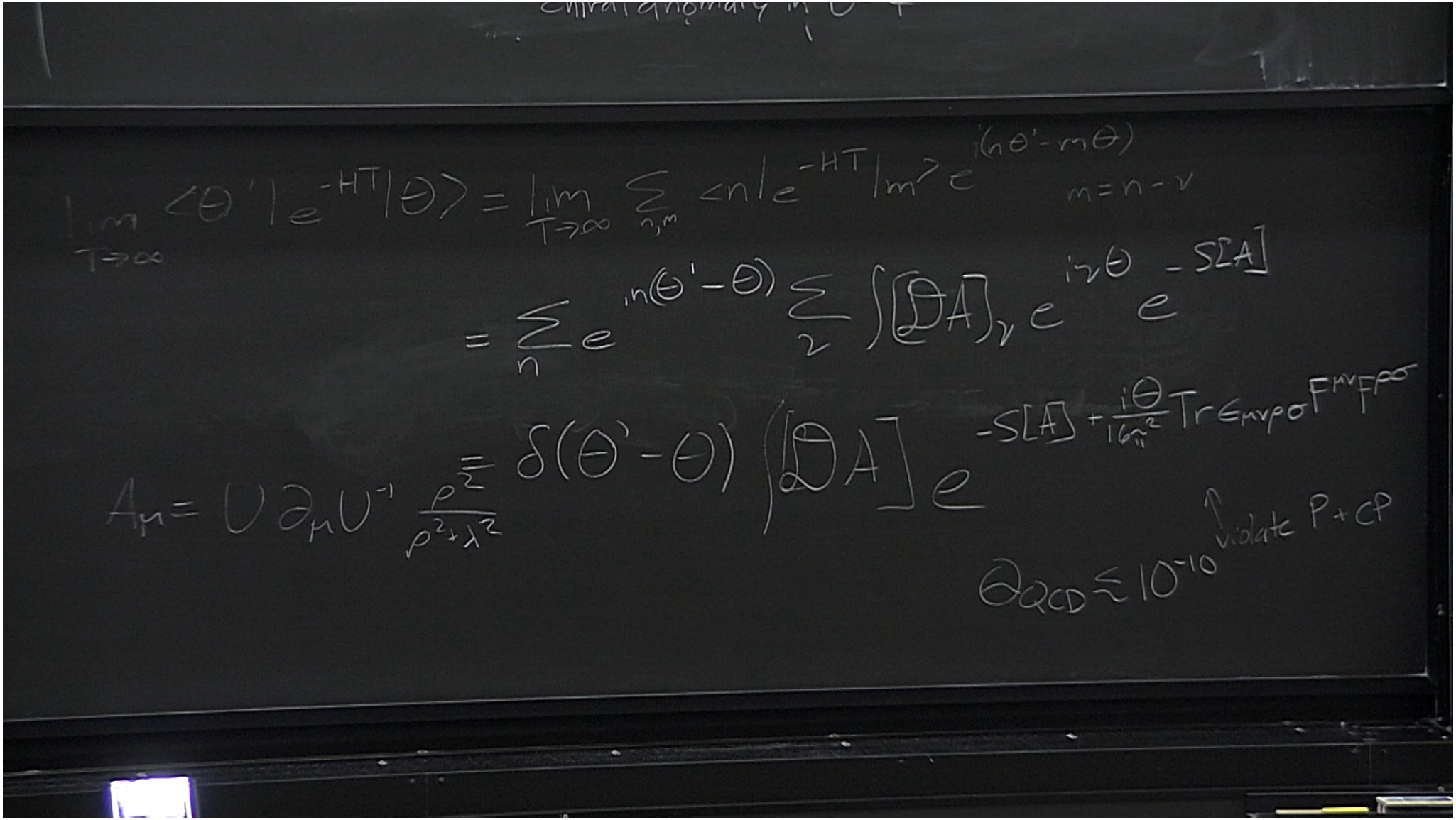
$$\lim_{T\to\infty} \langle\theta'|e^{-HT}|\theta\rangle = \lim_{T\to\infty} \sum_{n,m} \langle n|e^{-HT}|m\rangle e^{i(n\theta' - m\theta)} \qquad m = n - \nu$$

$$= \sum_n e^{in(\theta'-\theta)} \sum_\nu \int [\mathcal{D}A]_\nu \, e^{i\nu\theta} \, e^{-S[A]}$$

$$= \delta(\theta'-\theta) \int [\mathcal{D}A] \, e^{-S[A]}$$


$\lim_{T\to\infty} \langle\theta'|e^{-HT}|\theta\rangle = \lim_{T\to\infty} \sum_{n,m} \langle n|e^{-HT}|m\rangle e^{i(n\theta' - m\theta)}$
$m = n - \nu$
$= \sum_n e^{in(\theta'-\theta)} \sum_\nu \int [\mathcal{D}A]_\nu e^{i\nu\theta} e^{-S[A]}$
$= \delta(\theta'-\theta) \int [\mathcal{D}A] e^{-S[A] + \frac{i\theta}{16\pi^2} \mathrm{Tr}\,\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}}$


$\lim_{T\to\infty} \langle\theta'|e^{-HT}|\theta\rangle = \lim_{T\to\infty} \sum_{n,m} \langle n|e^{-HT}|m\rangle e^{i(n\theta' - m\theta)}$
$m = n - \nu$
$= \sum_n e^{in(\theta'-\theta)} \sum_\nu \int [\mathcal{D}A]_\nu e^{i\nu\theta} e^{-S[A]}$
$= \delta(\theta'-\theta) \int [\mathcal{D}A] e^{-S[A] + \frac{i\theta}{16\pi^2} \mathrm{Tr}\,\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}}$
violate P + CP
$\theta_{QCD} \lesssim 10^{-10}$


$\lim_{T\to\infty} \langle\theta'|e^{-HT}|\theta\rangle = \lim_{T\to\infty} \sum_{n,m} \langle n|e^{-HT}|m\rangle e^{i(n\theta' - m\theta)}$
$m = n - \nu$
$= \sum_n e^{in(\theta'-\theta)} \sum_\nu \int [\mathcal{D}A]_\nu e^{i\nu\theta} e^{-S[A]}$
$= \delta(\theta'-\theta) \int [\mathcal{D}A] e^{-S[A] + \frac{i\theta}{16\pi^2} \mathrm{Tr}\,\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}}$
$A_\mu = U \partial_\mu U^{-1} \frac{\rho^2}{\rho^2 + \lambda^2}$
$\theta_{QCD} \lesssim 10^{-10}$
violate P + CP

Anomalies

Quantum effects that spoil classical symmetries

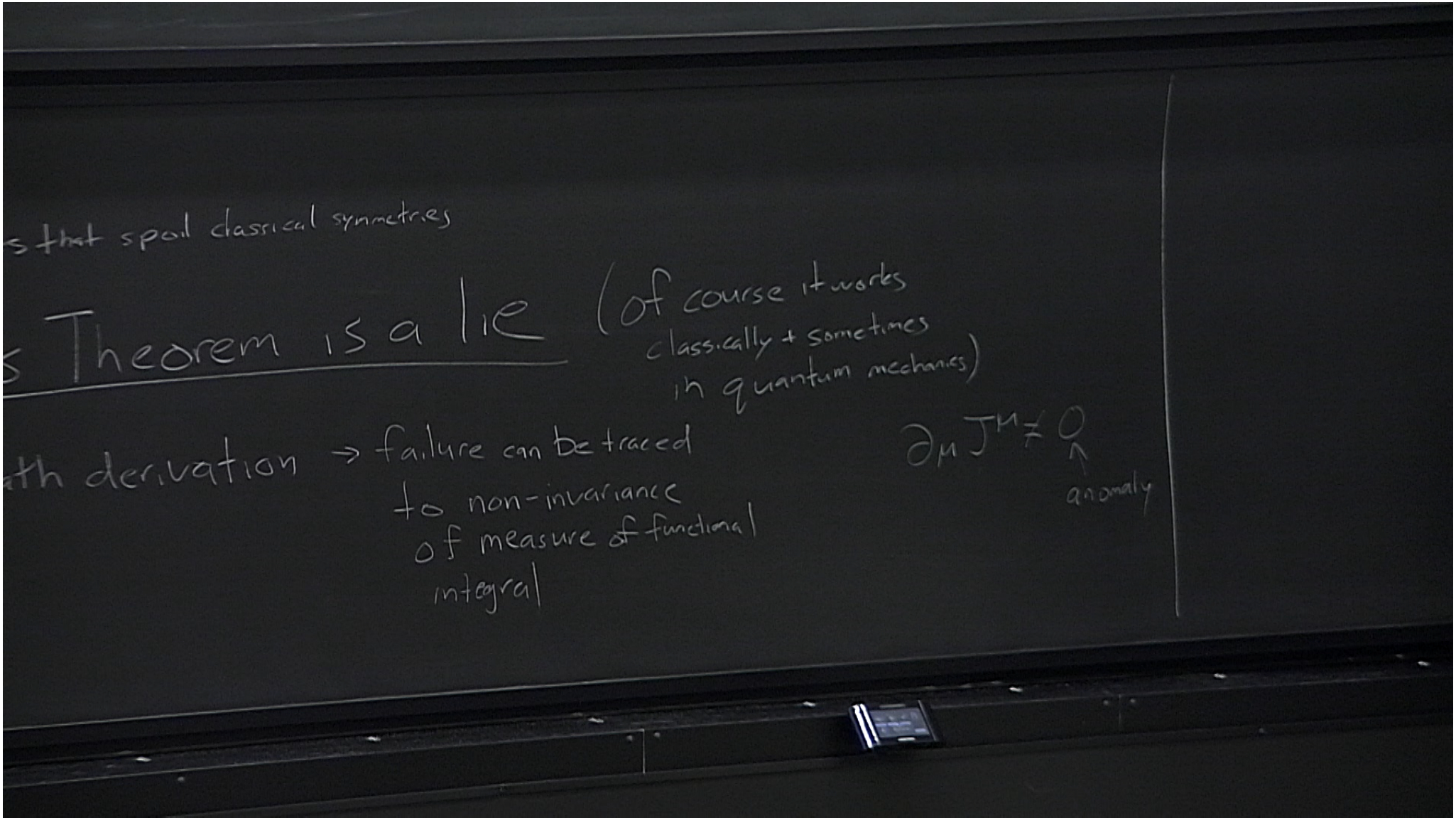
Noether's Theorem is a lie (of course it works classically + sometimes in quantum mechanics)

Anomalies

Quantum effects that spoil classical symmetries

Noether's Theorem is a lie (of course it works classically + sometimes in quantum mechanics)

focus on path derivation → failure can be traced to non-invariance of measure of functional integral


s that spoil classical symmetries
s Theorem is a lie (of course it works classically + sometimes in quantum mechanics)
th derivation → failure can be traced to non-invariance of measure of functional integral
$\partial_\mu J^\mu \neq 0$
anomaly

Anomalies can be observed in IR ($\pi^0 \to \gamma\gamma$)

Arise when it is impossible to find UV regulator that respects all symmetries

Outline: Massless QED in 2d

Path Integral Derivation of Chiral Anomaly