Title: Quantum Statistical Comparison, Quantum Majorization, and Their Applications to Generalized Resource Theories

Speakers: Francesco Buscemi

Series: Quantum Foundations

Date: February 26, 2019 - 3:30 PM

URL: http://pirsa.org/19020084

Abstract: The theory of statistical comparison was formulated (chiefly by David Blackwell in the 1950s) in order to extend the theory of majorization to objects beyond probability distributions, like multivariate statistical models and stochastic transitions, and has played an important role in mathematical statistics ever since. The central concept in statistical comparison is the so-called "information ordering," according to which information need not always be a totally ordered quantity, but often takes on a multi-faceted form whose content may vary depending on its use. In this talk, after reviewing the basic ideas of statistical comparison with an emphasis on their operational character, I will discuss various generalizations to quantum theory (and beyond). I will then argue that quantum statistical comparison provides a natural framework, somehow complementary to semi-definite programming, to study quantum resource theories, with explicit examples given by the resource theories of quantum nonlocality, quantum communication, and quantum thermodynamics.

Quantum Statistical Comparison and Majorization

(and their applications to generalized resource theories)

Francesco Buscemi (Nagoya University)

Quantum Foundations Seminar Series, Perimeter Institute
26 February 2019

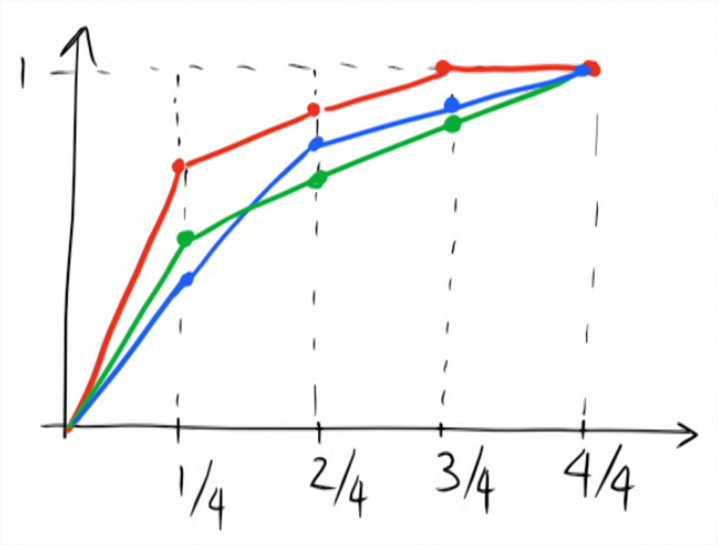
Guiding idea:
generalized resource theories as order theories for stochastic (probabilistic) structures

The Precursor: Majorization

Lorenz Curves and Majorization

- two probability distributions, $\boldsymbol{p} = (p_1, \dots, p_n)$ and $\boldsymbol{q} = (q_1, \dots, q_n)$
- truncated sums $P(k) = \sum_{i=1}^{k} p_i^{\downarrow}$ and $Q(k) = \sum_{i=1}^{k} q_i^{\downarrow}$, for all $k = 1, \dots, n$
- $\boldsymbol{p}$ majorizes $\boldsymbol{q}$, i.e., $\boldsymbol{p} \succeq \boldsymbol{q}$, whenever $P(k) \geq Q(k)$, for all k
- minimal element: uniform distribution $\boldsymbol{e} = n^{-1}(1, 1, \cdots, 1)$

Lorenz curve for probability distribution $\boldsymbol{p} = (p_1, \cdots, p_n)$:



$$(x_k, y_k) = (k/n, P(k)), \quad 1 \leq k \leq n$$

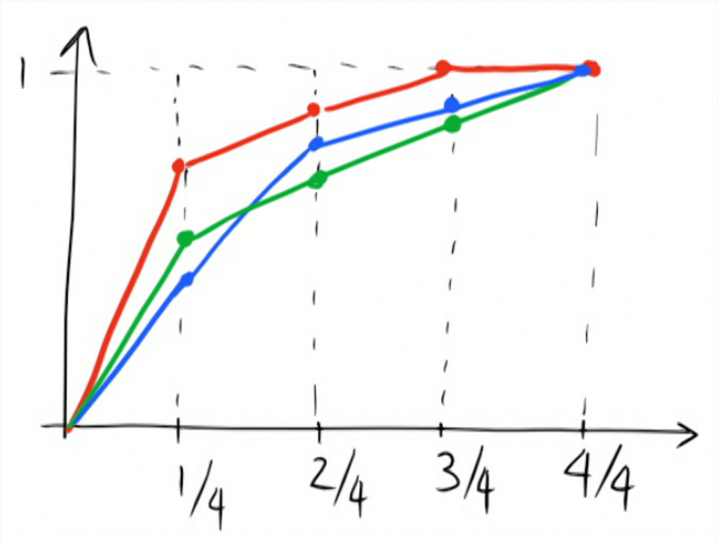
Lorenz Curves and Majorization

- two probability distributions, $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$
- truncated sums $P(k) = \sum_{i=1}^{k} p_i^{\downarrow}$ and $Q(k) = \sum_{i=1}^{k} q_i^{\downarrow}$, for all $k = 1, \ldots, n$
- $\boldsymbol{p}$ majorizes $\boldsymbol{q}$, i.e., $\boldsymbol{p} \succeq \boldsymbol{q}$, whenever $P(k) \geq Q(k)$, for all k
- minimal element: uniform distribution $\boldsymbol{e} = n^{-1}(1, 1, \cdots, 1)$

Hardy, Littlewood, and Pólya (1929)

$\boldsymbol{p} \succeq \boldsymbol{q} \iff \boldsymbol{q} = M\boldsymbol{p}$, for some bistochastic matrix M.

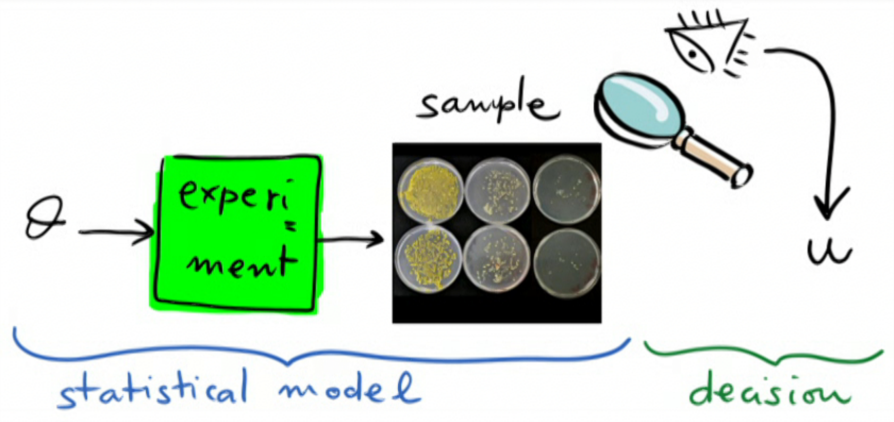
Lorenz curve for probability distribution $\boldsymbol{p} = (p_1, \cdots, p_n)$:



$$(x_k, y_k) = (k/n, P(k)), \quad 1 \leq k \leq n$$

Blackwell's Extensions

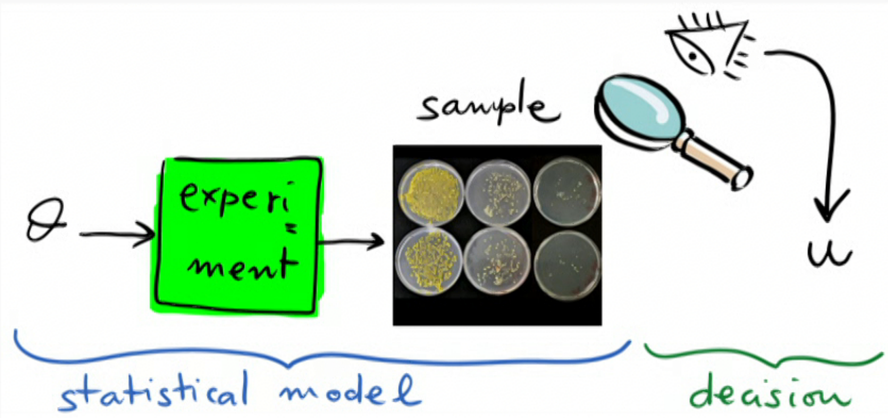
Statistical Decision Problems



$$\begin{array}{ccccc} \Theta & \xrightarrow{\text{experiment}} & \mathcal{X} & \xrightarrow{\text{decision}} & \mathcal{U} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u \end{array}$$

payoff is $\ell(\theta, u) \in \mathbb{R}$

Statistical Decision Problems



$$\begin{array}{ccccc} \Theta & \xrightarrow{\text{experiment}} & \mathcal{X} & \xrightarrow{\text{decision}} & \mathcal{U} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u \end{array}$$

payoff is $\ell(\theta, u) \in \mathbb{R}$

Definition

A **statistical model** (or *experiment*) is a triple $\mathbf{w} = \langle \Theta, \mathcal{X}, w \rangle$, a **statistical decision problem** (or *game*) is a triple $\mathbf{g} = \langle \Theta, \mathcal{U}, \ell \rangle$.

Playing Games with Experiments

- the experiment (model) *is given*, i.e., it is the "resource"
- the decision instead *can be optimized*

$$
\begin{array}{ccccc}
\Theta & \xrightarrow{\text{experiment}} & \mathcal{X} & \xrightarrow{\text{decision}} & \mathcal{U} \\
\wr & & \wr & & \wr \\
\theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u
\end{array}
$$

Playing Games with Experiments

- the experiment (model) *is given*, i.e., it is the "resource"
- the decision instead *can be optimized*

$$
\begin{array}{ccccc}
\Theta & \xrightarrow{\text{experiment}} & \mathcal{X} & \xrightarrow{\text{decision}} & \mathcal{U} \\
\rotatebox{-90}{$\leadsto$} & & \rotatebox{-90}{$\leadsto$} & & \rotatebox{-90}{$\leadsto$} \\
\theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u
\end{array}
$$

Definition

The **expected payoff** of a statistical model $\mathbf{w} = \langle \Theta, \mathcal{X}, w \rangle$ w.r.t. a decision problem $\mathbf{g} = \langle \Theta, \mathcal{U}, \ell \rangle$ is given by

$$\mathbb{E}_{\mathbf{g}}[\mathbf{w}] \stackrel{\text{def}}{=} \max_{d(u|x)} \sum_{u,x,\theta} \ell(\theta, u) d(u|x) w(x|\theta) |\Theta|^{-1} .$$

Comparing Statistical Models 2/2

Definition (Information Preorder)

If the model $\mathbf{w} = \langle \Theta, \mathcal{X}, w \rangle$ is better than model $\mathbf{w}' = \langle \Theta, \mathcal{Y}, w' \rangle$ **for all** decision problems $\mathbf{g} = \langle \Theta, \mathcal{U}, \ell \rangle$, then we say that $\mathbf{w}$ is *more informative* than $\mathbf{w}'$, and write

$$\mathbf{w} \succeq \mathbf{w}' \ .$$

Comparing Statistical Models 1/2

First model: $\mathbf{w} = \langle \Theta, \mathcal{X}, w(x|\theta) \rangle$

$$\begin{array}{ccccc} \Theta & \xrightarrow{\text{experiment}} & \mathcal{X} & \xrightarrow{\text{decision}} & \mathcal{U} \\ \downarrow & & \downarrow & & \downarrow \\ \theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u \end{array}$$

Second model: $\mathbf{w}' = \langle \Theta, \mathcal{Y}, w'(y|\theta) \rangle$

$$\begin{array}{ccccc} \Theta & \xrightarrow{\text{experiment}} & \mathcal{Y} & \xrightarrow{\text{decision}} & \mathcal{U} \\ \downarrow & & \downarrow & & \downarrow \\ \theta & \xrightarrow[w'(y|\theta)]{} & x & \xrightarrow[d'(u|y)]{} & u \end{array}$$

For a fixed decision problem $\mathbf{g} = \langle \Theta, \mathcal{U}, \ell \rangle$, the expected payoffs $\mathbb{E}_{\mathbf{g}}[\mathbf{w}]$ and $\mathbb{E}_{\mathbf{g}}[\mathbf{w}']$ can always be ordered.

Comparing Statistical Models 2/2

Definition (Information Preorder)

If the model $\mathbf{w} = \langle \Theta, \mathcal{X}, w \rangle$ is better than model $\mathbf{w}' = \langle \Theta, \mathcal{Y}, w' \rangle$ **for all** decision problems $\mathbf{g} = \langle \Theta, \mathcal{U}, \ell \rangle$, then we say that $\mathbf{w}$ is *more informative* than $\mathbf{w}'$, and write

$$\mathbf{w} \succeq \mathbf{w}' \ .$$

Comparing Statistical Models 2/2

Definition (Information Preorder)

If the model $\mathbf{w} = \langle \Theta, \mathcal{X}, w \rangle$ is better than model $\mathbf{w}' = \langle \Theta, \mathcal{Y}, w' \rangle$ **for all** decision problems $\mathbf{g} = \langle \Theta, \mathcal{U}, \ell \rangle$, then we say that $\mathbf{w}$ is *more informative* than $\mathbf{w}'$, and write

$$\mathbf{w} \succeq \mathbf{w}' \ .$$

Problem. Can we visualize the information morphism $\succeq$ more concretely?

Information Morphism = Statistical Sufficiency

Blackwell-Sherman-Stein Theorem (1948-1953)

Given two experiments with the same parameter space, $\mathbf{w} = \langle \Theta, \mathcal{X}, w \rangle$ and $\mathbf{w}' = \langle \Theta, \mathcal{Y}, w' \rangle$, the condition $\mathbf{w} \succeq \mathbf{w}'$ holds *iff* there exists a conditional probability $\varphi(y|x)$ such that $w'(y|\theta) = \sum_x \varphi(y|x) w(x|\theta)$.

David H. Blackwell (1919-2010)

Information Morphism = Statistical Sufficiency

Blackwell-Sherman-Stein Theorem (1948-1953)

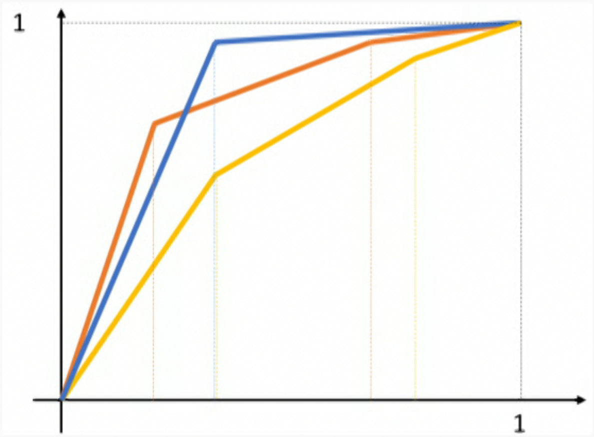
Given two experiments with the same parameter space, $\mathbf{w} = \langle \Theta, \mathcal{X}, w \rangle$ and $\mathbf{w}' = \langle \Theta, \mathcal{Y}, w' \rangle$, the condition $\mathbf{w} \succeq \mathbf{w}'$ holds *iff* there exists a conditional probability $\varphi(y|x)$ such that $w'(y|\theta) = \sum_x \varphi(y|x) w(x|\theta)$.

$$
\begin{array}{ccccccccc}
\Theta & \longrightarrow & \mathcal{Y} & & \Theta & \longrightarrow & \mathcal{X} & \overset{\text{noise}}{\longrightarrow} & \mathcal{Y} \\
\rightsquigarrow & & \rightsquigarrow & = & \rightsquigarrow & & \rightsquigarrow & & \rightsquigarrow \\
\theta & \underset{w'(y|\theta)}{\longrightarrow} & y & & \theta & \underset{w(x|\theta)}{\longrightarrow} & x & \underset{\varphi(y|x)}{\longrightarrow} & y
\end{array}
$$

David H. Blackwell (1919-2010)

Special Case: Dichotomies

- two *pairs* of probability distributions, i.e., two *dichotomies*, $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ and $(\boldsymbol{q}_1, \boldsymbol{q}_2)$, of dimension m and n, respectively
- relabel entries such that ratios p_1^i / p_2^i and q_1^j / q_2^j are nonincreasing
- construct the truncated sums $P_{1,2}(k) = \sum_{i=1}^{k} p_{1,2}^i$ and $Q_{1,2}(k)$
- $(\boldsymbol{p}_1, \boldsymbol{p}_2) \succeq (\boldsymbol{q}_1, \boldsymbol{q}_2)$ iff the relative Lorenz curve of the former is never below that of the latter

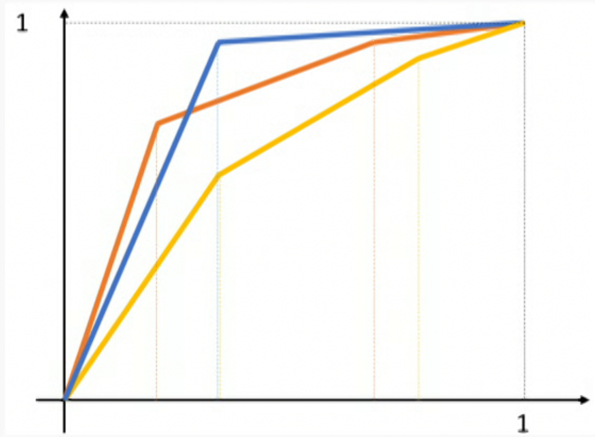


Relative Lorenz curves:

$(x_k, y_k) = (P_2(k), P_1(k))$

Special Case: Dichotomies

- two *pairs* of probability distributions, i.e., two *dichotomies*, $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ and $(\boldsymbol{q}_1, \boldsymbol{q}_2)$, of dimension m and n, respectively
- relabel entries such that ratios p_1^i/p_2^i and q_1^j/q_2^j are nonincreasing
- construct the truncated sums $P_{1,2}(k) = \sum_{i=1}^{k} p_{1,2}^i$ and $Q_{1,2}(k)$
- $(\boldsymbol{p}_1, \boldsymbol{p}_2) \succeq (\boldsymbol{q}_1, \boldsymbol{q}_2)$ iff the relative Lorenz curve of the former is never below that of the latter



Relative Lorenz curves:

$(x_k, y_k) = (P_2(k), P_1(k))$

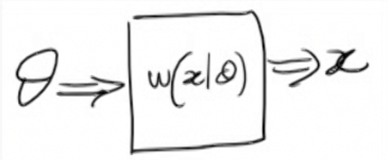
Blackwell's Theorem for Dichotomies (1953)

$(\boldsymbol{p}_1, \boldsymbol{p}_2) \succeq (\boldsymbol{q}_1, \boldsymbol{q}_2) \iff \boldsymbol{q}_i = M\boldsymbol{p}_i$, for some stochastic matrix M.

The Viewpoint of Communication Theory

Statistics vs Information Theory

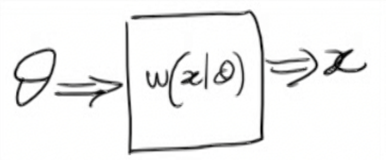
- Statistical models are mathematically equivalent to noisy channels:



8/31

Statistics vs Information Theory

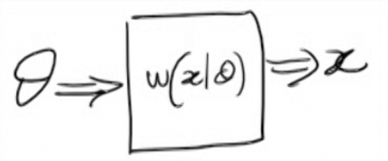
- Statistical models are mathematically equivalent to noisy channels:



- However: while in statistics the input is inaccessible (Nature does not bother with coding!)

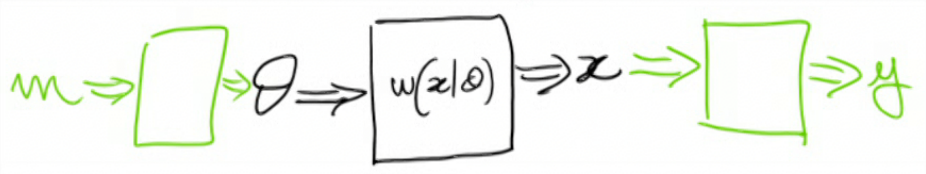
Statistics vs Information Theory

- Statistical models are mathematically equivalent to noisy channels:

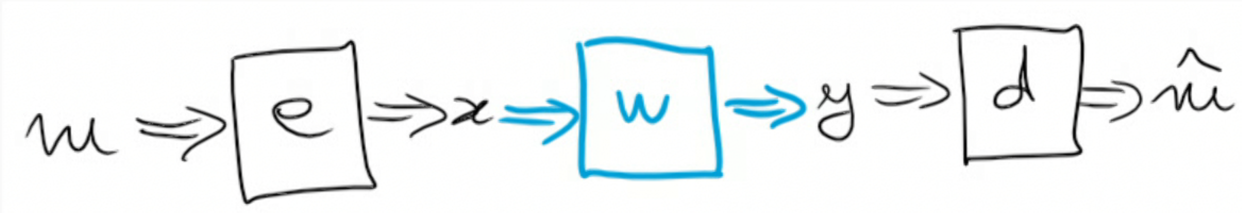


- However: while in statistics the input is inaccessible (Nature does not bother with coding!)

- in communication theory a sender *does code*!



From Decision Problems to Decoding Problems



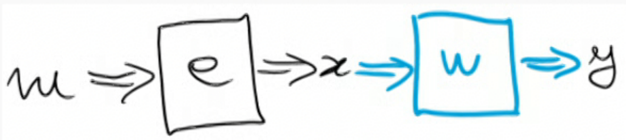
Definition (Decoding Problems)

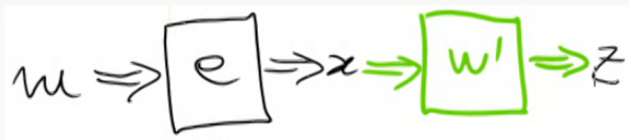
Given a channel $\mathbf{w} = \langle \mathcal{X}, \mathcal{Y}, w(y|x) \rangle$, a decoding problem is defined by an encoding $\mathbf{e} = \langle \mathcal{M}, \mathcal{X}, e(x|m) \rangle$ and the payoff function is the optimum guessing probability:

$$\mathbb{E}_{\mathbf{e}}[\mathbf{w}] \stackrel{\text{def}}{=} \max_{d(m|y)} \sum_{m,x,y} d(m|y) w(y|x) e(x|m) |\mathcal{M}|^{-1} = 2^{-H_{\min}(M|Y)}$$

Comparison of Classical Noisy Channels

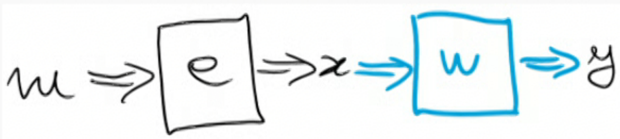
Consider two discrete noisy channels **w** and **w**$'$ with the same input alphabet

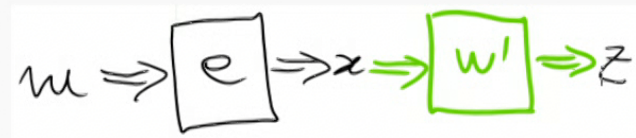




Comparison of Classical Noisy Channels

Consider two discrete noisy channels **w** and **w**$'$ with the same input alphabet





Theorem

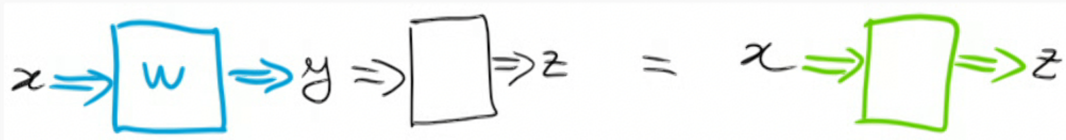
Given the following pre-orders

1. degradability: there exists $\varphi(z|y)$: $w'(x|z) = \sum_y \varphi(z|y) w(y|x)$
2. noisiness: for all encodings $\mathbf{e} = \langle \mathcal{M}, \mathcal{X}, e(x|m) \rangle$, $H(M|Y) \le H(M|Z)$
3. ambiguity: for all encodings $\mathbf{e} = \langle \mathcal{M}, \mathcal{X}, e(x|m) \rangle$, $H_{\min}(M|Y) \le H_{\min}(M|Z)$

we have: (1) $\Longrightarrow$ (2) (data-processing inequality), (2) $\not\Longrightarrow$ (1) (Körner and Marton, 1977), but (1) $\Longleftrightarrow$ (3) (FB, 2016).

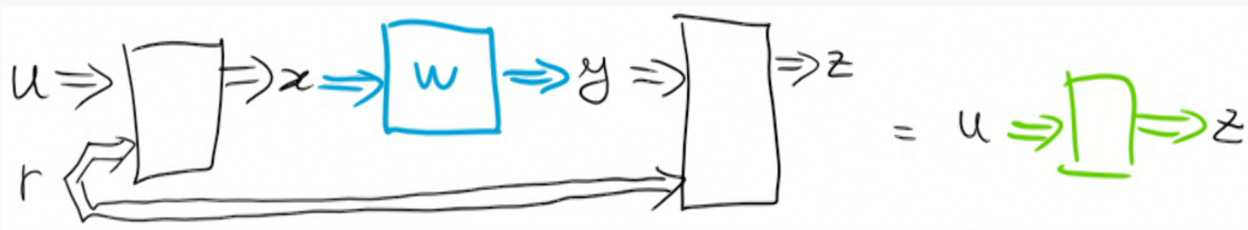
Some Classical Channel Morphisms

Output degrading:



Input degrading:

Full coding (Shannon's "channel inclusion", 1958):



Extensions to the Quantum Case

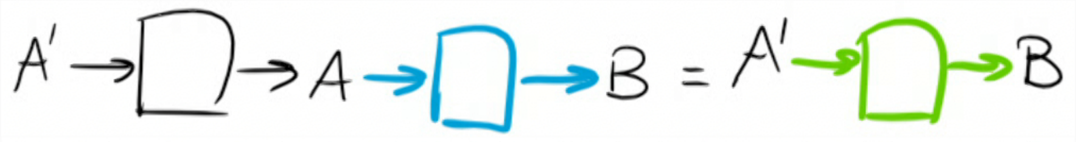
Some Quantum Channel Morphisms

Output degrading:

Some Quantum Channel Morphisms

Output degrading:

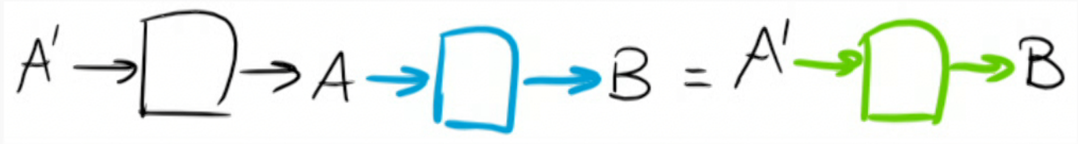
Input degrading:



Some Quantum Channel Morphisms

Output degrading:

Input degrading:



Quantum coding with forward CC:

Output Degradability

Comparison of Quantum Statistical Models 1/2

Quantum statistical models as cq-channels:

Comparison of Quantum Statistical Models 1/2

Quantum statistical models as cq-channels:

Formulation below from: A.S. Holevo, *Statistical Decision Theory for Quantum Systems*, 1973.

classical case	**quantum case**
• decision problems $\mathbf{g} = \langle \Theta, \mathcal{U}, \ell \rangle$	• decision problems $\mathbf{g} = \langle \Theta, \mathcal{U}, \ell \rangle$
• experiments $\mathbf{w} = \langle \Theta, \mathcal{X}, \{w(x\|\theta)\} \rangle$	• quantum experiments $\mathcal{E} = \big\langle \Theta, \mathcal{H}_S, \{\rho_S^\theta\} \big\rangle$
• decisions $d(u\|x)$	• POVMs $\{P_S^u : u \in \mathcal{U}\}$
• $p_c(u,\theta) = \sum_x d(u\|x) w(x\|\theta) \|\Theta\|^{-1}$	• $p_q(u,\theta) = \mathrm{Tr}\big[\rho_S^\theta \; P_S^u\big] \, \|\Theta\|^{-1}$
• $\mathbb{E}_{\mathbf{g}}[\mathbf{w}] = \max_{d(u\|x)} \sum \ell(\theta,u) p_c(u,\theta)$	• $\mathbb{E}_{\mathbf{g}}[\mathcal{E}] = \max_{\{P_S^u\}} \sum \ell(\theta,u) p_q(u,\theta)$

Comparison of Quantum Statistical Models 2/2

What follows is from: FB, Comm. Math. Phys., 2012

- consider two quantum statistical models $\mathcal{E} = \left\langle \Theta, \mathcal{H}_S, \{\rho_S^\theta\} \right\rangle$ and $\mathcal{E}' = \left\langle \Theta, \mathcal{H}_{S'}, \{\sigma_{S'}^\theta\} \right\rangle$

Comparison of Quantum Statistical Models 2/2

What follows is from: FB, Comm. Math. Phys., 2012

- consider two quantum statistical models $\mathcal{E} = \langle \Theta, \mathcal{H}_S, \{\rho_S^\theta\} \rangle$ and $\mathcal{E}' = \langle \Theta, \mathcal{H}_{S'}, \{\sigma_{S'}^\theta\} \rangle$
- **information ordering**: $\mathcal{E} \succeq \mathcal{E}'$ iff $\mathbb{E}_{\mathbf{g}}[\mathcal{E}] \geq \mathbb{E}_{\mathbf{g}}[\mathcal{E}']$ for all $\mathbf{g}$
- $\mathcal{E} \succeq \mathcal{E}'$ iff there exists a **quantum statistical morphism** (essentially, a PTP map) $\mathcal{M} : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that $\mathcal{M}(\rho_S^\theta) = \sigma_{S'}^\theta$ for all θ

Comparison of Quantum Statistical Models 2/2

What follows is from: FB, Comm. Math. Phys., 2012

- consider two quantum statistical models $\mathcal{E} = \left\langle \Theta, \mathcal{H}_S, \{\rho_S^\theta\} \right\rangle$ and $\mathcal{E}' = \left\langle \Theta, \mathcal{H}_{S'}, \{\sigma_{S'}^\theta\} \right\rangle$
- **information ordering**: $\mathcal{E} \succeq \mathcal{E}'$ iff $\mathbb{E}_{\mathbf{g}}[\mathcal{E}] \geq \mathbb{E}_{\mathbf{g}}[\mathcal{E}']$ for all $\mathbf{g}$
- $\mathcal{E} \succeq \mathcal{E}'$ iff there exists a **quantum statistical morphism** (essentially, a PTP map) $\mathcal{M} : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that $\mathcal{M}(\rho_S^\theta) = \sigma_{S'}^\theta$ for all θ
- **complete information ordering**: $\mathcal{E} \succeq_c \mathcal{E}'$ iff $\mathcal{E} \otimes \mathcal{F} \succeq \mathcal{E}' \otimes \mathcal{F}$ for all ancillary models $\mathcal{F}$ (in fact, one informationally complete model suffices)

Comparison of Quantum Statistical Models 2/2

What follows is from: FB, Comm. Math. Phys., 2012

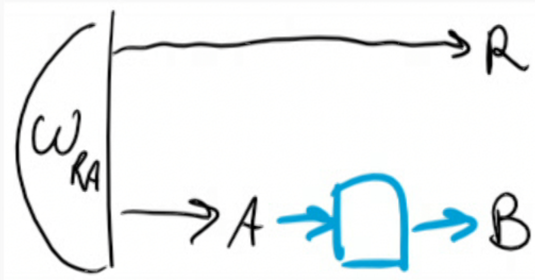
- consider two quantum statistical models $\mathcal{E} = \left\langle \Theta, \mathcal{H}_S, \{\rho_S^\theta\} \right\rangle$ and $\mathcal{E}' = \left\langle \Theta, \mathcal{H}_{S'}, \{\sigma_{S'}^\theta\} \right\rangle$
- **information ordering**: $\mathcal{E} \succeq \mathcal{E}'$ iff $\mathbb{E}_{\mathbf{g}}[\mathcal{E}] \geq \mathbb{E}_{\mathbf{g}}[\mathcal{E}']$ for all $\mathbf{g}$
- $\mathcal{E} \succeq \mathcal{E}'$ iff there exists a **quantum statistical morphism** (essentially, a PTP map) $\mathcal{M} : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that $\mathcal{M}(\rho_S^\theta) = \sigma_{S'}^\theta$ for all θ
- **complete information ordering**: $\mathcal{E} \succeq_c \mathcal{E}'$ iff $\mathcal{E} \otimes \mathcal{F} \succeq \mathcal{E}' \otimes \mathcal{F}$ for all ancillary models $\mathcal{F}$ (in fact, one informationally complete model suffices)
- $\mathcal{E} \succeq_c \mathcal{E}'$ iff there exists a **CPTP map** $\mathcal{N} : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that $\mathcal{N}(\rho_S^\theta) = \sigma_{S'}^\theta$ for all θ

Comparison of Quantum Statistical Models 2/2

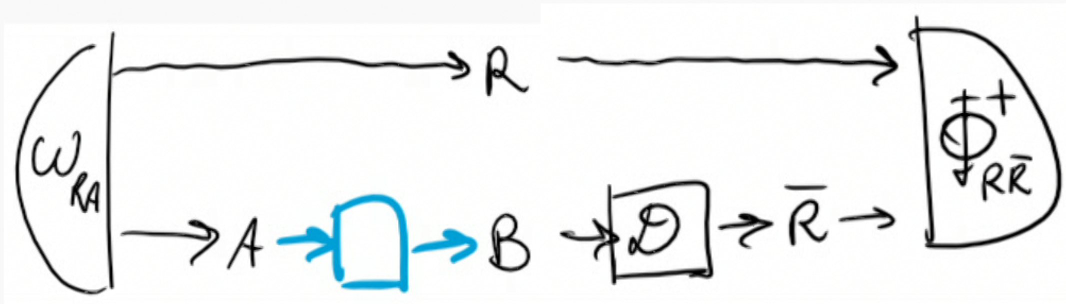
What follows is from: FB, Comm. Math. Phys., 2012

- consider two quantum statistical models $\mathcal{E} = \left\langle \Theta, \mathcal{H}_S, \{\rho_S^\theta\} \right\rangle$ and $\mathcal{E}' = \left\langle \Theta, \mathcal{H}_{S'}, \{\sigma_{S'}^\theta\} \right\rangle$
- **information ordering**: $\mathcal{E} \succeq \mathcal{E}'$ iff $\mathbb{E}_{\mathbf{g}}[\mathcal{E}] \geq \mathbb{E}_{\mathbf{g}}[\mathcal{E}']$ for all $\mathbf{g}$
- $\mathcal{E} \succeq \mathcal{E}'$ iff there exists a **quantum statistical morphism** (essentially, a PTP map) $\mathcal{M} : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that $\mathcal{M}(\rho_S^\theta) = \sigma_{S'}^\theta$ for all θ
- **complete information ordering**: $\mathcal{E} \succeq_c \mathcal{E}'$ iff $\mathcal{E} \otimes \mathcal{F} \succeq \mathcal{E}' \otimes \mathcal{F}$ for all ancillary models $\mathcal{F}$ (in fact, one informationally complete model suffices)
- $\mathcal{E} \succeq_c \mathcal{E}'$ iff there exists a **CPTP map** $\mathcal{N} : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that $\mathcal{N}(\rho_S^\theta) = \sigma_{S'}^\theta$ for all θ
- if $\mathcal{E}'$ is **abelian**, then $\mathcal{E} \succeq_c \mathcal{E}'$ iff $\mathcal{E} \succeq \mathcal{E}'$

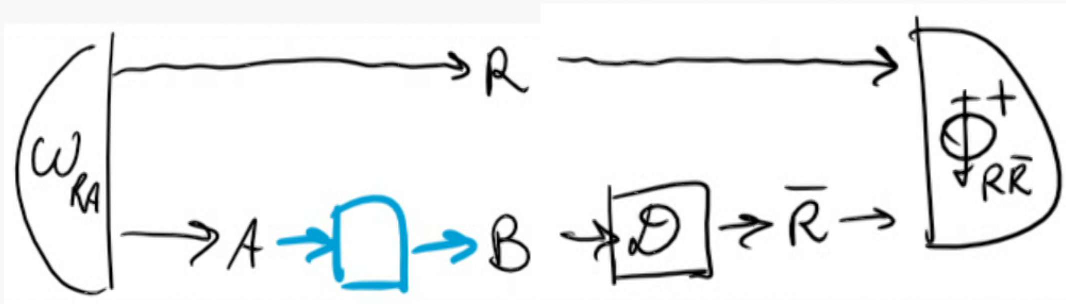
Comparison of Quantum Channels 1/2



Comparison of Quantum Channels 1/2



Comparison of Quantum Channels 1/2



Definition (Quantum Decoding Problems)

Given a quantum channel $\mathcal{N} : A \to B$, a quantum decoding problem is defined by a bipartite state ω_{RA} and the payoff function is the optimum singlet fraction:

$$\mathbb{E}_\omega[\mathcal{N}] \stackrel{\text{def}}{=} \max_{\mathcal{D}} \langle \Phi^+_{R\bar{R}} | (\mathrm{id}_R \otimes \mathcal{D}_{B\to\bar{R}} \circ \mathcal{N}_{A\to B})(\omega_{RA}) | \Phi^+_{R\bar{R}} \rangle$$

Comparison of Quantum Channels 2/2

Theorem (FB, 2016)

Given two quantum channels $\mathcal{N} : A \to B$ and $\mathcal{N}' : A \to B'$, the following are equivalent:

1. **output degradability**: *there exists CPTP map $\mathcal{C}$: $\mathcal{N}' = \mathcal{C} \circ \mathcal{N}$;*
2. **coherence preorder**: *for any bipartite state ω_{RA}, $\mathbb{E}_\omega[\mathcal{N}] \geq \mathbb{E}_\omega[\mathcal{N}']$, that is,* $H_{\min}(R|B)_{(\mathsf{id}\otimes\mathcal{N})(\omega)} \leq H_{\min}(R|B')_{(\mathsf{id}\otimes\mathcal{N}')(\omega)}$.

Comparison of Quantum Channels 2/2

Theorem (FB, 2016)

Given two quantum channels $\mathcal{N}: A \to B$ and $\mathcal{N}': A \to B'$, the following are equivalent:

1. **output degradability**: *there exists CPTP map $\mathcal{C}$: $\mathcal{N}' = \mathcal{C} \circ \mathcal{N}$;*
2. **coherence preorder**: *for any bipartite state ω_{RA}, $\mathbb{E}_\omega[\mathcal{N}] \geq \mathbb{E}_\omega[\mathcal{N}']$, that is,* $H_{\min}(R|B)_{(\mathrm{id}\otimes\mathcal{N})(\omega)} \leq H_{\min}(R|B')_{(\mathrm{id}\otimes\mathcal{N}')(\omega)}$.

$\rightsquigarrow$ applications to the theory of open quantum systems dynamics and, by adding symmetry constraints, to quantum thermodynamics

Discrete-Time Stochastic Processes

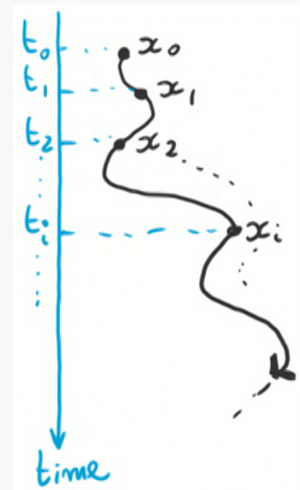
- Let x_i, for $i = 0, 1, \ldots$, index the state of a system at time $t = t_i$

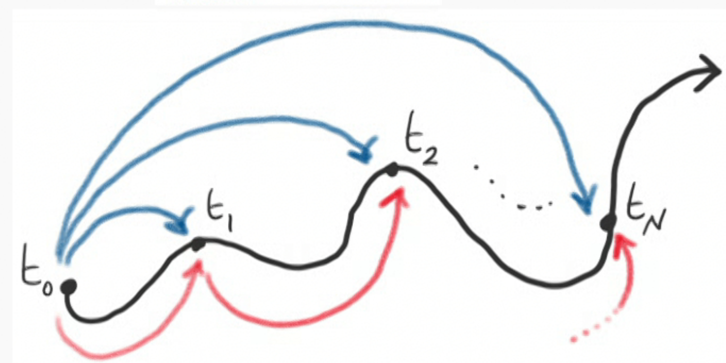
Discrete-Time Stochastic Processes

- Let x_i, for $i = 0, 1, \ldots$, index the state of a system at time $t = t_i$
- if the system can be initialized at time $t = t_0$, the process is fully described by the conditional distribution $p(x_N, \ldots, x_1 | x_0)$

Discrete-Time Stochastic Processes

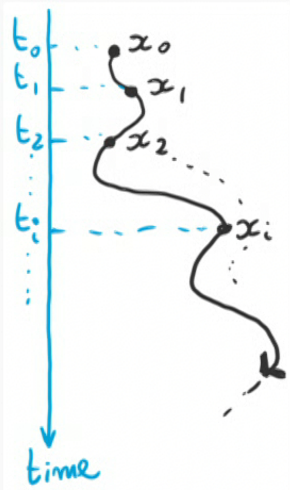
- Let x_i, for $i = 0, 1, \dots$, index the state of a system at time $t = t_i$
- if the system can be initialized at time $t = t_0$, the process is fully described by the conditional distribution $p(x_N, \dots, x_1 | x_0)$
- if the system evolving is quantum, we only have a quantum dynamical mapping $\left\{ \mathcal{N}^{(i)}_{Q_0 \to Q_i} \right\}_{i=1,\dots,N}$

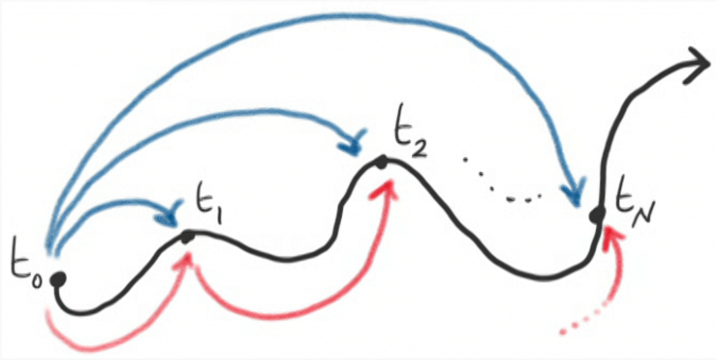




Discrete-Time Stochastic Processes

- Let x_i, for $i = 0, 1, \dots$, index the state of a system at time $t = t_i$
- if the system can be initialized at time $t = t_0$, the process is fully described by the conditional distribution $p(x_N, \dots, x_1 | x_0)$
- if the system evolving is quantum, we only have a quantum dynamical mapping $\left\{\mathcal{N}^{(i)}_{Q_0 \to Q_i}\right\}_{i=1,\dots,N}$
- the process is divisible if there exist channels $\mathcal{D}^{(i)}$ such that $\mathcal{N}^{(i+1)} = \mathcal{D}^{(i)} \circ \mathcal{N}^{(i)}$ for all i

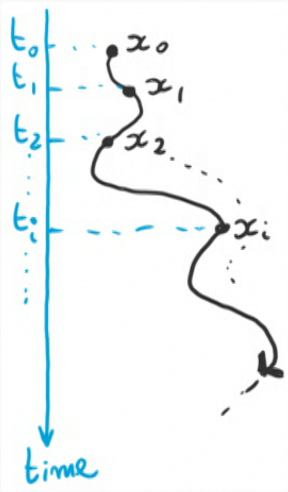


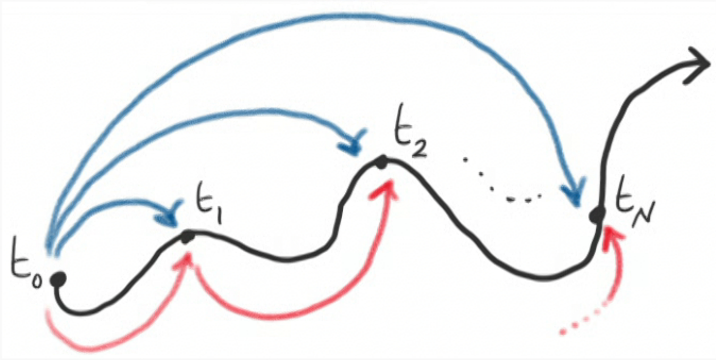


17/31

Discrete-Time Stochastic Processes

- Let x_i, for $i = 0, 1, \ldots$, index the state of a system at time $t = t_i$
- if the system can be initialized at time $t = t_0$, the process is fully described by the conditional distribution $p(x_N, \ldots, x_1 | x_0)$
- if the system evolving is quantum, we only have a quantum dynamical mapping $\left\{\mathcal{N}^{(i)}_{Q_0 \to Q_i}\right\}_{i=1,\ldots,N}$
- the process is divisible if there exist channels $\mathcal{D}^{(i)}$ such that $\mathcal{N}^{(i+1)} = \mathcal{D}^{(i)} \circ \mathcal{N}^{(i)}$ for all i
- problem: to provide a fully information-theoretic characterization of divisibility



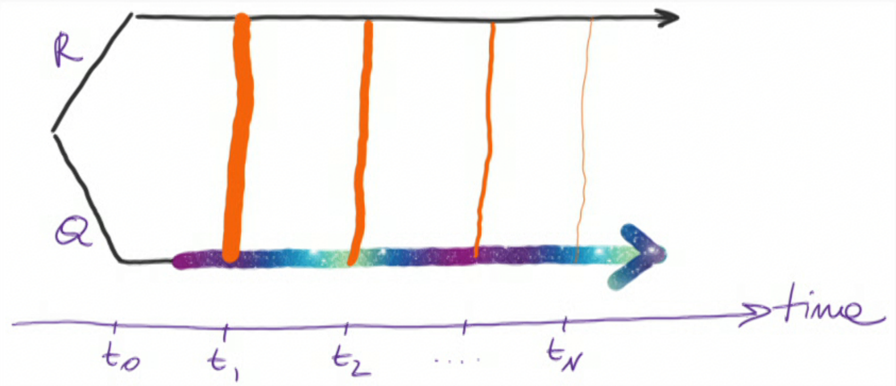


Divisibility as "Quantum Information Flow"

Theorem (2016-2018)

Given an initial open quantum system Q_0, a quantum dynamical mapping $\left\{\mathcal{N}^{(i)}_{Q_0\to Q_i}\right\}_{i\geq 1}$ is divisibile if and only if, for any initial state ω_{RQ_0},

$$H_{\min}(R|Q_1) \leq H_{\min}(R|Q_2) \leq \cdots \leq H_{\min}(R|Q_N) \ .$$



Application 2:
Quantum Thermodynamics

Resource Theory of Athermality and Asymmetry

From [FB, arXiv:1505.00535], [FB and Gour, Phys. Rev. A 95, 012110 (2017)], and [Gour, Jennings, FB, Duan, and Marvian, Nat. Comm. 9, 5352 (2018)]

- idea: to characterize thermal accessibility $\rho \to \sigma$ by comparing the dichotomies (ρ, γ) and (σ, γ), for γ thermal state

Resource Theory of Athermality and Asymmetry

From [FB, arXiv:1505.00535], [FB and Gour, Phys. Rev. A 95, 012110 (2017)], and [Gour, Jennings, FB, Duan, and Marvian, Nat. Comm. 9, 5352 (2018)]

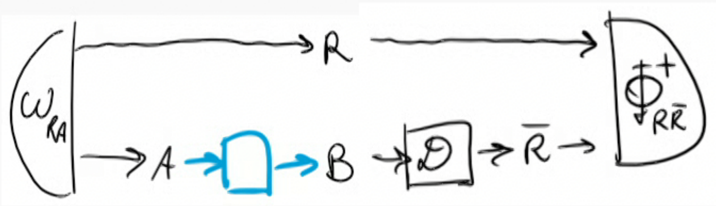
- idea: to characterize thermal accessibility $\rho \to \sigma$ by comparing the dichotomies (ρ, γ) and (σ, γ), for γ thermal state
- classically, Blackwell's theorem implies the thermomajorization relation

Resource Theory of Athermality and Asymmetry

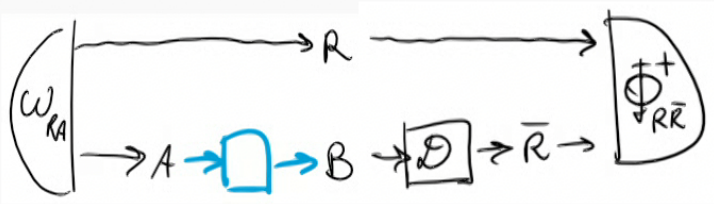
From [FB, arXiv:1505.00535], [FB and Gour, Phys. Rev. A 95, 012110 (2017)], and [Gour, Jennings, FB, Duan, and Marvian, Nat. Comm. 9, 5352 (2018)]

- idea: to characterize thermal accessibility $\rho \to \sigma$ by comparing the dichotomies (ρ, γ) and (σ, γ), for γ thermal state
- classically, Blackwell's theorem implies the thermomajorization relation
- in the quantum case, in order to account for coherence, symmetry constraints can also be added to the Gibbs-preserving map

Sketch Idea

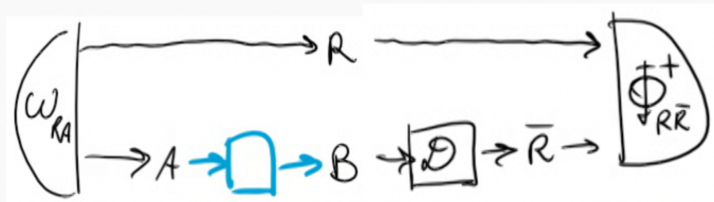


Sketch Idea

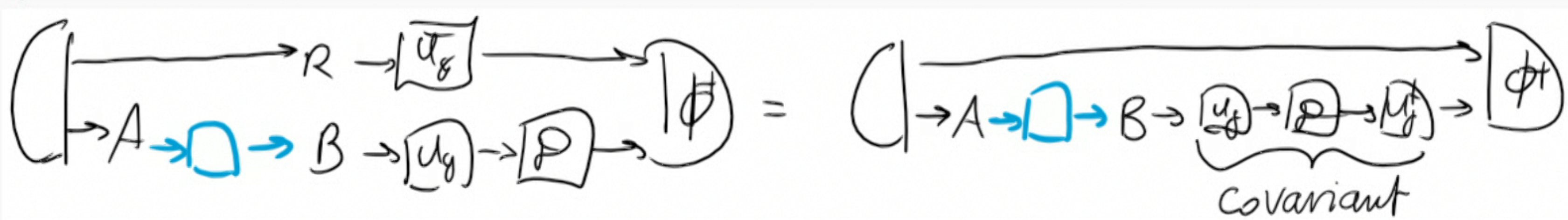


- we compare the singlet fraction of two channels, $\mathcal{N}^i_{A\to B}(\bullet) = \langle 0|\bullet|0\rangle\gamma + \langle 1|\bullet|1\rangle\rho^i$, with $\rho^1 \equiv \rho$ and $\rho^2 \equiv \sigma$

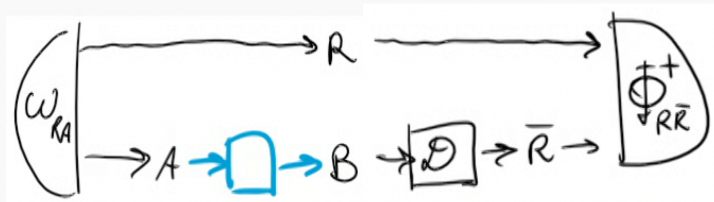
Sketch Idea



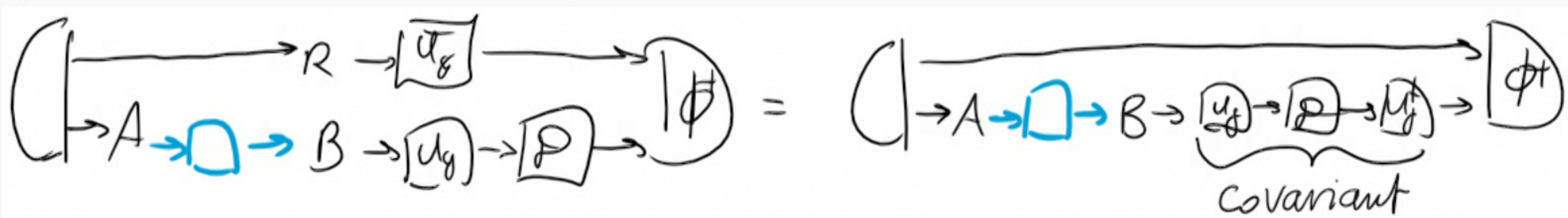
- we compare the singlet fraction of two channels, $\mathcal{N}^i_{A\to B}(\bullet) = \langle 0| \bullet |0\rangle\gamma + \langle 1| \bullet |1\rangle\rho^i$, with $\rho^1 \equiv \rho$ and $\rho^2 \equiv \sigma$
- to add symmetry constraints, we compare the two channels for the *twirled* quantum codes:



Sketch Idea



- we compare the singlet fraction of two channels, $\mathcal{N}^i_{A\to B}(\bullet) = \langle 0| \bullet |0\rangle \gamma + \langle 1| \bullet |1\rangle \rho^i$, with $\rho^1 \equiv \rho$ and $\rho^2 \equiv \sigma$
- to add symmetry constraints, we compare the two channels for the *twirled* quantum codes:

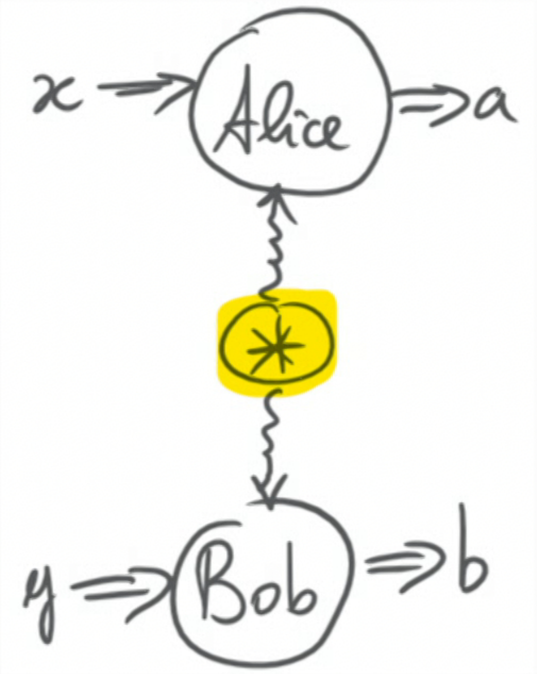


- by varying the input quantum code, we obtain a complete set of entropic monotones

Quantum Coding:
Probing Quantum Correlations in Space-Time

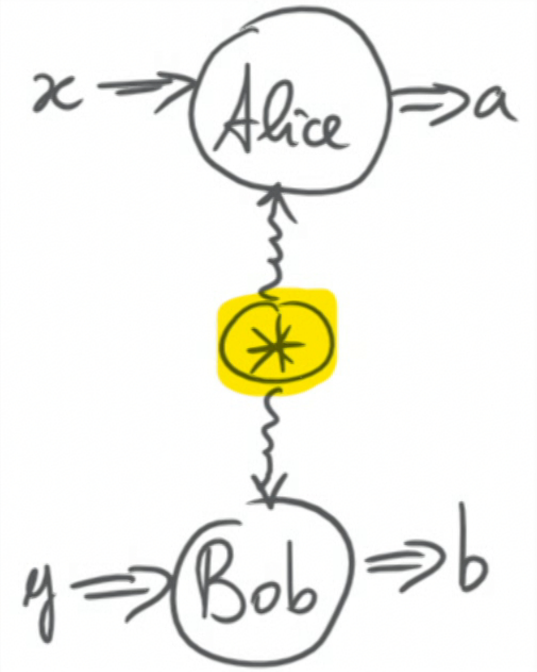
Part One: Quantum Space-Like Correlations

- nonlocal games (Bell tests) can be seen here as bipartite decision problems $\mathbf{ng} = \langle \mathcal{X}, \mathcal{Y}; \mathcal{A}, \mathcal{B}; \ell \rangle$ played "in parallel" by non-communicating players



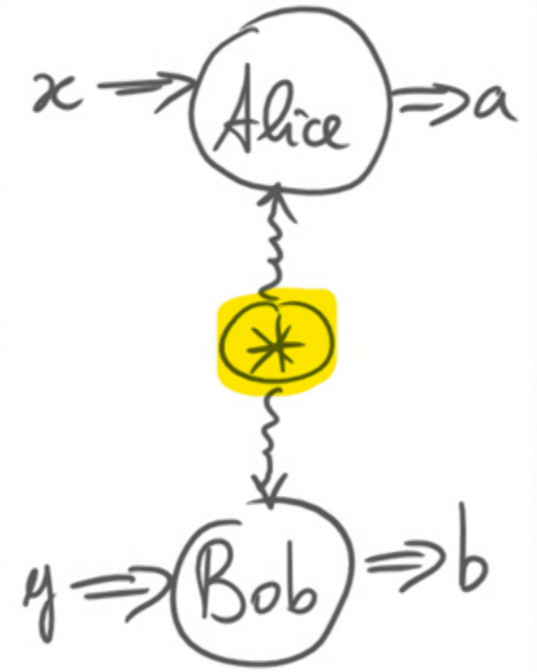
Part One: Quantum Space-Like Correlations

- nonlocal games (Bell tests) can be seen here as bipartite decision problems **ng** $= \langle \mathcal{X}, \mathcal{Y}; \mathcal{A}, \mathcal{B}; \ell \rangle$ played "in parallel" by non-communicating players
- with a classical source,
  $p_c(a,b|x,y) = \sum_\lambda \pi(\lambda) d_A(a|x,\lambda) d_B(b|y,\lambda)$
- with a quantum source,
  $p_q(a,b|x,y) = \mathrm{Tr}\left[\rho_{AB}\ (P_A^{a|x} \otimes Q_B^{b|y})\right]$



Part One: Quantum Space-Like Correlations

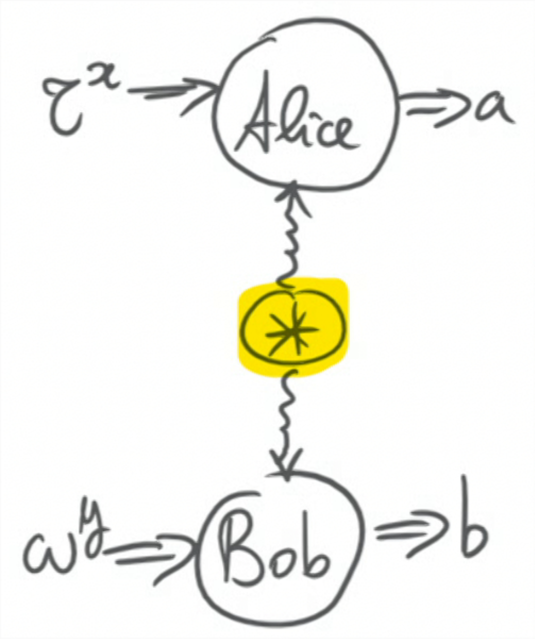
- nonlocal games (Bell tests) can be seen here as bipartite decision problems **ng** $= \langle \mathcal{X}, \mathcal{Y}; \mathcal{A}, \mathcal{B}; \ell \rangle$ played "in parallel" by non-communicating players
- with a classical source, $p_c(a, b|x, y) = \sum_\lambda \pi(\lambda) d_A(a|x, \lambda) d_B(b|y, \lambda)$
- with a quantum source, $p_q(a, b|x, y) = \mathrm{Tr}\Big[\rho_{AB}\ (P_A^{a|x} \otimes Q_B^{b|y})\Big]$



$$\mathbb{E}_{\mathbf{nl}}[*] \stackrel{\text{def}}{=} \max \sum_{x,y,a,b} \ell(x, y; a, b) p_{c/q}(a, b|x, y) |\mathcal{X}|^{-1} |\mathcal{Y}|^{-1}$$

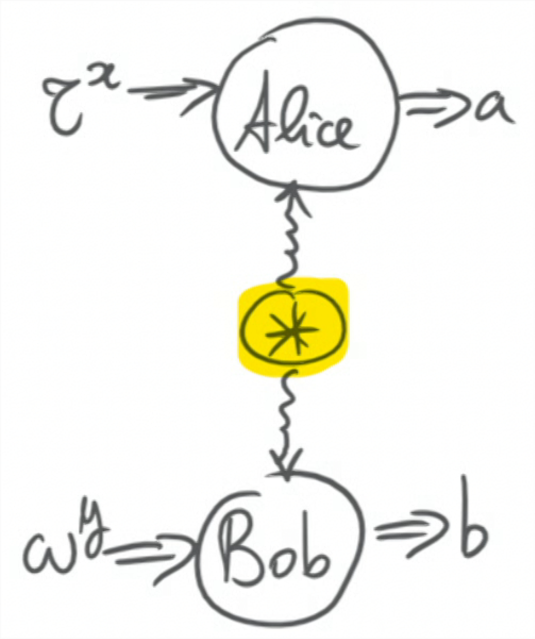
Semiquantum Nonlocal Games

- **semiquantum nonlocal games** replace classical inputs with quantum inputs: **sqnl** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$



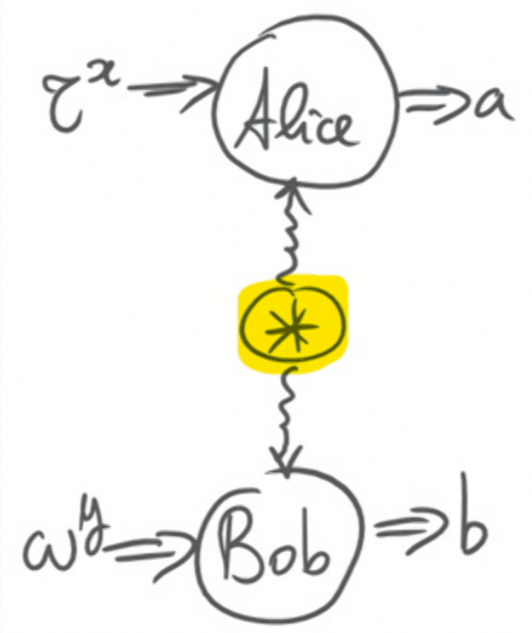
Semiquantum Nonlocal Games

- **semiquantum nonlocal games** replace classical inputs with quantum inputs: **sqnl** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$
- with a classical source, $p_c(a,b|x,y) = \sum_\lambda \pi(\lambda)\, \mathrm{Tr}\left[(\tau^x_X \otimes \omega^y_Y)\ (P^{a|\lambda}_X \otimes Q^{b|\lambda}_Y) \right]$
- with a quantum source, $p_q(a,b|x,y) = \mathrm{Tr}\left[(\tau^x_X \otimes \rho_{AB} \otimes \omega^y_Y)\ (P^a_{XA} \otimes Q^b_{BY}) \right]$



Semiquantum Nonlocal Games

- **semiquantum nonlocal games** replace classical inputs with quantum inputs: **sqnl** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$
- with a classical source,
  $p_c(a,b|x,y) = \sum_\lambda \pi(\lambda) \operatorname{Tr}\left[(\tau^x_X \otimes \omega^y_Y)\ (P^{a|\lambda}_X \otimes Q^{b|\lambda}_Y)\right]$
- with a quantum source,
  $p_q(a,b|x,y) = \operatorname{Tr}\left[(\tau^x_X \otimes \rho_{AB} \otimes \omega^y_Y)\ (P^a_{XA} \otimes Q^b_{BY})\right]$



$$\mathbb{E}_{\textsf{sqnl}}[*] \stackrel{\text{def}}{=} \max \sum_{x,y,a,b} \ell(x,y;a,b) p_{c/q}(a,b|x,y) |\mathcal{X}|^{-1} |\mathcal{Y}|^{-1}$$

LOSR Morphisms of Quantum Correlations

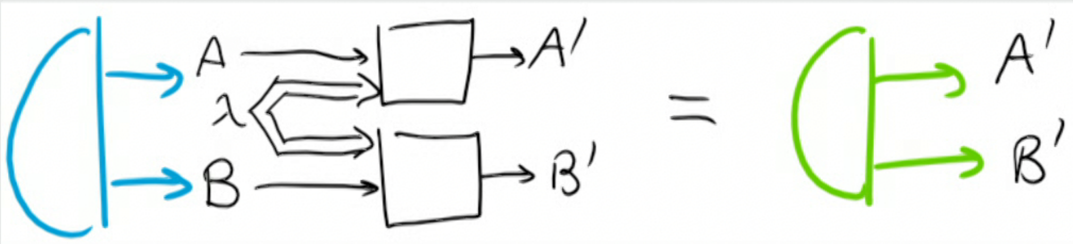
Theorem (FB, 2012)

Given two bipartite states ρ_{AB} and $\sigma_{A'B'}$, the condition (i.e., "nonlocality preorder")

$$\mathbb{E}_{\textsf{sqnl}}[\rho_{AB}] \geq \mathbb{E}_{\textsf{sqnl}}[\sigma_{A'B'}]$$

holds for all semiquantum nonlocal games **sqnl** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$,
iff there exist CPTP maps $\{\Phi^\lambda_{A\to A'}\}$, $\{\Psi^\lambda_{B\to B'}\}$, *and distribution* $\pi(\lambda)$ *such that*

$$\sigma_{A'B'} = \sum_\lambda \pi(\lambda)(\Phi^\lambda_{A\to A'} \otimes \Psi^\lambda_{B\to B'})(\rho_{AB}) \ .$$



Corollaries

- For any separable state ρ_{AB},

$$\mathbb{E}_{\textsf{\textbf{sqnl}}}[\rho_{AB}] = \mathbb{E}_{\textsf{\textbf{sqnl}}}[\rho_A \otimes \rho_B] = \mathbb{E}^{\textsf{sep}}_{\textsf{\textbf{sqnl}}} \ ,$$

for all semiquantum nonlocal games **sqnl** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$.

Corollaries

- For any separable state ρ_{AB},

$$\mathbb{E}_{\textsf{sqnl}}[\rho_{AB}] = \mathbb{E}_{\textsf{sqnl}}[\rho_A \otimes \rho_B] = \mathbb{E}_{\textsf{sqnl}}^{\textsf{sep}} ,$$

  for all semiquantum nonlocal games **sqnl** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$.
- For any entangled state ρ_{AB}, **there exists** a semiquantum nonlocal game **sqnl** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$ such that

$$\mathbb{E}_{\textsf{sqnl}}[\rho_{AB}] > \mathbb{E}_{\textsf{sqnl}}^{\textsf{sep}} .$$

Other Properties of Semiquantum Nonlocal Games

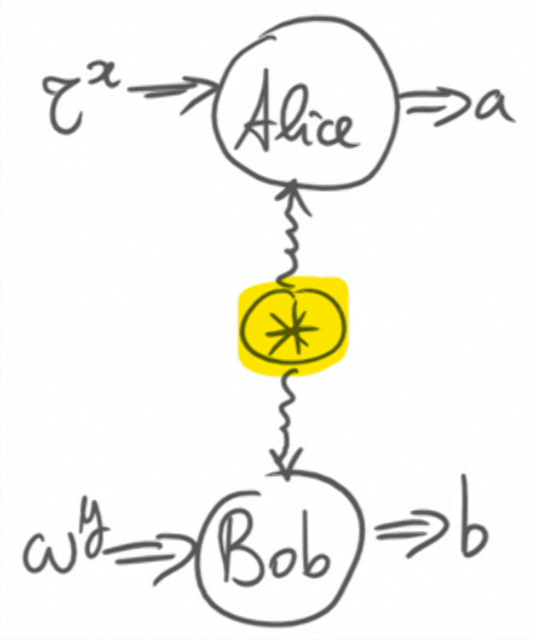
From [Branciard, Rosset, Liang, and Gisin, Phys. Rev. Lett. 110, 060405 (2013)]

Other Properties of Semiquantum Nonlocal Games

From [Branciard, Rosset, Liang, and Gisin, Phys. Rev. Lett. 110, 060405 (2013)]

Semiquantum nonlocal games:

- can be considered as measurement device-independent entanglement witnesses (i.e., MDI-EW)

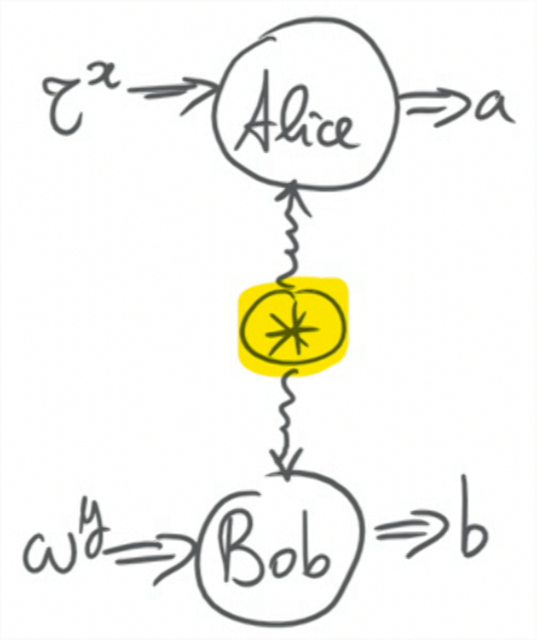


Other Properties of Semiquantum Nonlocal Games

From [Branciard, Rosset, Liang, and Gisin, Phys. Rev. Lett. 110, 060405 (2013)]

Semiquantum nonlocal games:

- can be considered as measurement device-independent entanglement witnesses (i.e., MDI-EW)
- can withstand losses in the detectors

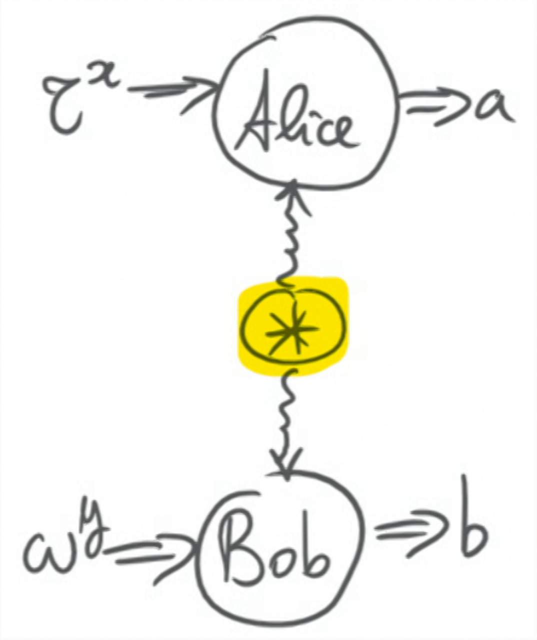


Other Properties of Semiquantum Nonlocal Games

From [Branciard, Rosset, Liang, and Gisin, Phys. Rev. Lett. 110, 060405 (2013)]

Semiquantum nonlocal games:

- can be considered as measurement device-independent entanglement witnesses (i.e., MDI-EW)
- can withstand losses in the detectors
- can withstand any amount of classical communication exchanged between Alice and Bob



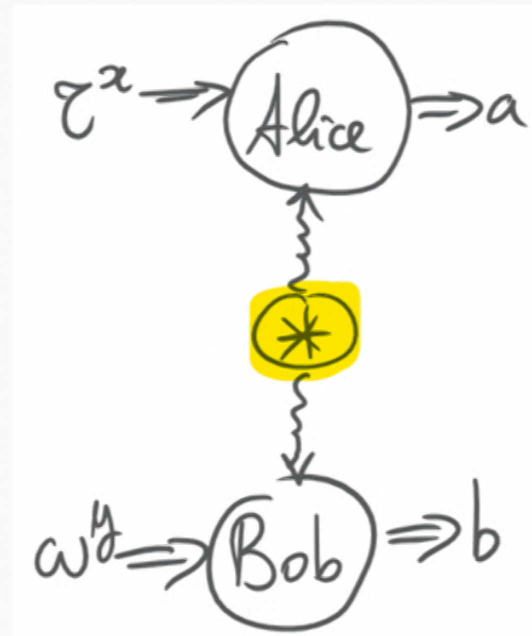
25/31

Other Properties of Semiquantum Nonlocal Games

From [Branciard, Rosset, Liang, and Gisin, Phys. Rev. Lett. 110, 060405 (2013)]

Semiquantum nonlocal games:

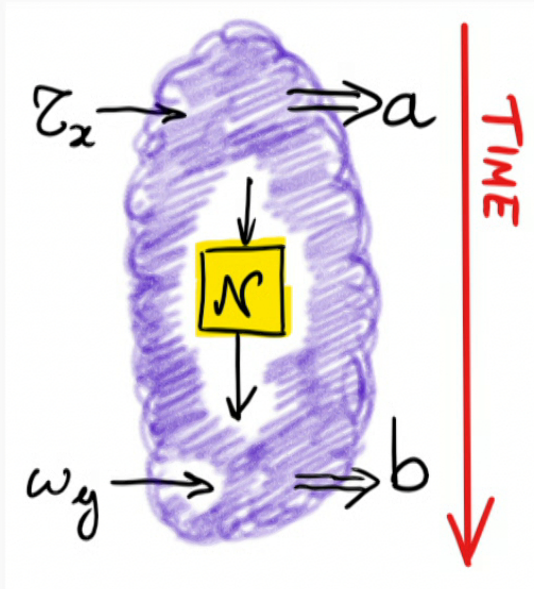
- can be considered as measurement device-independent entanglement witnesses (i.e., MDI-EW)
- can withstand losses in the detectors
- can withstand any amount of classical communication exchanged between Alice and Bob
- hence, contrarily to conventional Bell tests, semiquantum nonlocal games are non trivial also when rearranged *in time*



Part Two: Quantum Time-Like Correlations

Semiquantum *signaling* games:

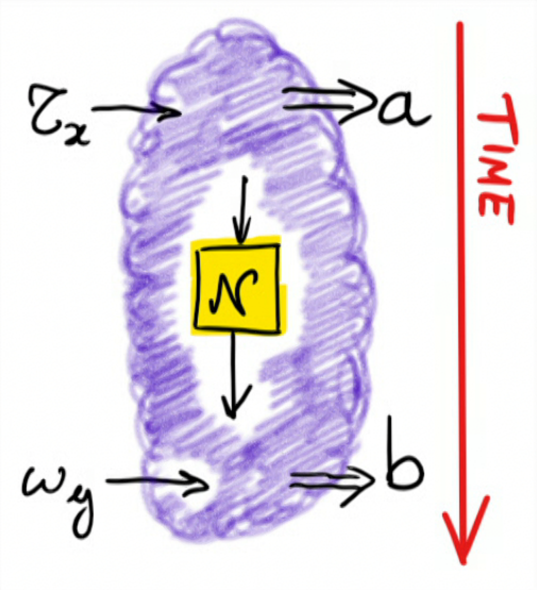
- the duo Alice–Bob becomes ‘Alice now’–‘Alice later’



Part Two: Quantum Time-Like Correlations

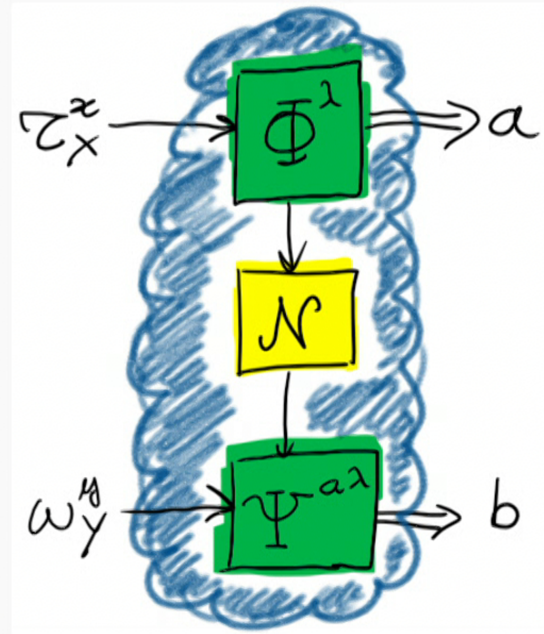
Semiquantum *signaling* games:

- the duo Alice–Bob becomes 'Alice now'–'Alice later'
- the semiquantum nonlocal game **sqnl** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$ is arranged in a time-like structure
- thus obtaining a semiquantum signaling game **sqsg**
- with unlimited classical memory, $p_c(a,b|x,y) = \sum_\lambda \pi(\lambda)\, \mathrm{Tr}\left[\tau_X^x\ P_X^{a|\lambda}\right] \mathrm{Tr}\left[\omega_Y^y\ Q_Y^{b|a,\lambda}\right]$



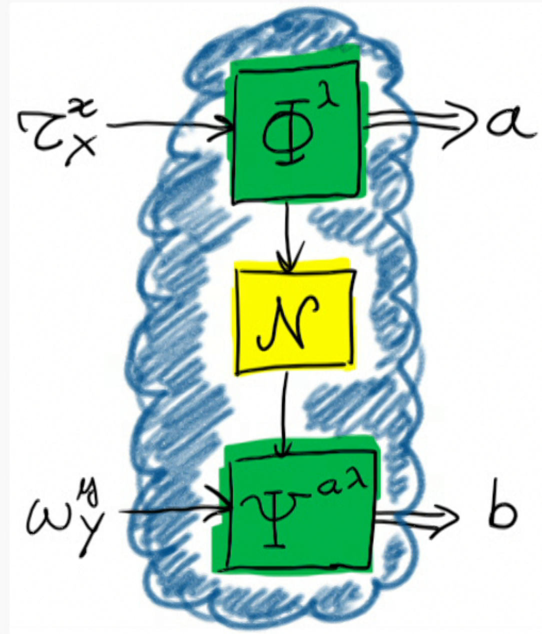
Admissible Quantum Strategies

- τ_X^x is fed through an *instrument* $\{\Phi_{X\to A}^{a|\lambda}\}$, and outcome a is recorded
- the quantum output of the instrument is fed through the quantum memory $\mathcal{N} : A \to B$
- the output of the memory, together with ω_Y^y, are fed into a final measurement $\{\Psi_{BY}^{b|a,\lambda}\}$, and output b is recorded



Admissible Quantum Strategies

- τ_X^x is fed through an *instrument* $\{\Phi_{X\to A}^{a|\lambda}\}$, and outcome a is recorded
- the quantum output of the instrument is fed through the quantum memory $\mathcal{N} : A \to B$
- the output of the memory, together with ω_Y^y, are fed into a final measurement $\{\Psi_{BY}^{b|a,\lambda}\}$, and output b is recorded



$$p_q(a,b|x,y) = \sum_\lambda \pi(\lambda)\,\mathrm{Tr}\Big[\Big(\{(\mathcal{N}_{A\to B} \circ \Phi_{X\to A}^{a|\lambda})(\tau_X^x)\} \otimes \omega_Y^y\Big)\,\Psi_{BY}^{b|a,\lambda}\Big]$$

Classical vs Quantum Strategies

Classical:

$$p_c(a,b|x,y) = \sum_\lambda \pi(\lambda)\,\mathrm{Tr}\left[\tau_X^x\; P_X^{a|\lambda}\right]\mathrm{Tr}\left[\omega_Y^y\; Q_Y^{b|a,\lambda}\right]$$

Quantum:

$$p_q(a,b|x,y) = \sum_\lambda \pi(\lambda)\,\mathrm{Tr}\left[\left(\{(\mathcal{N}_{A\to B}\circ\Phi_{X\to A}^{a|\lambda})(\tau_X^x)\}\otimes\omega_Y^y\right)\Psi_{BY}^{b|a,\lambda}\right]$$

Classical vs Quantum

Classical strategies correspond to the case in which the channel $\mathcal{N}$ is entanglement-breaking (i.e., "measure and prepare" form): $\mathcal{N}(\cdot) = \sum_i \rho_i\,\mathrm{Tr}[\cdot\; P_i]$.

EB Morphisms of Quantum Channels

Theorem (Rosset, FB, Liang, 2018)

Given two channels $\mathcal{N} : A \to B$ and $\mathcal{N}' : A' \to B'$, the condition (i.e., "quantum signaling preorder")

$$\mathbb{E}_{\mathbf{sqsg}}[\mathcal{N}] \geq \mathbb{E}_{\mathbf{sqsg}}[\mathcal{N}']$$

holds for all semiquantum signaling games **sqsg** $= \langle \{\tau^x\}, \{\omega^y\}; \mathcal{A}, \mathcal{B}; \ell \rangle$,
iff there exist a quantum instrument $\{\Phi^a_{A' \to A}\}$ *and CPTP maps* $\{\Psi^a_{B \to B'}\}$ *such that*

$$\mathcal{N}'_{A' \to B'} = \sum_a \Psi^a_{B \to B'} \circ \mathcal{N}_{A \to B} \circ \Phi^a_{A' \to A} .$$

A Resource Theory of Quantum Memories: Some Remarks

- formulation of a resource theory where all and only measure-and-prepare channels are "free"

A Resource Theory of Quantum Memories: Some Remarks

- formulation of a resource theory where all and only measure-and-prepare channels are "free"
- any non entanglement-breaking channel can be witnessed
- perfect analogy between separable states and entanglement-breaking channels

A Resource Theory of Quantum Memories: Some Remarks

- formulation of a resource theory where all and only measure-and-prepare channels are "free"
- any non entanglement-breaking channel can be witnessed
- perfect analogy between separable states and entanglement-breaking channels
- relation with Leggett-Garg inequalities: the "clumsiness loophole" (time-like analogue of communication loophole) can be closed with semiquantum games

A Resource Theory of Quantum Memories: Some Remarks

- formulation of a resource theory where all and only measure-and-prepare channels are "free"
- any non entanglement-breaking channel can be witnessed
- perfect analogy between separable states and entanglement-breaking channels
- relation with Leggett-Garg inequalities: the "clumsiness loophole" (time-like analogue of communication loophole) can be closed with semiquantum games
- semiquantum games can treat space-like and time-like correlations on an equal footing

Conclusions

Conclusions

- the theory of statistical comparison studies morphisms (preorders) of one "statistical structure" X into another "statistical structure" Y

Conclusions

- the theory of statistical comparison studies morphisms (preorders) of one "statistical structure" X into another "statistical structure" Y
- equivalent conditions are given in terms of (finitely or infinitely many) *monotones*, e.g., $f_i(X) \geq f_i(Y)$
- such monotones shed light on the "resources" at stake in the operational framework at hand

Conclusions

- the theory of statistical comparison studies morphisms (preorders) of one "statistical structure" X into another "statistical structure" Y
- equivalent conditions are given in terms of (finitely or infinitely many) *monotones*, e.g., $f_i(X) \geq f_i(Y)$
- such monotones shed light on the "resources" at stake in the operational framework at hand
- in a sense, statistical comparison is complementary to SDP, which instead searches for *efficiently computable* functions like $f(X, Y)$

Conclusions

- the theory of statistical comparison studies morphisms (preorders) of one "statistical structure" X into another "statistical structure" Y
- equivalent conditions are given in terms of (finitely or infinitely many) *monotones*, e.g., $f_i(X) \geq f_i(Y)$
- such monotones shed light on the "resources" at stake in the operational framework at hand
- in a sense, statistical comparison is complementary to SDP, which instead searches for *efficiently computable* functions like $f(X, Y)$
- however, SDP does not provide much insight into the resources at stake (and not all statistical comparisons are equivalent to SDP!)

Conclusions

- the theory of statistical comparison studies morphisms (preorders) of one "statistical structure" X into another "statistical structure" Y
- equivalent conditions are given in terms of (finitely or infinitely many) *monotones*, e.g., $f_i(X) \geq f_i(Y)$
- such monotones shed light on the "resources" at stake in the operational framework at hand
- in a sense, statistical comparison is complementary to SDP, which instead searches for *efficiently computable* functions like $f(X, Y)$
- however, SDP does not provide much insight into the resources at stake (and not all statistical comparisons are equivalent to SDP!)

Thank you