Title: Complexity and RG flow

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Series: Quantum Fields and Strings

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Abstract: Motivated by recent interesting holographic results, several attempts have been made to study complexity (rather " Circuit Complexity") for quantum field theories using Nielsen's geometric method. But most of the studies so far have been limited to free quantum field theory. In this talk we will take a baby step towards understanding the circuit complexity for interacting quantum field theories. We will consider \lambda \phi^4 theory and discuss in detail how to set up the computation perturbatively in coupling. Our method enables us to study circuit complexity in the epsilon expansion for the Wilson-Fisher fixed point. We find that with increasing dimensionality the circuit depth increases in the presence of the \phi^4 interaction eventually causing the perturbative calculation to breakdown. We discuss how circuit complexity relates with the renormalization group. Finally we discuss several possible generalization and compare our results with other approaches.

Complexity and RG flow

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A.Bhattacharyya, A. Shekar, A. Sinha,
``Circuit complexity in interacting QFTs and RG flows,''
JHEP 1810 (2018) [arXiv: 1808.03105[hep-th]].

Outline

– A (very) brief Introduction:

– Circuit Complexity for interacting QFT

Setup

Assumptions

Results (Relation with RG flows)

– Conclusions

An interetsing generalization

Other approach ?

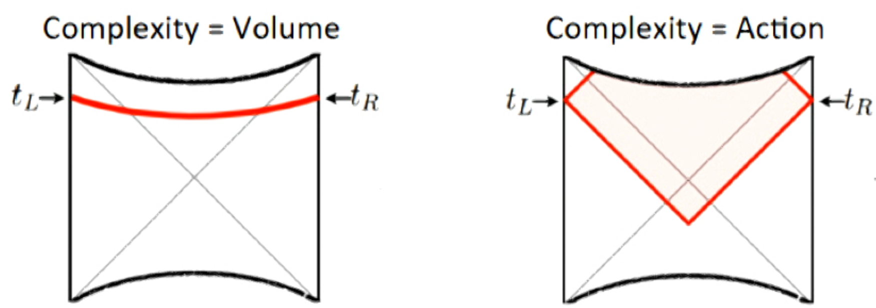
Future directions

Introduction

Holography: Provided us useful objects which connect various
various aspects of geometry with field theory

Ryu-Takayanagi Surface for EE ⟷ Emergence of spacetime from microscopic degrees of freedom

Two interesting objects probing the interior of black hole



(Brown, Roberts, Swingle, Susskind & Zhao)

(Detail calculations done by,
Carmi, Chapman, Lehner, Myers,
Marrochio, Poisson, Sorkin, Sugishita)

$$\mathcal{C}_V(\Sigma) = \max \left[\frac{\mathcal{V}(\mathcal{B})}{G_N \, l}\right] \qquad \mathcal{C}_A(\Sigma) = \frac{I_{WDW}}{\pi \hbar}$$

(picture courtesy Jefferson-Myers, 1707.08570 [hep-th])

Grows with time and keep growing even after the thermalization time

"Complexity" is dual to these two objects ?
Can we compute it field theory ?

Computational complexity : How difficult is to implement a task ?

Important applications in QI:

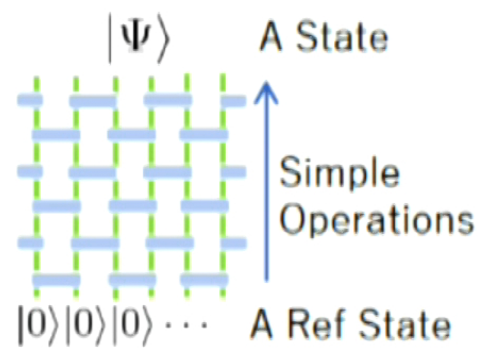
Provides a meaningful comparison between classical and quantum algorithm

Various applications in Quantum Many body physics (Vidal '03, '04, F. Verstraete and I.Cirac '06,09 N. Schuch, I. Cirac, and F. Verstraete '08, D. Aharonov, I. Arad, Z. Landau, and U. Vazirani '11)

Here we will use the notion of "Circuit complexity"

how difficult is to prepare a particular state ?

$$|\psi >= U|\psi_0 >$$



Quantum circuit model: "minimize the number of operations":

Depend on choice of reference state

Free QFT computation: Jefferson Myers ‘17 using Nilesen approach

(for other approaches refer to Chapman, Heller, Marrochio, Pastawski (arXiv:1707.08582), Caputa, Kundu, Miyaji, Takyanagi,Watanabe arXiv: 1706.07056)

But to make contact with holography we need to understand this interacting QFT.

Jordan-Lee-Preskill (2012): Non-perturbative computation of n-particle scattering for ϕ^4 theory by a quantum computer provides an exponential advantage over perturbative method which uses Feynman Diagrams.

Then question naturally arises how a quantum computer would compute other interesting quantities that are calculated by conventional means

Motivated by all these we ask what other important aspects of QFTs can be capture of “Complexity”

RG flow is one important aspects: what we can we say about it in terms complexity ?

Circuit complexity for Interacting QFT

AB, A.Shekar, A. Sinha,
JHEP 1810 (2018) 140,
arXiv: 1808.03105[hep-th]

$\lambda\phi^4$ theory:

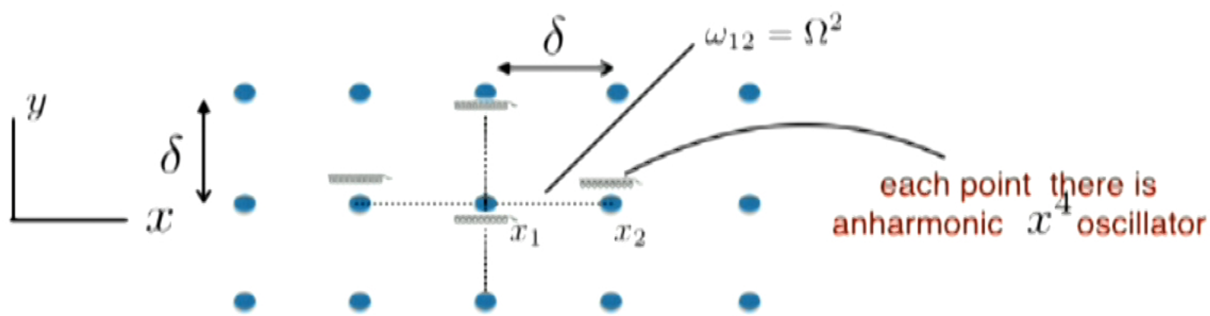
$$\mathcal{H} = \frac{1}{2}\int d^{d-1}x\left[\pi(x)^2 + (\nabla\phi(x))^2 + m^2\phi(x)^2 + \frac{\hat{\lambda}}{12}\phi(x)^4\right]$$

Discretize:

$$\mathcal{H} = \frac{1}{2}\sum_{\vec{n}}\left\{\frac{\pi(\vec{n})^2}{\delta^{d-1}} + \delta^{d-1}\left[\frac{1}{\delta^2}\sum_i(\phi(\vec{n}) - \phi(\vec{n}-\hat{x}_i))^2 + m^2\phi(\vec{n})^2 + \frac{\hat{\lambda}}{12}\phi(\vec{n})^4\right]\right\}.$$

Use the following redefinition:

$$X(\vec{n}) = \delta^{d/2}\phi(\vec{n}),\ P(\vec{n}) = \pi(\vec{n})/\delta^{d/2},\ M = \frac{1}{\delta}, \omega = m,\ \Omega = \frac{1}{\delta},\ \lambda = \frac{\hat{\lambda}}{24}\delta^{-d}.$$



Eigenstates for Hamiltonian for : $\lambda = 0$

$$\langle \tilde{x}_0, \tilde{x}_1 | \psi^0(n_1, n_2)\rangle = \psi^0_{n_1,n_2}(\tilde{x}_0, \tilde{x}_1)$$
$$= \frac{1}{\sqrt{2^{n_1+n_2} n_1! n_2!}} \frac{(\tilde{\omega}_0\tilde{\omega}_1)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\tilde{\omega}_0\tilde{x}_0^2 - \frac{1}{2}\tilde{\omega}_1\tilde{x}_1^2} H_{n_1}(\sqrt{\tilde{\omega}_0}\tilde{x}_0) H_{n_2}(\sqrt{\tilde{\omega}_1}\tilde{x}_1),$$

Given this, the ground state for full Hamiltonian perturbatively in λ

$$\psi_{0,0}(\tilde{x}_0, \tilde{x}_1) = \psi^0_{0,0}(\tilde{x}_0, \tilde{x}_1) + \lambda\, \psi^1_{0,0}(\tilde{x}_0, \tilde{x}_1)$$

$$\psi^0_{0,0}(\tilde{x}_0, \tilde{x}_1) = \frac{(\tilde{\omega}_0\tilde{\omega}_1)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\tilde{\omega}_0\tilde{x}_0^2 - \frac{1}{2}\tilde{\omega}_1\tilde{x}_1^2}$$

It involves terms like

$$\tilde{x}_0^4, \tilde{x}_1^4, \tilde{x}_0^2\tilde{x}_1^2$$

$$\psi^1_{0,0}(\tilde{x}_0, \tilde{x}_1) = -\frac{3(\tilde{\omega}_0 + \tilde{\omega}_1)}{4\sqrt{2}\,\tilde{\omega}_0^3\tilde{\omega}_1}\psi^0_{2,0}(\tilde{x}_0, \tilde{x}_1) - \frac{3(\tilde{\omega}_0 + \tilde{\omega}_1)}{4\sqrt{2}\,\tilde{\omega}_0\tilde{\omega}_1^3}\psi^0_{0,2}(\tilde{x}_0, \tilde{x}_1)$$
$$- \frac{3}{4\tilde{\omega}_0\tilde{\omega}_1(\tilde{\omega}_0 + \tilde{\omega}_1)}\psi^0_{2,2}(\tilde{x}_0, \tilde{x}_1) - \frac{\sqrt{3}}{8\sqrt{2}\,\tilde{\omega}_0^3}\psi^0_{4,0}(\tilde{x}_0, \tilde{x}_1) - \frac{\sqrt{3}}{8\sqrt{2}\,\tilde{\omega}_1^3}\psi^0_{0,4}(\tilde{x}_0, \tilde{x}_1).$$

Still the expression is little bit involved.
From here we will now progress step by step

Eigenstates for Hamiltonian for : $\lambda = 0$

$$\langle \tilde{x}_0, \tilde{x}_1 | \psi^0(n_1, n_2)\rangle = \psi^0_{n_1,n_2}(\tilde{x}_0, \tilde{x}_1)$$
$$= \frac{1}{\sqrt{2^{n_1+n_2} n_1! n_2!}} \frac{(\tilde{\omega}_0\tilde{\omega}_1)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\tilde{\omega}_0\tilde{x}_0^2 - \frac{1}{2}\tilde{\omega}_1\tilde{x}_1^2} H_{n_1}(\sqrt{\tilde{\omega}_0}\tilde{x}_0) H_{n_2}(\sqrt{\tilde{\omega}_1}\tilde{x}_1)\,,$$

Given this, the ground state for full Hamiltonian perturbatively in λ

$$\psi_{0,0}(\tilde{x}_0, \tilde{x}_1) = \psi^0_{0,0}(\tilde{x}_0, \tilde{x}_1) + \lambda\, \psi^1_{0,0}(\tilde{x}_0, \tilde{x}_1)$$

$$\psi^0_{0,0}(\tilde{x}_0, \tilde{x}_1) = \frac{(\tilde{\omega}_0\tilde{\omega}_1)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}\tilde{\omega}_0\tilde{x}_0^2 - \frac{1}{2}\tilde{\omega}_1\tilde{x}_1^2}$$

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$$- \frac{3}{4\tilde{\omega}_0\tilde{\omega}_1(\tilde{\omega}_0 + \tilde{\omega}_1)}\psi^0_{2,2}(\tilde{x}_0, \tilde{x}_1) - \frac{\sqrt{3}}{8\sqrt{2}\,\tilde{\omega}_0^3}\psi^0_{4,0}(\tilde{x}_0, \tilde{x}_1) - \frac{\sqrt{3}}{8\sqrt{2}\,\tilde{\omega}_1^3}\psi^0_{0,4}(\tilde{x}_0, \tilde{x}_1).$$

Still the expression is little bit involved.
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Version I:

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As we are working perturbatively lets make the following approximation

$$\widetilde{\psi_{0,0}}(\tilde{x}_0, \tilde{x}_1) \approx \frac{(\tilde{\omega}_0 \tilde{\omega}_1)^{1/4}}{\sqrt{\pi}} \exp(a_0) \exp\Big[-\frac{1}{2}\Big(a_1\, \tilde{x}_0^2 + a_2\, \tilde{x}_1^2\Big)\Big]$$

Define the Fidelity :

$$F(1,2) = 1 - \frac{|\langle 1|2\rangle|^2}{\langle 1|1\rangle\langle 2|2\rangle},$$

$$a_0 = \frac{3\lambda}{8}\Big(\frac{3}{4\bar{\omega}_0^3} + \frac{3}{4\bar{\omega}_1^3} + \frac{\bar{\omega}_0\bar{\omega}_1 + \bar{\omega}_0^2 + \bar{\omega}_1^2}{\bar{\omega}_0^2\bar{\omega}_1^2(\bar{\omega}_0 + \bar{\omega}_1)}\Big),$$

$$a_1 = \bar{\omega}_0 + \frac{1}{\bar{\omega}_0}\Big(3a_3 + \frac{a_5}{2}\Big), \quad a_2 = \bar{\omega}_1 + \frac{1}{\bar{\omega}_1}\Big(3a_4 + \frac{a_5}{2}\Big),$$

$$a_3 = \frac{\lambda}{4\bar{\omega}_0}, \quad a_4 = \frac{\lambda}{4\bar{\omega}_1}, \quad a_5 = \frac{3\lambda}{(\bar{\omega}_1 + \bar{\omega}_0)}$$

$$F(\psi, \widetilde{\psi}) = \frac{3\,\lambda}{32}\left(\frac{1}{\bar{\omega}_0^3} + \frac{4}{\bar{\omega}_0\bar{\omega}_1(\bar{\omega}_0 + \bar{\omega}_1)} + \frac{1}{\bar{\omega}_1^3}\right)$$

In term of original coordinates:

$$\widetilde{\psi_{0,0}}(x_1, x_2) \approx \frac{(\tilde{\omega}_0 \tilde{\omega}_1)^{1/4}}{\sqrt{\pi}} \exp(a_0) \exp\Big[-\frac{1}{2}\Big(\frac{a_1 + a_2}{2}(x_1^2 + x_2^2) + (a_1 - a_2)\, x_1\, x_2\Big)\Big]$$

We will refer to as "Target State" $\psi^T(x_1, x_2)$

This is a Gaussian State !!!!

(we can borrow the techniques of Jefferson-Myers, 1707.08570 [hep-th])

Lets compute the circuit complexity ("Circuit Depth") for the state

The reference state:

No entanglement in the original basis i.e in the position space

$$\psi^R(x_1, x_2) = \mathcal{N} \exp\Big[- \frac{\tilde{\omega}_{ref}}{2}(x_1^2 + x_2^2)\Big]$$

Now lets build the circuit:

We choose the following natural set of gates:

$$Q_{11} = \exp\Big[\frac{i\,\epsilon}{2}\underbrace{(x_1\,p_1 + p_1\,x_1)}_{O_{11}}\Big], Q_{22} = \exp\Big[\frac{i\,\epsilon}{2}\underbrace{(x_2\,p_2 + p_2\,x_2)}_{O_{22}}\Big], Q_{12} = \exp\Big[i\,\epsilon\,\underbrace{x_1\,p_2}_{O_{12}}\Big], Q_{21} = \exp\Big[i\,\epsilon\,\underbrace{x_2\,p_1}_{O_{21}}\Big]$$

scaling, scaling, entangling, entangling

Note that the operators O's form a closed algebra.
We will utilize this underline group structure later

Then we construct the circuit U with these gates

$$|\psi^T(x_1, x_2) - U\psi^R(x_1, x_2)|^2 < \epsilon$$

We can tune $\epsilon \to 0$ to get a very precise match

Now one possible circuit :

$$\psi^T(x_1, x_2) = Q_{22}^{\alpha_3} Q_{21}^{\alpha_2} Q_{11}^{\alpha_1} \psi^R(x_1, x_2)$$

Now the circuit depth (total number of gates) :

$$\begin{aligned}\mathcal{D}(U) &= |\alpha_1| + |\alpha_2| + |\alpha_3|, \\ &= \frac{1}{2\epsilon} \log\left[\frac{a_1\, a_2}{\tilde{\omega}_{ref}^2}\right] + \frac{|a_1 - a_2|}{\epsilon} \sqrt{\frac{2\,\tilde{\omega}_{ref}}{(a_1 + a_2) a_1 a_2}}\end{aligned}$$

Note that these operators do not commute each other

This is just one possible circuit. What is the optimal one ?

To find a possible optimal circuit we use the ***Nielsen approach***

(Nielsen quant-ph/0502070, Nielsen, Dowling, Gu, and Doherty,quant-ph/0603161 M.~A. Nielsen and M.~R. Dowling,quant-ph/0701004)

Question is given a set of operators (gates) satisfying a closed algebra how to find the optimal circuit ?

Nielsen approach:

(Nielsen quant-ph/0502070,
Nielsen, Dowling, Gu, Doherty,quant-ph/0603161
M.~A. Nielsen and M.~R. Dowling,quant-ph/0701004)

To achieve the optimal circuit instead of working in discrete picture we work in continuous picture, the circuit is parametrized by continuous parameter "s" and consists of continuous functions

$$U(s) = \overleftarrow{\mathcal{P}} \exp(i \int_0^s ds\, Y^I(s)\, O_I(s)),$$

Path ordering

In the discrete picture

$$\Delta s = \epsilon$$

$$O_I = \{O_{11}, O_{22}, O_{12}, O_{21}\}$$

$Y^I(s)$ control functions

Boundary conditions:

$$\psi^T(x_1, x_2) = U(s=1)\psi^R(x_1, x_2),$$
$$U(s=0) = I$$

(note: there is a freedom in choosing end point for "s", we fix it to be at s=1)

Optimal Circuit: We need to find optimal $Y^I(s)$

This can be typically achieved by minimizing some kind of action "***Cost function***" $\mathcal{F}(U, \dot{U})$ for these $Y^I(s)$

Complexity: $$\mathcal{D}(U) = \int_0^1 \mathcal{F}(U, \dot{U})\, ds.$$

Some desirable properties of cost functions:

1. Continous
2. Positivity
3. Homogeneity
4. Satisfy triangle inequality

(Nielsen quant-ph/0502070, Nielsen, Dowling, Gu, Doherty,quant-ph/0603161 M.~A. Nielsen and M.~R. Dowling,quant-ph/0701004, Jefferson-Myers, 1707.08570 [hep-th])

These help us to identify these functions as distance function between two point on a given manifold

Several Choice:

Jefferson-Myers, 1707.08570 [hep-th], Hackl-Myers, 1803.10638 [hep-th], Guo-Hernandez-Myers-Ruan, 1807.07677[hep-th]

$$\mathcal{F}_2(U,Y) = \sqrt{\sum_I p_I (Y^I)^2}, \mathcal{F}_\kappa(U,Y) = \sum_I p_I\, |Y^I|^\kappa, \quad \kappa \quad \text{is an integer and }, \kappa \geq 1,$$

$$\mathcal{F}_p(U,Y) = (Tr(V^\dagger V)^{p/2}))^{1/p}, V^I = Y^I(s) M_I, \quad p \quad \text{is an integer}$$

$\mathcal{F}_{\kappa=1}(\text{for } p_I = 1, \forall I)$ counts number of gates (for us case we will prefer this the others, reason we will explain later)

$\mathcal{F}_2$ is the distance on Riemannian manifold

Now the strategy is to minimize these cost functions.

For this we first solve for the geodesics

Remember : $Y^I(s)O_I = \partial_s U(s).U(s)^{-1}$

Next we define a metric (right invariant !)

$$ds^2 = G_{IJ} dY^I dY^J$$

Find the Geodesic and evaluate the the action on it

$$\mathcal{D}(U) = \int_0^1 ds \sqrt{\sum_{I,J} G_{IJ} Y^I Y^J}$$ (here we have used the $\mathcal{F}_2$)

G_{IJ} Penalty factor

To practically compute this metric we first note: *Jefferson-Myers, 1707.08570 [hep-th],*

Our wavefunction can be written in the following way:

$$\psi^s(x_1, x_2) = \mathcal{N}^s \exp\left[-\frac{1}{2}\vec{v}.A(s).\vec{v}\right] \qquad \vec{v} = \{x_1, x_0\}$$

And: $s = 0$, Reference State

$s = 1$, Target State

For our case:

$$A(s=0) = \begin{pmatrix} \tilde{\omega}_{\rm ref} & 0 \\ 0 & \tilde{\omega}_{\rm ref} \end{pmatrix}$$

$$A(s=1) = \begin{pmatrix} \frac{1}{2}(a_1 + a_2) & \frac{1}{2}(a_1 - a_2) \\ \frac{1}{2}(a_1 - a_2) & \frac{1}{2}(a_1 + a_2) \end{pmatrix}$$

Line element:

$$ds^2 = \delta_{IJ} Tr(dU.U^{-1}.M_I^T) Tr(dU.U^{-1}.M_J^T),$$
$$= 2dy^2 + 2d\rho^2 + 2\cosh(2\rho)\cosh(\rho)^2 d\tau^2 + 2\cosh(2\rho)\sinh(\rho)^2 d\theta^2 - 2\sinh(2\rho)^2$$

We now solve for the geodesic on this background

Boundary conditions:

We observe that the unitary operator acts on the wavefunction in the following way $A(s) = U(s).A(s=0).U(s)^T$

arbitrary, either θ or τ can be uniquely determined

$$s := 0, \{y(0) = 0, \rho(0) = 0, \theta(0) + \tau(0) = c_0\}$$

$$s := 1, \exp(2y(1)) = \sqrt{\frac{a_1\, a_2}{\tilde{\omega}_{ref}^2}}, \cosh(2\rho(1)) = \frac{a_1 + a_2}{2\sqrt{a_1 a_2}}, \tan(\theta(1) + \tau(1)) = 0.$$

Choosing: $c_0 = 0$ Geodesic became straight line

$$y(s) = y(1)\, s, \rho(s) = \rho(1)\, s$$
$$\tau(s) = 0, \theta(s) = \theta_0$$

We can show $\mathcal{D}(U) = \int_0^1 ds \sqrt{\sum_{I,J} G_{IJ} Y^I Y^J}$ gets minimized on this geodesic

In fact one can check that it is the global minimum

Complexity: $\mathcal{D}(U) = \sqrt{y(1)^2 + \rho(1)^2}$

In fact one can check $\mathcal{F}_{\kappa=1} (\sum_I |Y^I|)$ and the associated complexity

$$\mathcal{C}_{\kappa=1} = \int_0^1 ds \mathcal{F}_{\kappa=1}$$

also gets minimized when evaluated on this geodesic

In terms of normal mode frequency:

$$\boxed{\mathcal{C}_{\kappa=1} = \frac{1}{2}\left[\left|\log\frac{\tilde{\omega}_0}{\tilde{\omega}_{ref}}\right| + \left|\log\frac{\tilde{\omega}_1}{\tilde{\omega}_{ref}}\right| + \frac{3\lambda\,\delta}{4}\left(\frac{1}{\tilde{\omega}_0^3} + \frac{1}{\tilde{\omega}_1^3}\right)\right] + \mathcal{O}(\lambda^2)}$$

Some useful points:

Optimal Circuit:

$$U^{opt}(s) = \exp\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y_1\, s + \begin{pmatrix} -\sin(\theta_0) & \cos(\theta_0) \\ \cos(\theta_0) & \sin(\theta_0) \end{pmatrix} \rho_1\, s \right],$$
$$= \exp\left[M_{11}(y_1 - \sin(\theta_0)\rho_1)\, s + M_{22}(y_1 + \sin(\theta_0)\rho_1)\, s + (M_{12} + M_{21}) \cos(\theta_0)\rho_1\, s \right]$$

Normal mode basis: $\tilde{x}_0 = \frac{1}{\sqrt{2}}(x_1 + x_2), \tilde{x}_1 = \frac{1}{\sqrt{2}}(x_1 - x_2)$

$$\vec{v} = \{\tilde{x}_0, \tilde{x}_1\}, A(s=0) = \begin{pmatrix} \tilde{\omega}_{\rm ref} & 0 \\ 0 & \tilde{\omega}_{\rm ref} \end{pmatrix}, A(s=1) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

and not hard to show that $U^{opt}(s) = \exp\left[M_{11}(y_1 - \rho_1)\, s + M_{22}(y_1 + \rho_1)\, s \right]$

optimal circuit is consist of only scaling operator and
the geodesic is always a straight line

Complexity:

$$C_{\kappa=1} = \frac{1}{2} \sum_i \left[\left| \log \frac{\lambda_i(s=1)}{\lambda_i(s=0)} \right| \right] \qquad \mathcal{D}(U) = \int_0^1 ds \mathcal{F}_2 = \sqrt{\sum_i \left(\log \left[\frac{\lambda_i(s=1)}{\lambda_i(s=0)} \right] \right)^2}$$
$$= \frac{1}{2} \left| \log \frac{\det A(s=1)}{\det A(s=0)} \right|$$

A lesson:

Whenever, we can simultaneously diagonalize the reference and the target state, the complexity will be given by (appropriate function) the ratio of the eigenvalues of $A(s=0)$ and $A(s=1)$

Some useful points:

Optimal Circuit:

$$U^{opt}(s) = \exp\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y_1\, s + \begin{pmatrix} -\sin(\theta_0) & \cos(\theta_0) \\ \cos(\theta_0) & \sin(\theta_0) \end{pmatrix} \rho_1\, s\right],$$
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Normal mode basis: $\tilde{x}_0 = \frac{1}{\sqrt{2}}(x_1 + x_2), \tilde{x}_1 = \frac{1}{\sqrt{2}}(x_1 - x_2)$

$$\vec{v} = \{\tilde{x}_0, \tilde{x}_1\}, A(s=0) = \begin{pmatrix} \tilde{\omega}_{\rm ref} & 0 \\ 0 & \tilde{\omega}_{\rm ref} \end{pmatrix}, A(s=1) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

and not hard to show that $U^{opt}(s) = \exp\left[M_{11}(y_1 - \rho_1)\, s + M_{22}(y_1 + \rho_1)\, s\right]$

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Complexity:

$$C_{\kappa=1} = \frac{1}{2}\sum_i \left[\left|\log\frac{\lambda_i(s=1)}{\lambda_i(s=0)}\right|\right] \qquad \mathcal{D}(U) = \int_0^1 ds \mathcal{F}_2 = \sqrt{\sum_i \left(\log\left[\frac{\lambda_i(s=1)}{\lambda_i(s=0)}\right]\right)^2}$$
$$= \frac{1}{2}\left|\log\frac{\det A(s=1)}{\det A(s=0)}\right|$$

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Whenever, we can simultaneously diagonalize the reference and the target state, the complexity will be given by (appropriate function) the ratio of the eigenvalues of $A(s=0)$ and $A(s=1)$

Version II:

AB, A.Shekar, A. Sinha, JHEP 1810 (2018) 140, arXiv: 1808.03105[hep-th]

Lets us now consider the full wavefunction, we work in normal model coordinates

For convenience we write in the following form:

$$\psi_{0,0}(\tilde{x}_0, \tilde{x}_1) \approx \frac{(\bar{\omega}_0 \bar{\omega}_1)^{1/4}}{\sqrt{\pi}} \exp(a_0) \exp\Big[-\frac{1}{2}\Big(a_1 \tilde{x}_0^2 + a_2 \tilde{x}_1^2 + a_3 \tilde{x}_0^4 + a_4 \tilde{x}_1^4 + a_5 \tilde{x}_0^2 \tilde{x}_1^2 \Big)\Big]$$

$$a_0 = \frac{3\lambda}{8}\Big(\frac{3}{4\bar{\omega}_0^3} + \frac{3}{4\bar{\omega}_1^3} + \frac{\bar{\omega}_0\bar{\omega}_1 + \bar{\omega}_0^2 + \bar{\omega}_1^2}{\bar{\omega}_0^2\bar{\omega}_1^2(\bar{\omega}_0 + \bar{\omega}_1)}\Big),$$

$$a_1 = \bar{\omega}_0 + \frac{1}{\bar{\omega}_0}\Big(3a_3 + \frac{a_5}{2}\Big), \quad a_2 = \bar{\omega}_1 + \frac{1}{\bar{\omega}_1}\Big(3a_4 + \frac{a_5}{2}\Big),$$

$$a_3 = \frac{\lambda}{4\bar{\omega}_0}, \quad a_4 = \frac{\lambda}{4\bar{\omega}_1}, \quad a_5 = \frac{3\lambda}{(\bar{\omega}_1 + \bar{\omega}_0)}$$

Now to set up our calculation as before we write the wavefunction in the following form

$$\psi^s(\tilde{x}_0, \tilde{x}_1) = \mathcal{N}^s \exp\Big[-\frac{1}{2}(v_a . A(s)_{a\,b} . v_b)\Big].$$

Now we need to extend the basis: $\vec{v} = \{\tilde{x}_0, \tilde{x}_1\} \to \{\tilde{x}_0, \tilde{x}_1, ??, \cdots ??\}$

Many possibilities and also many ways to write A(s) for eg :

$$\vec{v} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_0^2, \tilde{x}_1^2\} \to A(s=1) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & \frac{a_5}{2} \\ 0 & 0 & \frac{a_5}{2} & a_4 \end{pmatrix}$$

$$\vec{v} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_0\tilde{x}_1\, \tilde{x}_0^2, \tilde{x}_1^2\} \to A(s=1) = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & \tilde{b}\, a_5 & 0 & 0 \\ 0 & 0 & 0 & a_3 & \frac{1}{2}(1-\tilde{b})a_5 \\ 0 & 0 & 0 & \frac{1}{2}(1-\tilde{b})a_5 & a_4 \end{pmatrix}$$

$\tilde{b}$ arbitrary.

which one to choose ?

Also what is the reference state?

In this basis Gaussian reference state will look like for eg:

$$\vec{v} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_0^2, \tilde{x}_1^2\} \to A(s=0) = \begin{pmatrix} \tilde{\omega}_{ref} & 0 & 0 & 0 \\ 0 & \tilde{\omega}_{ref} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Determinant is zero. This will be create problem

Remember our previous calculations. We will restrict ourselves to GL(R) unitaries.

These are matrices with non-zero determinant

We also have: $A(s=1) = U(s=1).A(s=0).U(s=1)^T$

It preserves the sign of determinant of A's and they have to non zero

We can have the following guiding principles :

1. $det(A(s=0)) \neq 0$ but still $|\psi^R\rangle$ is unentangled in the original position basis

2. Given a basis $\vec{v}$ both $\det(A(s=0))$ and $\det(A(s=1)) > 0(\text{or} < 0)$

3. Also remember that the value of the complexity depend on of number of eigenvalues of A, hence on the dimensions of $\vec{v}$, So a minimum possible extension is desirable

These guiding principles will help us to choose the minimal $\vec{v}$ and partially fix the form of A

Given these guidelines for this two oscillator case we have following :

1. Reference state (normal mode basis):

$$\psi^{s=0}(\tilde{x}_0, \tilde{x}_1) = \mathcal{N}^{s=0} \exp\left(-\frac{\tilde{\omega}_{ref}}{2}\left[\tilde{x}_0^2 + \tilde{x}_1^2 + \frac{\lambda_0}{2}(\tilde{x}_0^4 + \tilde{x}_1^4 + 6\tilde{x}_0^2\tilde{x}_1^2)\right]\right)$$

(in the position basis: $\psi^{s=0}(x_1, x_2) = \mathcal{N}^{s=0} \exp\left[-\frac{\tilde{\omega}_{ref}}{2}(x_1^2 + x_2^2 + \lambda_0(x_1^4 + x_2^4))\right]$

no entanglement)

arbitrary

2. $\vec{v} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_0\,\tilde{x}_1, \tilde{x}_0^2, \tilde{x}_1^2\}$

$$A(s=0) = \begin{pmatrix} \tilde{\omega}_{ref} & 0 & 0 & 0 & 0 \\ 0 & \tilde{\omega}_{ref} & 0 & 0 & 0 \\ 0 & 0 & b\lambda_0\tilde{\omega}_{ref} & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda_0\tilde{\omega}_{ref}}{2} & \frac{1}{2}(3-b)\lambda_0\tilde{\omega}_{ref} \\ 0 & 0 & 0 & \frac{1}{2}(3-b)\lambda_0\tilde{\omega}_{ref} & \frac{\lambda_0\tilde{\omega}_{ref}}{2} \end{pmatrix} \quad A(s=1) = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & \bar{b}\,a_5 & 0 & 0 \\ 0 & 0 & 0 & a_3 & \frac{1}{2}(1-\bar{b})a_5 \\ 0 & 0 & 0 & \frac{1}{2}(1-\bar{b})a_5 & a_4 \end{pmatrix}$$

assuming $\det(A(s=0)) > 0, \det(A(s=1)) > 0$

$$2 < b < 4,\; 0 < \bar{b} < 1 + \frac{1}{6}\sqrt{\frac{(\tilde{\omega}_0 + \tilde{\omega}_0)^2}{\tilde{\omega}_0\tilde{\omega}_1}}\; (\tilde{\omega}_1 > \tilde{\omega}_0)$$

For simplicity we choose: $b = 3$ to kill the off-diagonal terms

Now what about the computation of complexity ?

To expedite the computation we make some useful observations

First notice the structure of A:

Gaussian

Non-Gaussian

$$\left(\begin{array}{cc:ccc}
a_1 & 0 & 0 & 0 & 0 \\
0 & a_2 & 0 & 0 & 0 \\
\hdashline
0 & 0 & \tilde{b}\,a_5 & 0 & 0 \\
0 & 0 & 0 & a_3 & \frac{1}{2}(1-\tilde{b})a_5 \\
0 & 0 & 0 & \frac{1}{2}(1-\tilde{b})a_5 & a_4
\end{array}\right)$$

Gaussian: $A^{(1)}(s) = A^{(1,0)}(s) + \lambda\, A^{(1,1)}(s)$ Non-Gaussian: $A^{(2)}(s) = \lambda\, A^{(2,1)}(s)$

Given these block structure it is convenient to parametrize the unitary as $R^3 \times GL(2,R)$

We will always look for these kind of pattern and it will be useful when we generalize for arbitrary number of oscillator

Secondly, both $A(s=0)$ and $A(s=1)$ can be simultaneously

so again, we can simply have $\mathcal{C}_{\kappa=1} = \frac{1}{2}\sum_i \left[\left|\log \frac{\lambda_i(s=1)}{\lambda_i(s=0)}\right|\right] = \frac{1}{2}\left|\log \frac{\det A(s=1)}{\det A(s=0)}\right|$ $\quad \mathcal{D}(U) = \int_0^1 ds\, \mathcal{F}_2 = \sqrt{\sum_i \left(\log\left[\frac{\lambda_i(s=1)}{\lambda_i(s=0)}\right]\right)^2}$

Geodesics are again straight line !!!

Now lets take closer look at the expression: eg:

$$\mathcal{C}_{\kappa=1} = \frac{1}{2}\sum_i \left[\left|\log\frac{\lambda_i(s=1)}{\lambda_i(s=0)}\right|\right] = \frac{1}{2}\left|\log\frac{\det A(s=1)}{\det A(s=0)}\right|$$

We expand this expression in λ keep only linear order terms

$$\mathcal{C}_{\kappa=1} = \mathcal{C}^{(1)}_{\kappa=1} + \mathcal{C}^{(2)}_{\kappa=1}$$

$$\mathcal{C}^{(1)}_{\kappa=1} = \frac{1}{2}\left[\left|\log\frac{\tilde{\omega}_0}{\tilde{\omega}_{ref}}\right| + \left|\log\frac{\tilde{\omega}_1}{\tilde{\omega}_{ref}}\right| + \frac{3\lambda\,\delta}{4}\left(\frac{1}{\tilde{\omega}_0^3} + \frac{1}{\tilde{\omega}_1^3}\right)\right] + \mathcal{O}(\lambda^2)$$

Gaussian part

$$\mathcal{C}^{(2)}_{\kappa=1} = \log\left[\left(\frac{\lambda}{2\,\tilde{\omega}_0\,\lambda_0}\right)\sqrt{\left(\frac{1}{\tilde{\omega}_0\,\tilde{\omega}_1} - \frac{36(\tilde{b}-1)^2}{(\tilde{\omega}_0+\tilde{\omega}_1)^2}\right)}\right]$$

Non- Gaussian part

$$\psi^{s=0}(x_1,x_2) = \mathcal{N}^{s=0}\exp\left[-\frac{\tilde{\omega}_{ref}}{2}(x_1^2+x_2^2+\lambda_0(x_1^4+x_2^4))\right]$$

$$A(s=1) = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & \tilde{b}\,a_5 & 0 & 0 \\ 0 & 0 & 0 & a_3 & \frac{1}{2}(1-\tilde{b})a_5 \\ 0 & 0 & 0 & \frac{1}{2}(1-\tilde{b})a_5 & a_4 \end{pmatrix}$$

Perturbation breaks down, can we fix it ?
We will see the this will help us
to further constrain the undetermined parameters

Now lets take closer look at the expression: eg:

$$\mathcal{C}_{\kappa=1} = \frac{1}{2}\sum_i \left[\left|\log \frac{\lambda_i(s=1)}{\lambda_i(s=0)}\right|\right]$$
$$= \frac{1}{2}\left|\log \frac{\det A(s=1)}{\det A(s=0)}\right|$$

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$$\mathcal{C}_{\kappa=1} = \mathcal{C}^{(1)}_{\kappa=1} + \mathcal{C}^{(2)}_{\kappa=1}$$

$$\mathcal{C}^{(1)}_{\kappa=1} = \frac{1}{2}\left[\left|\log \frac{\tilde{\omega}_0}{\tilde{\omega}_{ref}}\right| + \left|\log \frac{\tilde{\omega}_1}{\tilde{\omega}_{ref}}\right| + \frac{3\lambda\,\delta}{4}\left(\frac{1}{\tilde{\omega}_0^3} + \frac{1}{\tilde{\omega}_1^3}\right)\right] + \mathcal{O}(\lambda^2)$$

Gaussian part

$$\mathcal{C}^{(2)}_{\kappa=1} = \log\left[\left(\frac{\lambda}{2\,\tilde{\omega}_0\,\lambda_0}\right)\sqrt{\left(\frac{1}{\tilde{\omega}_0\,\tilde{\omega}_1} - \frac{36(\bar{b}-1)^2}{(\tilde{\omega}_0+\tilde{\omega}_1)^2}\right)}\right]$$

$$\psi^{s=0}(x_1,x_2) = \mathcal{N}^{s=0}\exp\left[-\frac{\tilde{\omega}_{ref}}{2}(x_1^2+x_2^2+\lambda_0(x_1^4+x_2^4))\right]$$

Non- Gaussian part

$$A(s=1) = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & \bar{b}\,a_5 & 0 & 0 \\ 0 & 0 & 0 & a_3 & \frac{1}{2}(1-\bar{b})a_5 \\ 0 & 0 & 0 & \frac{1}{2}(1-\bar{b})a_5 & a_4 \end{pmatrix}$$

Perturbation breaks down, can we fix it ?
We will see the this will help us
to further constrain the undetermined parameters

$$\mathcal{C}^{(2)}_{\kappa=1} = \log\Big[\frac{\lambda}{2\,\bar{\omega}_0\,\lambda_0}\sqrt{\Big(\frac{1}{\bar{\omega}_0\,\bar{\omega}_1} - \frac{36(\bar{b}-1)^2}{(\bar{\omega}_0+\bar{\omega}_1)^2}\Big)}\Big]$$

We fix it in two steps:

Step I: $\lambda_0 \propto \lambda$

Step II : After Step I, $\mathcal{C}^{(2)}_{\kappa=1}$ will become $\mathcal{O}(1)$

So this will be of the same order as that of free theory part.
But we are only solving the wavefunction perturbatively,

$\mathcal{C}^{(2)}_{\kappa=1}$ comes from the $\mathcal{O}(\lambda)$ part of the wavefunction

So it will be natural if we can make this
complexity expression perturbative

Now the penalty factor comes to rescue:

Remember: $\mathcal{C}_{\kappa=1} = \sum_I p_I \left|Y^I\right|$

We do the following $\mathcal{C}_{\kappa=1} = \frac{1}{2}\Big(\log\Big(\frac{\det(A^{(1)}(s=1))}{\det(A^{(1)}(s=0))}\Big) + A\log\Big(\frac{\det(A^{(2)}(s=1))}{\det(A^{(2)}(s=0))}\Big)\Big)$

At least have to $\mathcal{O}(\lambda)$

Optimal Circuit: $U(s) = \exp\left[\tilde{M}\, s\right]$

$$\tilde{M} = \alpha M_{11} + \beta M_{22} + \gamma M_{33} + \delta M_{44} + \zeta M_{55} + \tau M_{66} + \kappa M_{77} + \mu M_{88}$$

Exact values of these coefficient are not important

$$M_{11} = diag(1,0,1,2,0),\, M_{22} = diag(0,1,1,0,2),\, M_{33} = diag(-1,-\frac{1}{4},-\frac{13}{4},-3,-\frac{1}{2}),$$

$$M_{44} = diag(-\frac{1}{4},-1,-\frac{13}{4},-\frac{1}{2},-3),\, M_{77} = diag(-\frac{13}{4},0,-\frac{13}{4},-\frac{33}{2},0),\, M_{88} = diag(0,-\frac{13}{4},-\frac{13}{4},0,-\frac{33}{2})$$

$$M_{55} = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & -5 & -1 \end{pmatrix},\; M_{66} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & -1 & -5 \end{pmatrix}$$

Given this: $\vec{v} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_0\,\tilde{x}_1, \tilde{x}_0^2, \tilde{x}_1^2\}$

$$\frac{i\,(\tilde{x}_0\tilde{p}_0 + \tilde{p}_0\,\tilde{x}_0)}{2} = i\,(\tilde{x}_0\,\tilde{p}_0 - \frac{i}{2}),\quad i\,\tilde{x}_0\,\tilde{p}_0 \to M_{11},\quad \frac{i\,(\tilde{x}_1\tilde{p}_1 + \tilde{p}_1\,\tilde{x}_1)}{2} = i\,(\tilde{x}_1\,\tilde{p}_1 - \frac{i}{2}),\quad i\,\tilde{x}_1\,\tilde{p}_1 \to M_{22},$$

$$\frac{i}{8}\,(\tilde{x}_1\tilde{p}_1 + \tilde{p}_1\tilde{x}_1)(\tilde{x}_0\tilde{p}_0 + \tilde{p}_0\tilde{x}_0)(\tilde{x}_0\tilde{p}_0 + \tilde{p}_0\tilde{x}_0)$$

$$= \frac{i}{8}(8\tilde{x}_1\tilde{p}_1(\tilde{x}_0\tilde{p}_0)^2 - 8\,i\,\tilde{x}_1\tilde{p}_1\tilde{x}_0\tilde{p}_0 - 2\tilde{x}_1\tilde{p}_1 - 4\,i\,(\tilde{x}_0\tilde{p}_0)^2 - 4\tilde{x}_0\tilde{p}_0 + i),$$

$$\frac{i}{8}(8\tilde{x}_1\tilde{p}_1(\tilde{x}_0\tilde{p}_0)^2 - 8\,i\,\tilde{x}_1\tilde{p}_1\tilde{x}_0\tilde{p}_0 - 2\tilde{x}_1\tilde{p}_1 - 4\,i\,(\tilde{x}_0\tilde{p}_0)^2 - 4\tilde{x}_0\tilde{p}_0) \to M_{33},$$

$$\frac{i}{8}(8\tilde{x}_0\tilde{p}_0(\tilde{x}_1\tilde{p}_1)^2 - 8\,i\,\tilde{x}_0\tilde{p}_0\tilde{x}_1\tilde{p}_1 - 2\tilde{x}_0\tilde{p}_0 - 4\,i\,(\tilde{x}_1\tilde{p}_1)^2 - 4\tilde{x}_1\tilde{p}_1) \to M_{44},$$

$$\frac{i}{8}\,(\tilde{x}_0\tilde{p}_0 + \tilde{p}_0\tilde{x}_0)(\tilde{x}_0\tilde{p}_1 + \tilde{x}_1\tilde{p}_0)(\tilde{x}_0\tilde{p}_1 + \tilde{x}_1\tilde{p}_0) \to M_{55},\quad \frac{i}{8}\,(\tilde{x}_1\tilde{p}_1 + \tilde{p}_1\tilde{x}_1)(\tilde{x}_0\tilde{p}_1 + \tilde{x}_1\tilde{p}_0)(\tilde{x}_0\tilde{p}_1 + \tilde{x}_1\tilde{p}_0) \to M_{66},$$

$$\frac{i}{8}(8\tilde{x}_0\tilde{p}_0(\tilde{x}_0\tilde{p}_0)^2 - 8\,i\,\tilde{x}_0\tilde{p}_0\tilde{x}_0\tilde{p}_0 - 2\tilde{x}_0\tilde{p}_0 - 4\,i\,(\tilde{x}_0\tilde{p}_0)^2 - 4\tilde{x}_0\tilde{p}_0) \to M_{77},$$

$$\frac{i}{8}(8\tilde{x}_1\tilde{p}_1(\tilde{x}_1\tilde{p}_1)^2 - 8\,i\,\tilde{x}_1\tilde{p}_1\tilde{x}_1\tilde{p}_1 - 2\tilde{x}_1\tilde{p}_1 - 4\,i\,(\tilde{x}_1\tilde{p}_1)^2 - 4\tilde{x}_1\tilde{p}_1) \to M_{88}$$

composite operator made up of usual scaling operators

Generalization for N oscillator in arbitrary dimensions

AB, A. Shekar, A. Sinha, JHEP 1810 (2018) 140, arXiv: 1808.03105[hep-th]

We are now in a position to generalize our previous results

For N- oscillator : $\mathcal{H} = \sum_{a=0}^{N-1} \frac{1}{2}\left[p_a^2 + \omega^2 x_a^2 + \Omega^2 (x_a - x_{a+1})^2 + 2\lambda\, x_a^4\right]$

Normal mode basis:

$$x_a = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(\frac{2\pi\, i\, k}{N}\, a\right) \tilde{x}_k ,$$

$$p_a = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(-\frac{2\pi\, i\, k}{N}\, a\right) \tilde{p}_k$$

$$\mathcal{H} = \frac{1}{2} \sum_{k=0}^{N-1} \left[|\tilde{p}_k|^2 + \tilde{\omega}_k^2 |\tilde{x}_k|^2\right] + \frac{\lambda}{N} \sum_{\alpha = N-k'-k_1-k_2 \bmod N,\, k',k_1,k_2=0}^{N-1} \tilde{x}_\alpha \tilde{x}_{k'} \tilde{x}_{k_1} \tilde{x}_{k_2} ,$$

with: $\tilde{\omega}_k^2 = \left(\omega^2 + 4\Omega^2 \sin^2\left(\frac{\pi k}{N}\right)\right)$

This can be straightforwardly generalized for higher dimensions (d):

$$\sum_{k=1}^{d-1} \tilde{\omega}_{i_k}^2 = m^2 + \frac{4}{\delta^2} \sum_{k=1}^{d-1} \sin^2\left(\frac{\pi\, i_k}{N}\right),$$

We again solve for the ground state wave functional and that will be our target state

$$\psi^s(\tilde{x}_0, \cdots \tilde{x}_{N-1}) \approx \exp\Big(-\frac{1}{2} v_a . A(s)_{ab} . v_b\Big)$$

Minimal choice for the basis:

$$\vec{v} = \{\underbrace{\tilde{x}_0, \tilde{x}_1, ..., \tilde{x}_{N-1}}_{N}, \underbrace{\tilde{x}_0^2, \tilde{x}_1^2, ..., \tilde{x}_{N-1}^2, .., \tilde{x}_i\tilde{x}_j, ...}_{\frac{N(N-1)}{2}}\}.$$

$N \to \infty$, it grows as N^2

Also, $A(s) = \begin{pmatrix} A^{(1)}(s) & 0 \\ 0 & A^{(2)}(s) \end{pmatrix}$ with,

$$A^{(1)}(s=0) = \tilde{\omega}_{ref} I_{N\times N}, A^{(2)}(s=0) = \tilde{\omega}_{ref}\lambda_0 I_{k\times k}.$$

k is of dimension $\frac{N(N-1)}{2}$

An important observations:

For the "ambiguous" block we can only make sure that the ratio

$$\frac{\det(A^{(2)}(s=1))}{\det(A^{(2)}(s=0))} > 0$$ (but individually they can be negative as well !!)

Hence: $\mathcal{C}_{\kappa=1} = \frac{1}{2}\Big(\log\Big(\frac{\det(A^{(1)}(s=1))}{\det(A^{(1)}(s=0))}\Big) + \mathcal{A}\log\Big(\frac{\det(A^{(2)}(s=1))}{\det(A^{(2)}(s=0))}\Big)\Big)$ this is always well defined

But for example: $\mathcal{D}(U) = \int_0^1 ds\mathcal{F}_2 = \sqrt{\sum_i \Big(\log\Big[\frac{\lambda_i(s=1)}{\lambda_i(s=0)}\Big]\Big)^2} \neq \log\Big(\frac{\det(A(s=1))}{\det(A(s=0))}\Big)^2$

In fact:

$$\mathcal{F}_2(U,Y) = \sqrt{\sum_I p_I (Y^I)^2}, \mathcal{F}_\kappa(U,Y) = \sum_I p_I\, |Y^I|^\kappa, \quad \kappa \quad \text{is an integer and }, \kappa \geq 1,$$

$$\mathcal{F}_p(U,Y) = (Tr(V^\dagger V)^{p/2}))^{1/p}, V^I = Y^I(s) M_I, \quad p \quad \text{is an integer}$$

Among these only : $\mathcal{F}_{\kappa=1}$ and $\mathcal{F}_{p=2}$ are being preferred

Armed with this we can now compute the complexity for arbitrary number of oscillator and for arbitrary dimensions

Continous Limit: $N \to \infty, \delta \to 0, N\,\delta \to \text{finite}$

$$\mathcal{C}_{\kappa=1} = \frac{1}{2}\Big(\underbrace{\log\Big(\frac{\det(A^{(1)}(s=1))}{\det(A^{(1)}(s=0))}\Big)}_{\mathcal{C}^{(1)}_{\kappa=1}} + \mathcal{A}\underbrace{\log\Big(\frac{\det(A^{(2)}(s=1))}{\det(A^{(2)}(s=0))}\Big)}_{\mathcal{C}^{(2)}_{\kappa=1}}\Big)$$

Fairly universal

A bit ambiguous. We will constrain it as much as possible

Results

AB, A. Shekar, A. Sinha,
JHEP 1810 (2018) 140,
arXiv: 1808.03105[hep-th]

$$\mathcal{C}_\kappa^{(1)} = \frac{1}{2^\kappa}\sum_{k=1}^{d-1}\Big[\sum_{i_k=0}^{N-1}\Big|\log\frac{\tilde\omega_{i_k}}{\tilde\omega_{ref}}\Big|^\kappa + \frac{3\lambda\,\kappa\,\delta}{2\,N}\sum_{i_k=0}^{N-1}\frac{1}{\tilde\omega_{i_k}^3}\Big|\log(\frac{\tilde\omega_{i_k}}{\tilde\omega_{ref}})\Big|^{\kappa-1}\Big] + \mathcal{O}(\lambda^2)$$

After taking continuous limit:

d=2:

$$\mathcal{C}_{\kappa=1}^{(1)} = \frac{V}{2\,\delta}\log\left(\frac{m}{\tilde\omega_{ref}}\right) + \frac{V}{2\delta}\log\left(\frac{1}{2}\sqrt{\frac{4}{(m\,\delta)^2}+1}+\frac{1}{2}\right) + \frac{\hat\lambda}{8\,\pi\,m^2}\frac{E\left(\frac{4}{(m\,\delta)^2+4}\right)}{\sqrt{(m\,\delta)^2+4}},$$
$$= \frac{V}{2\delta}\log\left(\frac{1}{\tilde\omega_{ref}\delta}\right) + \frac{V}{\delta}\Big(a_1 + c_1\,(m\,\delta) + c_3(m\,\delta)^3 + \mathcal{O}((m\,\delta)^5)\Big)$$
$$+ \frac{\hat\lambda\,\delta^2}{16}\Big(\frac{f_1}{(m\delta)^2} + f_{1,log}\,\log(m\,\delta) + f_0 + \cdots\Big)$$

$$\hat\lambda = 24\,\delta^d\,\lambda$$

$$\frac{V}{\delta^{d-1}} = N^{d-1}$$

we have expanded in δ

$$a_1 = 0, c_1 = 1/4, c_3 = -\frac{1}{96}, f_1 = \frac{1}{\pi}, f_{1,log} = -\frac{1}{8\,\pi}, f_0 = 0.02.$$

d=3:

$$\mathcal{C}_{\kappa=1}^{(1)} = \frac{V}{2\,\delta^2}\log\left(\frac{1}{\tilde\omega_{ref}\,\delta}\right) + \frac{V}{\delta^2}\Big(a_2 + b_2\,(m\,\delta)^2\log(m\delta) + c_2(m\,\delta)^2 + \mathcal{O}((m\delta)^3)\Big) + f_1\,\frac{\hat\lambda\,V^{1/2}}{16\,(m\,\delta)} + \cdots$$

New logarithmic divergence

Fractional volume scaling

$$a_1 = 0, c_1 = 1/4, c_3 = -\frac{1}{96}, f_1 = \frac{1}{\pi}, f_{1,log} = -\frac{1}{8\,\pi}, f_0 = 0.02.$$

There is an extra factor of δ
in denominator compared to d=2
so for fixed $\hat\lambda$ complexity increases

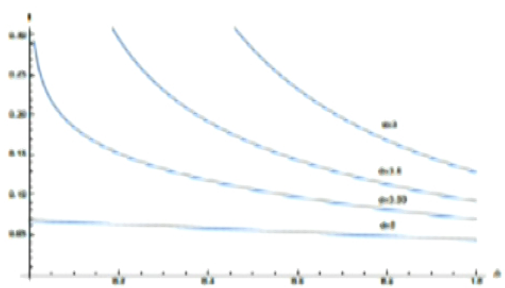
General d:

$$
\mathcal{C}^{(1)}_{\kappa=1} = \frac{V}{2\,\delta^{d-1}} \log\left(\frac{1}{\tilde{\omega}_{ref}\,\delta}\right) + \frac{V}{\delta^{d-1}}\left(a_{d-1} + \log(m\delta)\left[\sum_{k=2} b_k (m\delta)^k\right] + \sum_{k=1} c_k (m\delta)^k\right)
$$
$$
+ \frac{\hat{\lambda}}{16}\delta^{6-2d} V^{\frac{d-2}{d-1}} \left(f_1\{(m\delta)^{d-4}|_{d\neq 4} + \log(m\delta)|_{d=4}\} + f_0 + \cdots\right)
$$

$$0 < a_{d-1} < 1, c_1 = 0 \text{ for d} \geq 3$$

$\Gamma(\frac{3}{2})f_i$	d=3	d=3.99	d=4	d=5
$\Gamma(\frac{3}{2})f_0$	-0.001	-4.75	0.07	0.07
$\Gamma(\frac{3}{2})f_1$	0.14	4.83	-0.05	-0.03

f's flip sign for d< 4 as opposed to d>4.
Will be important when we make contact with RG flow



We plot the interaction part:

Divergent for small mass for d<4 while approaches to a constant value for d > 4.

Now lets rewrite everything in terms of Renormalized quantities

At 1-loop Renormalization order:

d=2:

$$(m\,\delta)^2 = (m_R\,\delta)^2 - \frac{\lambda_R\,\delta^2}{2}\left[C_0 - 2\,C_1\log(m_R\,\delta) - C_2(m_R\,\delta)^2 + \frac{1}{32\,\pi}(m_R\,\delta)^2\log((m_R\,\delta)^2) + \mathcal{O}((m_R\,\delta)^4)\right]$$

$$C_0 = 0.28, C_1 = 0.08, C_2 = 0.02.$$

$d \geq 3$

$$(m\,\delta)^2 = (m_R\,\delta)^2 - \frac{\lambda_R\,\delta^{4-d}}{2}\left[C_0 - C_2(m_R\,\delta)^2 + \frac{1}{16\pi^2}(m_R\,\delta)^2\log((m_R\,\delta)^2)|_{d=4} + \mathcal{O}((m_R\,\delta)^4)\right]$$

C_i	d=3	d=3.99	d=4	d=5
C_0	0.21	0.15	0.15	0.11
C_2	0.06	0.03	0.03	0.015

extra log term for d=4

also: $\hat{\lambda} = \lambda_R$

Using these we write our complexity expressions

d=2: $\mathcal{C}^{(1)}_{\kappa=1} \approx \frac{V}{2\delta}\left[\log\left(\frac{1}{\tilde{\omega}_{ref}\delta}\right) + 2a_1 - \lambda_R\delta^2\frac{C_0 - 2C_1\log(m_R\delta)}{2m_R\delta}c_1\right] + \cdots$

$d \geq 3$ $\mathcal{C}^{(1)}_{\kappa=1} \approx \frac{V}{2\delta^{d-1}}\left[\log\left(\frac{1}{\tilde{\omega}_{ref}\delta}\right) + 2a_{d-1} - \lambda_R\delta^{4-d}C_0(c_2 + b_2\log(m_R\delta)) + \frac{b_2}{2}\right]$

$+\frac{\lambda_R}{16}\delta^{6-2d}V^{\frac{d-2}{d-1}}(f_1\{(m_R\delta)^{d-4}|_{d\neq 4} + \log(m_R\delta)|_{d=4}\} + f_0) + \cdots$

vanishes for $d \geq 4$

fractional volume dependence

For large V this is the leading contribution at linear of order of λ_R Perturbation theory breaks down for $d > 4$

We can understand this break down of perturbation theory intitutively invoking RG picture, as for $d > 4$ Gaussian fixed points are stable compared to the Wilson-Fisher fixed point.

Lets define: $\Delta\mathcal{C}^{(1)}_{\kappa=1} = \mathcal{C}^{(1)}_{\kappa=1} - \mathcal{C}^{(1)}_{\kappa=1}|_{\lambda_R=0} \approx -\frac{V}{2\delta^{d-1}}\lambda_R\delta^{4-d}C_0c_2$ → sign will be important

Change in complexity from free to interacting theory with same mass parameter m_R *(for fixed points* $m_R = 0$ *)*

Structure of the "Ambiguous Block"

To compute $\mathcal{C}^{(2)}_{\kappa=1}$ we need to compute eigenvalues of $A^{(2)}$

There are ambiguities in it and we can only constrain the overall form

Also we need to take care of the penalty factors

$$\mathcal{C}^{(2)}_{\kappa=1} = \mathcal{A}\sum_j \left|\log\left(\frac{\lambda_j^{(2)}\delta^2}{\tilde{\omega}_{ref}\lambda_0}\right)\right|$$

$\lambda_j^{(2)}$ are the eigenvalues $A^{(2)}$ and there are N^2 of them

Leading order: $\lambda_j^{(2)} = \frac{b_j\,\lambda_R\,\delta^{-d}}{V^{\frac{1}{d-1}}\,\tilde{\omega}_j}$; $j \in \{0, 1, ..., (\text{Dim}\ \mathcal{A}_2) - 1\}$

we can choose (on dimensional ground) $\lambda_0 = a\,\lambda_R\,\tilde{\omega}^{d-3}_{ref}$, or $\lambda_0 = a\,\lambda_R\,m_R^{d-3}$

As explained before, we demand that, for perturbation theory to be valid

$$\mathcal{A} \propto \lambda_R$$

Also we demand the leading divergence that will come out will be at most same same as that of being generated from the $\mathcal{C}^{(1)}_{\kappa=1}$

This gives: $\mathcal{A} = (\lambda_R\delta^{4-d})\delta^d V^{\frac{d}{1-d}}$

But then: $\mathcal{C}^{(2)}_{\kappa=1} \propto \lambda_R\delta^{6-2d} V^{\frac{d-2}{d-1}} \longrightarrow$ fractional volume dependence

A Flow equation for Complexity

Now armed with all these we derive a flow equation for complexity:

Define: $\widetilde{\Delta \mathcal{C}} = (\mathcal{C}_{\kappa=1} - \mathcal{C}_{\kappa=1}|_{\lambda_R=0})\frac{\delta^{d-1}}{V}$ change of complexity per unit degree of freedom

Scale Transformations: $\lambda_R \to b^{d-4}\lambda'_R, \quad \delta \to b\,\delta,$
$b = 1 + db, \lambda_R = \lambda_R + d\lambda_R$

Now in large volume "V" keeping only leading order term in small δ

$$\frac{d\widetilde{\Delta \mathcal{C}}}{db} = 2(4-d)\widetilde{\Delta \mathcal{C}} + \mathcal{O}(\lambda_R^2)$$

Similar to the flow equation for coupling: $\frac{d\lambda_R}{db} = (4-d)\lambda_R + O(\lambda_R^2)$

For $d < 4$: *Wilson Fisher fixed point is favored in term of complexity*

$d > 4$ *Gaussian fixed point is favored in term of complexity*

This matches nicely with the intuitive idea of RG flow!!!

Brief Comparison with Holography

For Free Theory: $\mathcal{C}_{\kappa=1} \propto V$

Provides a quantitative matching with the Holographic result

But note that: $\mathcal{O}(\lambda_R)$ has a fractional volume dependence

$\lambda_R\, \delta^{6-2d} V^{\frac{d-2}{d-1}}$ (except for d=2)

Complexity= Volume *cannot produce such volume dependence*

Complexity= Action *gives a logarithmic enhancement but still do not give fractional volume dependence*

Many issues:

locality conditions ?

Different set of gates: They have recently appeared in some version of interacting cMERA (Cotler, M. R. Mohammadi Mozaffar, A. Mollabashi and A. Naseh '06)

(Cotler-Myers)

$$\tilde{O}_1 = \tilde{x}_0 . \tilde{p}_0 - \frac{i}{2}, \;\; \tilde{O}_2 = \tilde{x}_1 . \tilde{p}_1 - \frac{i}{2},$$

$$\tilde{O}_3 = \hat{\lambda}\left(\tilde{x}_0^3\,\tilde{p}_0 - \frac{3\,i\,\tilde{x}_0^2}{2}\right), \;\; \tilde{O}_4 = \hat{\lambda}\left(\tilde{x}_1^3\,\tilde{p}_1 - \frac{3\,i\,\tilde{x}_1^2}{2}\right),$$

$$\tilde{O}_5 = \hat{\lambda}\,\tilde{x}_1^2\left(\tilde{x}_0\,\tilde{p}_0 - \frac{i}{2}\right), \quad \tilde{O}_6 = \hat{\lambda}\,\tilde{x}_0^2\left(\tilde{x}_1\,\tilde{p}_1 - \frac{i}{2}\right)$$

$$[\tilde{O}_1, \tilde{O}_2] = 0, [\tilde{O}_3, \tilde{O}_4] = 0, [\tilde{O}_1, \tilde{O}_3] = -2\,i\,\tilde{O}_3, [\tilde{O}_1, \tilde{O}_4] = 0,$$
$$[\tilde{O}_2, \tilde{O}_3] = 0, [\tilde{O}_2, \tilde{O}_4] = -2\,i\,\tilde{O}_4, [\tilde{O}_1, \tilde{O}_5] = 0, [\tilde{O}_2, \tilde{O}_5] = -2\,i\,\tilde{O}_5,$$
$$[\tilde{O}_3, \tilde{O}_5] = \mathcal{O}(\hat{\lambda}^2), [\tilde{O}_4, \tilde{O}_5] = \mathcal{O}(\hat{\lambda}^2).$$

$$\mathcal{D}(U) = \frac{1}{2\,\epsilon}\Big[\log\Big|\frac{\bar{\omega}_0}{\bar{\omega}_{ref}}\Big| + \log\Big|\frac{\bar{\omega}_1}{\bar{\omega}_{ref}}\Big| + \frac{\hat{\lambda}}{32\,\delta}\Big(\frac{1}{\bar{\omega}_0^3} + \frac{1}{\bar{\omega}_1^3} + \frac{2}{\bar{\omega}_0^2(\bar{\omega}_0 + \bar{\omega}_1)} + \frac{2}{\bar{\omega}_1^2(\bar{\omega}_0 + \bar{\omega}_1)}\Big)$$
$$+ \frac{x_0}{p_0}\Big(\frac{(\bar{\omega}_0^2 + \bar{\omega}_1^2)}{96\,\delta^2\bar{\omega}_0^2\bar{\omega}_1^2} + \frac{1}{8\,\delta^2\,(\bar{\omega}_1 + \bar{\omega}_0)\bar{\omega}_0}\Big) + \Big(\frac{\hat{\lambda}(\bar{\omega}_0^3 + \bar{\omega}_1^3)}{64\,\delta\,\bar{\omega}_0^3\,\bar{\omega}_1^3} + \frac{\hat{\lambda}}{16\,\delta\,(\bar{\omega}_1 + \bar{\omega}_0)\bar{\omega}_0\bar{\omega}_1}\Big)\Big].$$

Can we give it a geometric interpretation ?