Title: Formulations of General Relativity (Part 3 of 4)

Speakers: Kirill Krasnov

Series: Quantum Gravity

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Abstract: The goal of this series is to collect various different formulations of General Relativity, with emphasis on four spacetime dimensions and formulations that use differential forms. The (non-exhaustive) list of formulations to be covered is per this plan:

Lecture 1): Motivations, followed by the usual Einstein-Hilbert to start with, first order Palatini, second order pure affine connection Eddington-Schroedinger.

Lecture 2) Cartan's geometry of soldering. First order Einstein-Cartan tetrad formulation, second order pure spin connection formulation, MacDowell-Mansouri formulation.

Lecture 3) Non-chiral BF-type formulations. Explicit pure spin connection Lagrangian, field redefinitions, BF plus potential for the 2-form field formulation.

Lecture 4) Chiral formulations in four dimensions. Chiral Einstein-Cartan, Plebanski formulation, pure SU(2) connection formulation, SU(2) BF plus potential for the 2-form field. Self-dual gravity. Concluding remarks.

Proof of $\mathrm{Tr}\sqrt{\sqrt{\epsilon}X\sqrt{\epsilon}} = \mathrm{Tr}\sqrt{\epsilon X}$

We prove more general statement

$$\mathrm{Tr}\sqrt{MM'} = \mathrm{Tr}\, M^{-1}\sqrt{MM'}\,M = \mathrm{Tr}\sqrt{M^{-1}MM'M} = \mathrm{Tr}\sqrt{M'M}$$

Because

$$\sqrt{A^{-1}BA} = A^{-1}\sqrt{B}\,A$$

Square root commutes with similarity transformations

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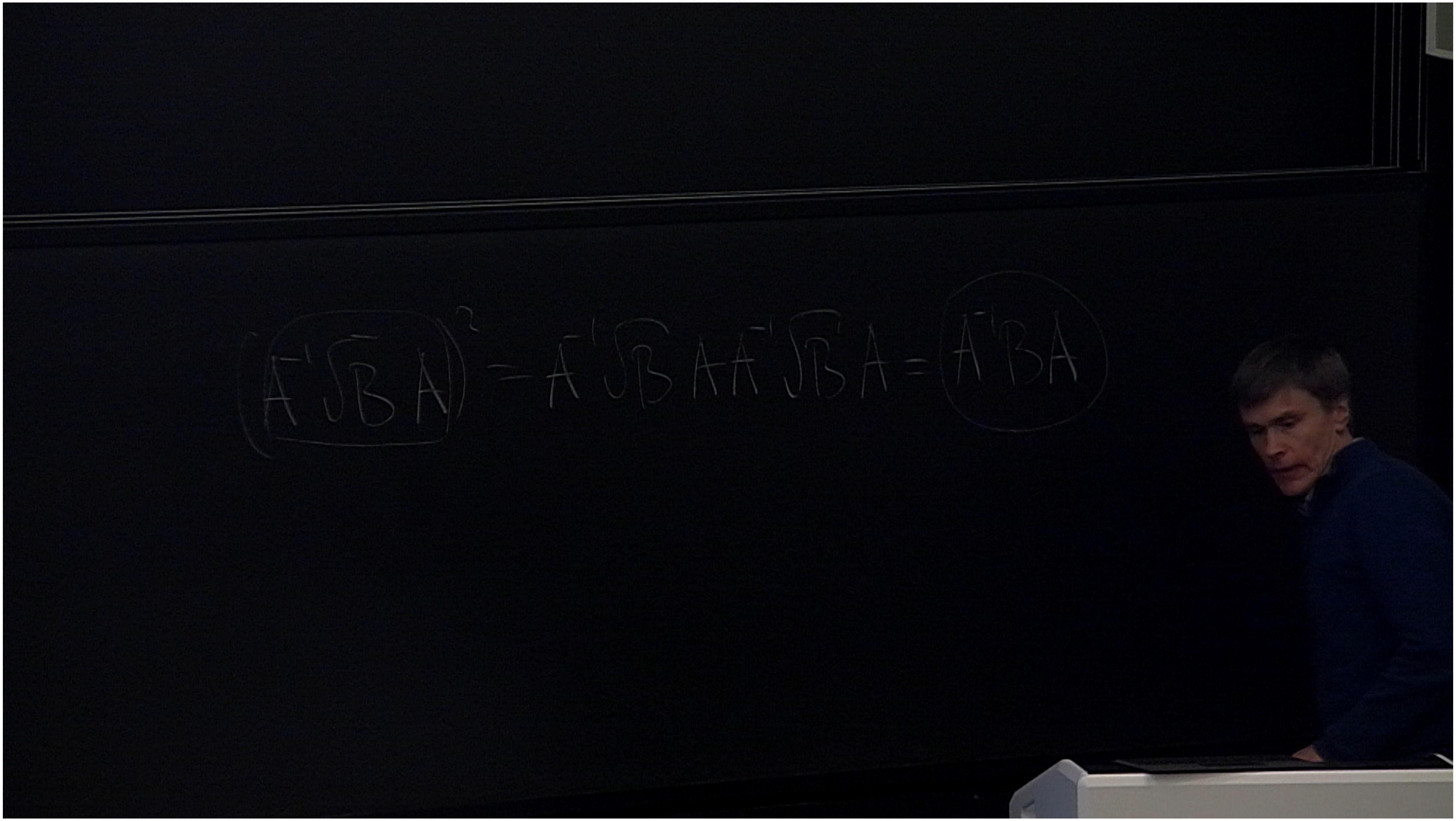
$(A^{-1}\sqrt{B}A)^2 = A^{-1}\sqrt{B}AA^{-1}\sqrt{B}A = A^{-1}BA$

Modifications of GR from BF formalism

Coming back to BF formulation, the action with Lagrange multipliers

$$S[B,\omega,M,\mu] = \frac{1}{16\pi G}\int B_{IJ}\, R^{IJ}(\omega) - \frac{1}{2} M^{IJKL} B_{IJ} B_{KL} + \frac{\mu}{2}\left(f_{GR}(M) - 4\Lambda\right)$$

where $f_{GR}(M) = M^{IJKL}\epsilon_{IJKL}$

As discussed, can replace this f_{GR} by more general trace

$$f_{GR}(M) = M^{IJKL}\left(\alpha\,\epsilon_{IJKL} + \beta\,\delta_{IK}\,\delta_{LJ}\right)$$

This still remains GR

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e.g. $\mathrm{Tr}\,M^2$ or $\det M$

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analysed by Smolin, Speziale and collaborators

It is interesting that there is a 2-parameter family of choices of f that still gives GR but is very different from the f_{GR} above.

Field redefinitions

Coming back to actions of BF type, we can obtain a very surprising formulation of GR that does not have Lagrange multiplier fields. The first step is to consider some field redefinitions that become possible in BF formalism

Let us form a new 2-form field $\tilde{B}^{IJ}$ given by a linear combination of the "old" B^{IJ} and $R^{IJ}(\omega)$

$$B = G\tilde{B} + HR$$

index-free notation, G, H are matrices, or linear maps acting on $so(1,3)$

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index-free notation, G, H are matrices, or linear maps acting on $so(1,3)$

The action in terms of $\tilde{B}$ will contain R^2 terms. Can choose G, H so that the R^2 terms generated are just the 2 topological terms

$$\int R^t R \quad \text{and} \quad \int R^t \epsilon R$$

transpose of a column vector

Thus, we require the R^2 terms to be of the form

B^{IJ} - vector with values in $so(1,3)$

$\tilde{B}^{IJ}, R^{IJ}$

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G^{IJKL}, H^{IJKL} - matrices 6×6

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transpose of a column vector

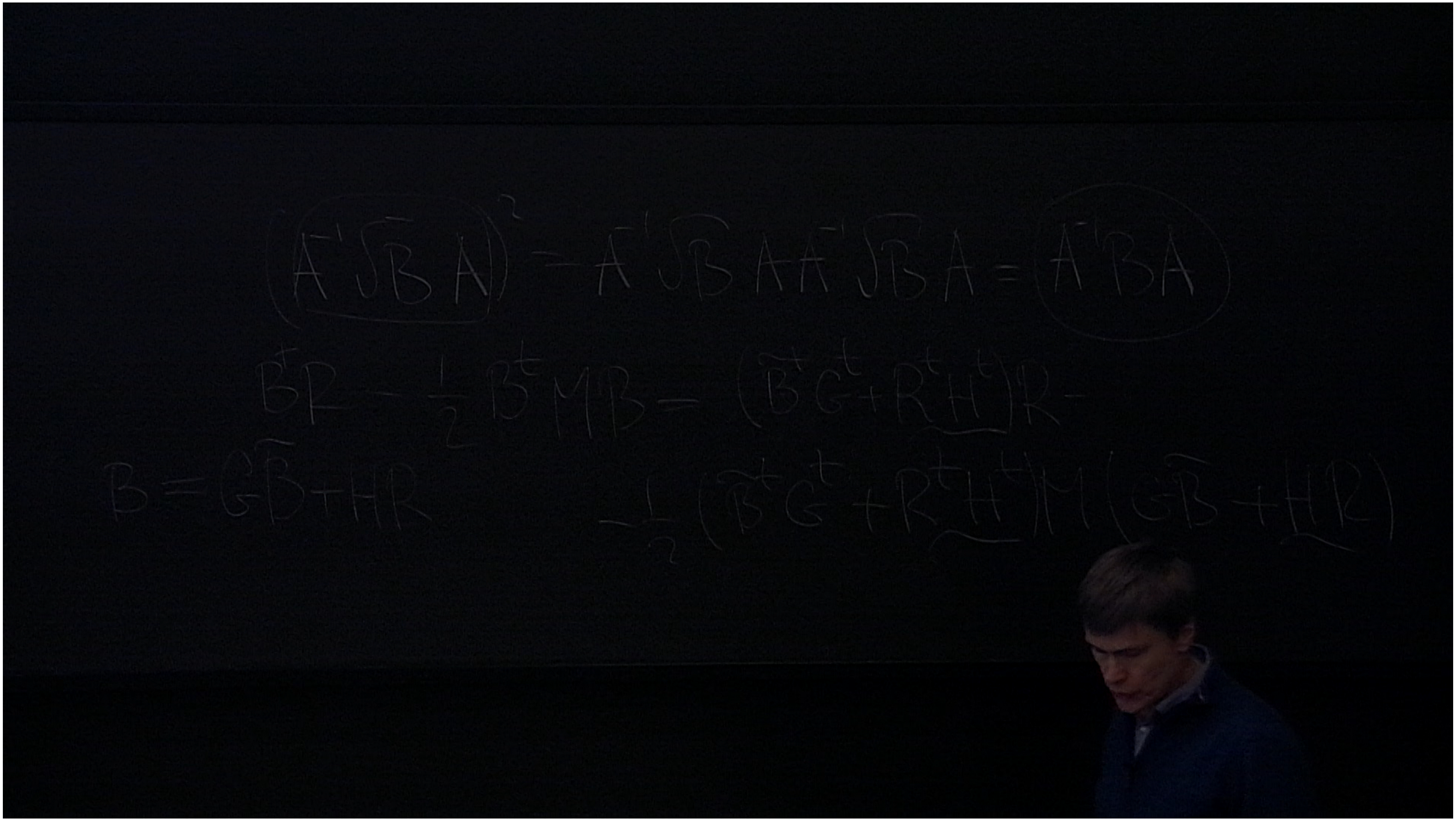
$$\int R^t R \qquad \text{and} \qquad \int R^t \epsilon R$$

$$R^{IJ} R_{IJ} \qquad\qquad R^{IJ} \epsilon_{IJ}{}^{KL} R_{KL}$$

Thus, we require the R^2 terms to be of the form

$$\int R^t \Gamma R \qquad \Gamma = t_1 \mathbb{1} + \tfrac{1}{2} t_2 \epsilon$$

$$\mathbb{1}^{IJ}{}_{KL} = \delta^I \delta^J$$


$(A^{-1}\sqrt{B}A)^2 = A^{-1}\sqrt{B}AA^{-1}\sqrt{B}A = A^{-1}BA$
$\tilde{B}^t R - \frac{1}{2} B^t M B = (\tilde{B}^t G^t + R^t H^t) R - \frac{1}{2} (\tilde{B}^t G^t + R^t H^t) M (G\tilde{B} + HR)$
$B = G\tilde{B} + HR$

$\int R^t \Gamma R$ $\qquad \Gamma = t_1 \mathbb{1} + \frac{1}{2} t_2 \epsilon$

$\mathbb{1}^{IJ}{}_{KL} = \delta^{I}_{[K}\delta^{J}_{L]}$

We will also require the $\tilde{B}R$ term to preserve its form
This gives 2 equations

$$H^t - \frac{1}{2} H^t M H = \Gamma$$
$$G^t - G^t M H = \mathbb{1}$$

After some algebra, these equations can be solved

$$G^t = \Gamma^{-1/2} \left(\mathbb{1} - 2\Gamma^{1/2} M \Gamma^{1/2}\right)^{-1/2} \Gamma^{1/2}$$

$$H = \Gamma^{1/2} \left(\mathbb{1} + \Gamma^{1/2} M \Gamma^{1/2}\right)^{-1} \Gamma^{1/2}$$

We denote the matrix arising in the $\tilde{B}\tilde{B}$ term by $\tilde{M}$.

$$G^t - G^t M H = \mathbb{1}$$

After some algebra, these equations can be solved

$$G^t = \mathcal{T}^{-1/2}\left(\mathbb{1} - 2\mathcal{T}^{1/2} M \mathcal{T}^{1/2}\right)^{-1/2} \mathcal{T}^{1/2}$$

$$H = \mathcal{T}^{1/2}\left(\mathbb{1} + \mathcal{T}^{1/2} M \mathcal{T}^{1/2}\right)^{-1} \mathcal{T}^{1/2}$$

We denote the matrix arising in the $\tilde{B}\tilde{B}$ term by $\tilde{M}$.

One then finds $\quad M = \tilde{M}\left(\mathbb{1} + 2\mathcal{T}\tilde{M}\right)^{-1}$

Even though $\mathcal{T}^{1/2}$ was present in intermediate expressions, it disappears on final formulas that matter

Omitting the tildas, the action in terms of new fields is

One then finds $\boxed{M} = \tilde{M}\left(\mathbb{1} + 2T\tilde{M}\right)^{-1}$

Even though T^{ij} was present in intermediate expressions, it disappears on final formulas that matter

Omitting the tildas, the action in terms of new fields is

$$S[B,A,M,\mu] = \frac{1}{16\pi G}\int B^t R - \frac{1}{2} B^t M B + R^t T R$$
$$+ \frac{\mu}{2}\left(\mathrm{Tr}\left(\epsilon M \left(\mathbb{1}+2TM\right)^{-1}\right) - 4\Lambda \right)$$

When $T=0$ gives the original BF type action.
Now have a 2-parameter family of its "versions".
They all describe the same theory, it is just now changed what the 2-form field is. The

$$\frac{\ }{2}\left(\mathrm{Tr}\left(\in M \left(1\!\!1 + 2TM\right)^{-1} \right) - 4\Lambda \right)$$

9 / 12

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$$\underline{f_{GR}} = \mathrm{Tr}\left(\in M \left(1\!\!1 + 2TM\right)^{-1} \right)$$

$$T = t_1 1\!\!1 + \frac{t_2}{2} \in$$

More general than could be expected family of theories still describing GR

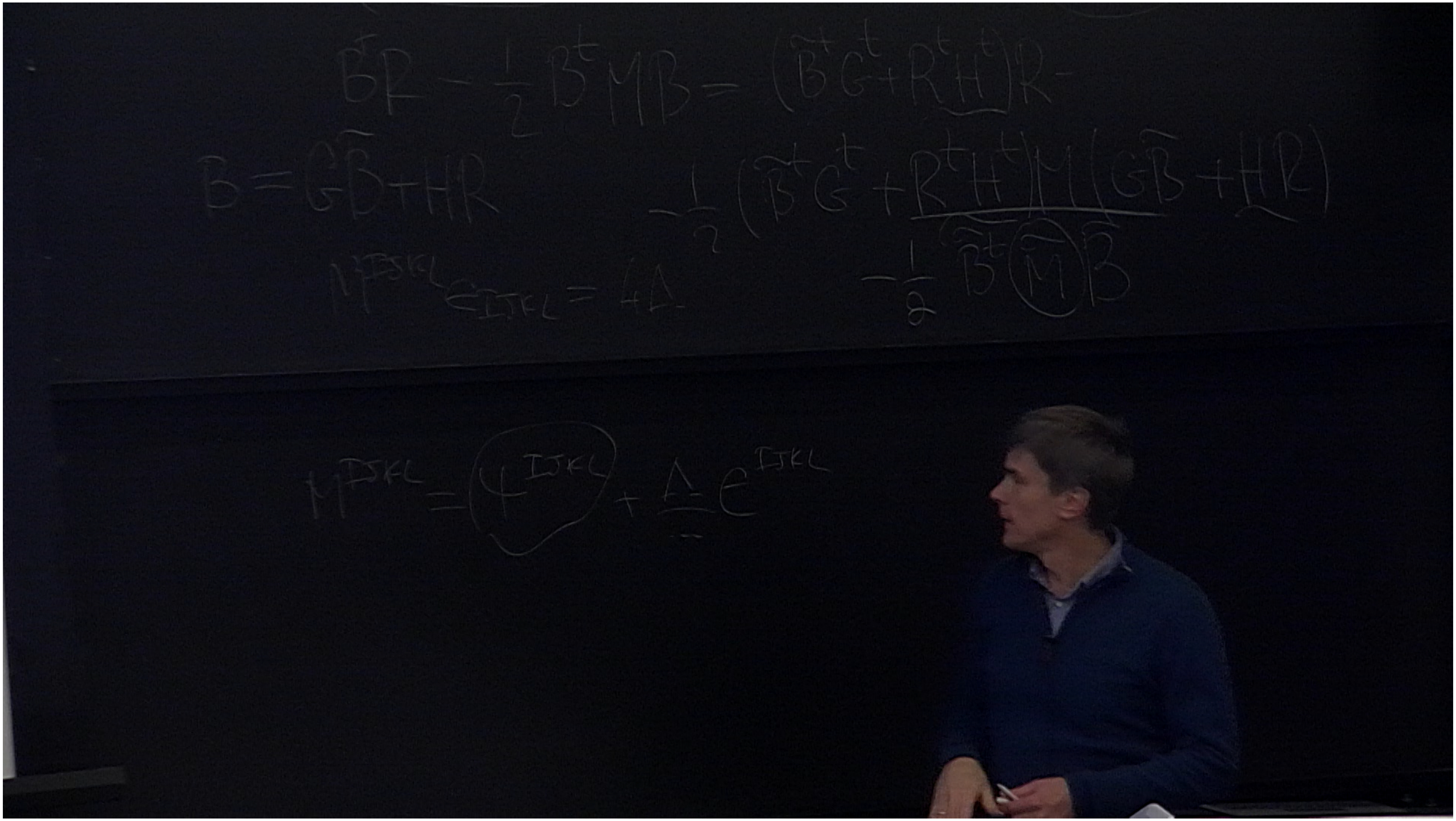
211 %

When $\tau = 0$ gives the original BF type action.
Now have a 2-parameter family of its "versions".
They all describe the same theory, it is just now changed what the 2-form field is. The new B does not satisfy the metricity constraints anymore!

$$f_{GR} = \mathrm{Tr}\left(\epsilon M \left(\mathbb{1} + 2\tau M \right)^{-1} \right)$$

$$\tau = t_1 \mathbb{1} + \frac{t_2}{2} \epsilon$$

More general than could be expected family of theories still describing GR


$\tilde{B}^t R - \frac{1}{2} B^t M B = (\tilde{B}^t G^t + R^t H^t) R -$
$B = G\tilde{B} + HR$
$-\frac{1}{2}(\tilde{B}^t G^t + R^t H^t) M (G\tilde{B} + HR)$
$M^{IJKL}\epsilon_{IJKL} = 4\Delta$
$-\frac{1}{2}\tilde{B}^t \tilde{M} \tilde{B}$
$M^{IJKL} = \psi^{IJKL} + \Delta e^{IJKL}$

$$\Upsilon = t_1 \mathbb{1} + \frac{t_2}{2} \epsilon$$

More general than could be expected family of theories still describing GR

BF type formulation with a potential for the 2-form field

As we will now show, one can eliminate all Lagrange multiplier fields from the above action. The result is a pure BF-type action, with no other fields apart from B^{IJ} and ω^{IJ}

The field equation obtained by varying with respect to M

$$(\mathbb{1} + 2\Upsilon M)\, X_B\, (\mathbb{1} + 2M\Upsilon) = \mu\, \epsilon$$

where $X_B = B \wedge B$

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where $X_B = B \wedge B$

Its solution can be written as

$$\Upsilon + 2\Upsilon M \Upsilon = \pm \sqrt{\mu}\, \sqrt{\epsilon}\, \left(\sqrt{\epsilon}\, \Upsilon^{-1} X_B \Upsilon^{-1} \sqrt{\epsilon}\right)^{-1/2} \sqrt{\epsilon}$$

One then determines M, then μ. The final arising action is

The field equation obtained by varying with respect to M

$$(\mathbb{1} + 2\tau M)\, X_B\, (\mathbb{1} + 2M\tau) = \mu\, \epsilon$$

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One then determines M, then μ. The final arising action is

$$S[B,\omega] = \frac{1}{16\pi G} \int B^t R + \frac{1}{4} B^t \tau^{-1} B$$

"potential"

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$$- \frac{\left(\mathrm{Tr} \sqrt{\epsilon \Upsilon^{-1} X_B \Upsilon^{-1}} \right)^2}{\ldots}$$

"potential"

for the

$$\mathcal{T} + 2\mathcal{T}M\mathcal{T} = \pm\sqrt{\mu}\,\sqrt{\epsilon}\left(\sqrt{\epsilon}\,\mathcal{T}^{-1}X_B\mathcal{T}^{-1}\sqrt{\epsilon}\right)^{-1/2}\sqrt{\epsilon}$$

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$$- \frac{\left(\mathrm{Tr}\sqrt{\epsilon\mathcal{T}^{-1}X_B\mathcal{T}^{-1}}\right)^2}{4\,\mathrm{Tr}(\mathcal{T}^{-1}\epsilon) - 32\Lambda}$$

"potential" for the B field

The first line is a topological theory. Only the last term makes this into a theory with propagating DOF

231 %

$$T + 2TMT = \pm \sqrt{\mu}\sqrt{\epsilon}\left(\sqrt{\epsilon}\,T^{-1}X_B T^{-1}\sqrt{\epsilon}\right)^{-1/2}\sqrt{\epsilon}$$

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The fact that such a formulation of GR should be possible is far from obvious! No Lagrange

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"potential" for the B field

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The fact that such a formulation of GR should be possible is far from obvious! No Lagrange multipliers left, but the action is non-polynomial

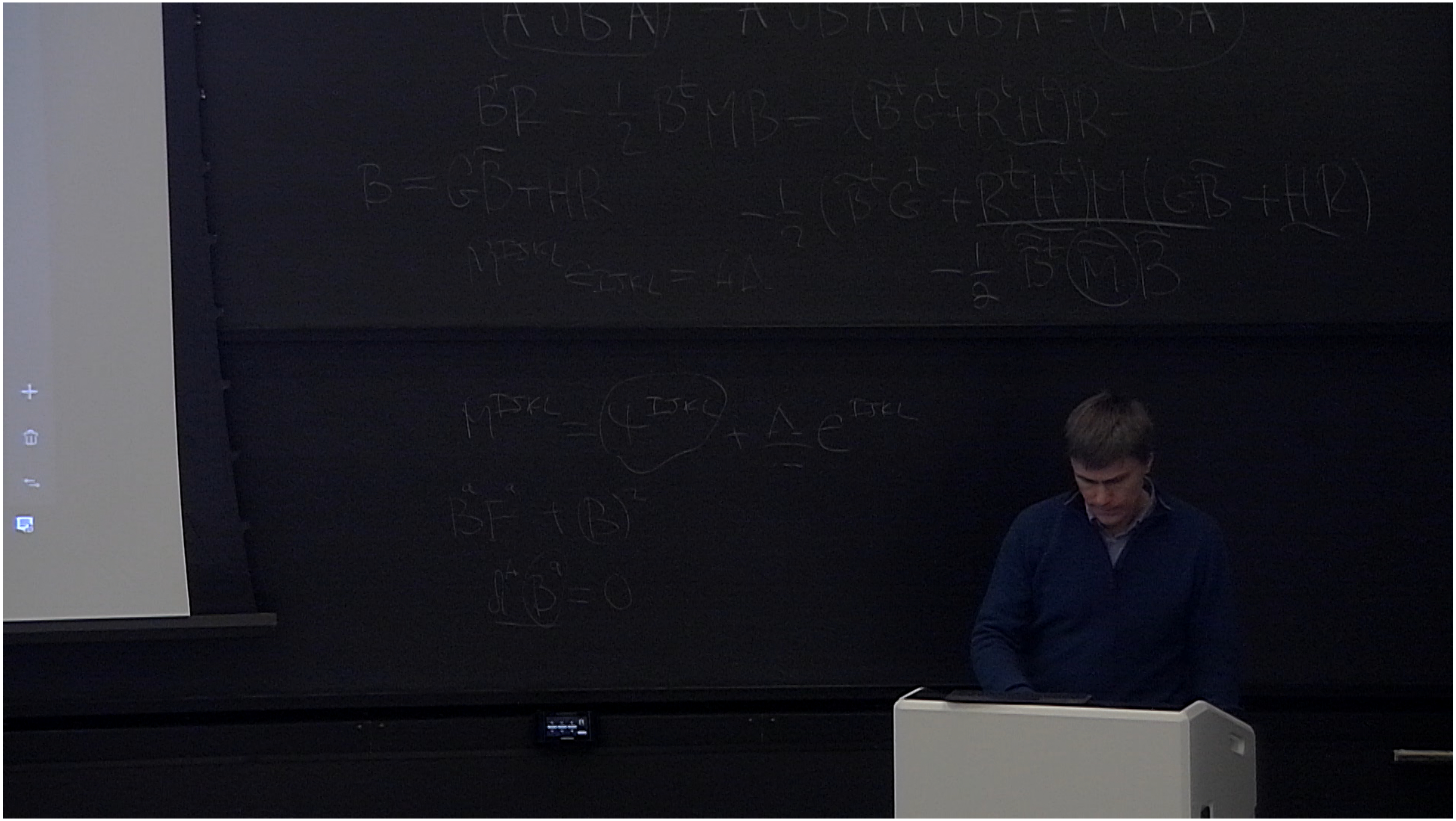
Second-order Lagrangian for the 2-form field

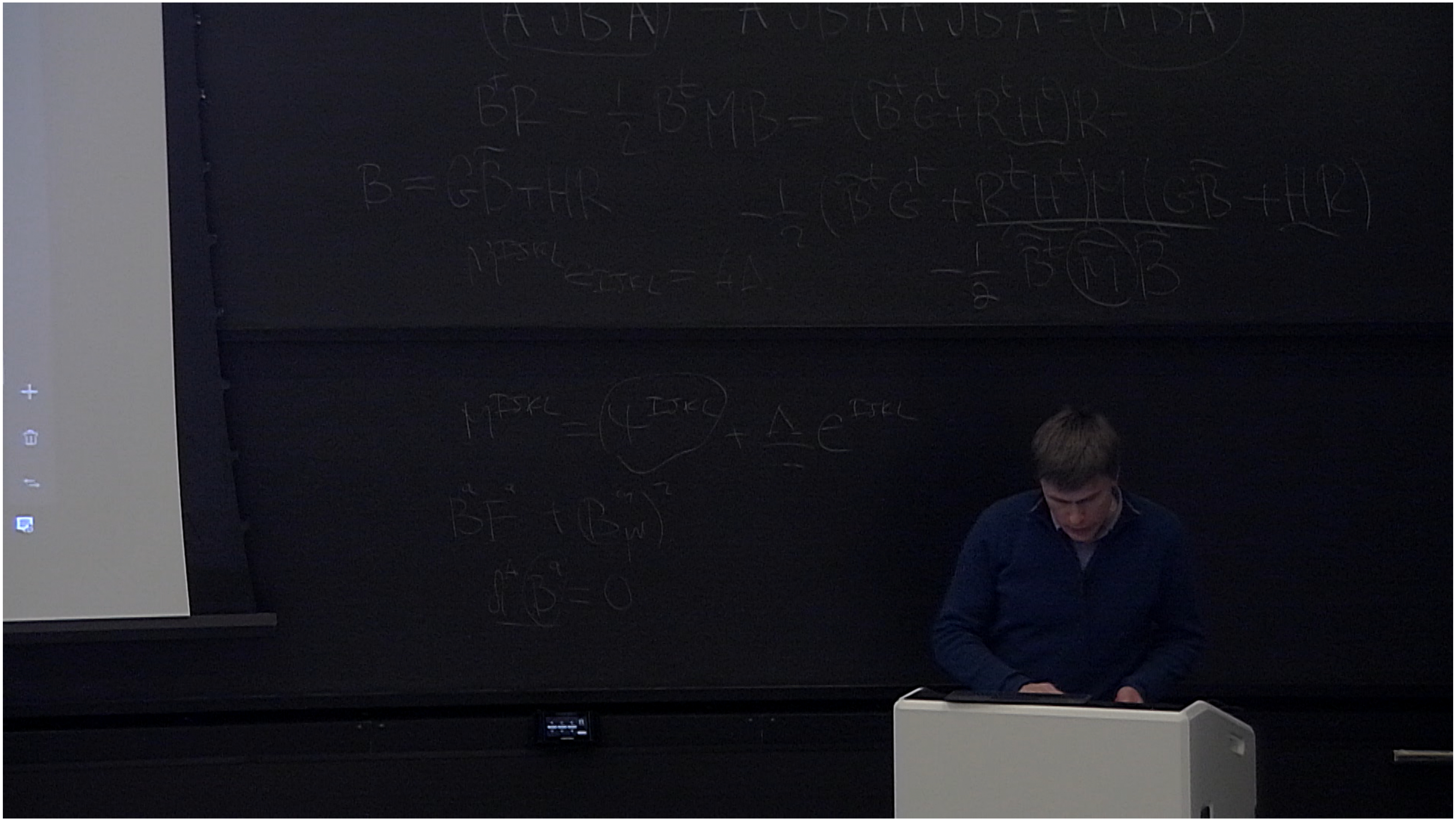
Varying the BF Lagrangian wrt the connection gives

$$d^{\omega} B^{IJ} = 0$$

The number of equations here matches the number of unknown connection components, so one expects to be able to solve for $\omega = \omega(B)$. Substituting this back to the action would give a second order formalism with $B^{IJ}_{\mu\nu}$ as the only field.

This exercise has only been carried out in the chiral case considered next





like a chiral formulation must be possible.

Lorentz group in 4D is not simple

At the level of Lie algebra

$$so(4) = su(2) \oplus su(2)$$
$$so(1,3) = sl(2,\mathbb{C}) \oplus \overline{sl(2,\mathbb{C})}$$
$$so(2,2) = sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$$

This means that thinking about the spin connection as the sum of its self-dual and anti-self-dual parts is a good idea

We already remarked that the full spin connection has too many "momentum" variables, some of them are eliminated by second class constraints. Here is the God-given

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Decomposition of the Riemann tensor

In any dimension, Riemann tensor splits into different irreducible (with respect to the Lorentz group) irreducible (with respect to the Lorentz group) components – Weyl, Ricci, scalar.

In four dimensions, there is another decomposition, which is into self- anti-self-dual pieces.

They encode the same information. But because Einstein equations can be stated as $\text{Ricci}\big|_{\text{tracefree}} = 0$ we get a completely new, specific only to 4D, way of encoding the Einstein condition

In details

Consider the Hodge operator (of a given metric)

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$$* : \underline{\Lambda}^2 \to \underline{\Lambda}^2 \qquad *B_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} B_{\rho\sigma}$$

It squares to $\pm$ id, depending on the signature

$$*^2 = \sigma \cdot \mathrm{id} \qquad \sigma = \begin{cases} +1 & \text{Euclidean, Split} \\ -1 & \text{Lorentzian} \end{cases}$$

This means that the space of 2-forms on M splits into its self- and anti-self-dual parts

$$\underline{\Lambda}^2 = \underline{\Lambda}^+ \oplus \underline{\Lambda}^-$$

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$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

$$B^\pm \in \Lambda^\pm \qquad *B^\pm = \pm\sqrt{\sigma}\, B^\pm$$

In Lorentzian signature $\Lambda^\pm$ are complex!

Concretely let

There is a deep relation between

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Concretely let

$$P^\pm = \frac{1}{2}\left(\mathbb{1} \pm \frac{1}{\sqrt{\sigma}} *\right)$$

There is a deep relation between necessity of using complex numbers and Lorentz signature!

be the projectors on $\Lambda^\pm$

Then any 2-form gets represented as the sum of its self-dual anti-self-dual parts

$$B = B^+ + B^- \qquad B^\pm = P^\pm B$$

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$$B = B^{+} + B^{-} \qquad B^{\pm} = P^{\pm} B$$

In Lorentzian signature, if B is real then $B^{+} = (B^{-})^{*}$

No relation between Λ^{+} and Λ^{-} in the other signa

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No relation between $\underline{\Lambda}^+$ and $\underline{\Lambda}^-$ in the other signatures

Riemann curvature can be viewed as a map

$$\text{Riemann} : \Lambda^2 \to \Lambda^2 \qquad R_{\mu\nu}{}^{\rho\sigma}$$

can decompose $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ and thus get SD/ASD parts of the Riemann curvature

Lemma: $A := P^+ \,\text{Riemann}\, P^+ = P_+ \left(\text{Weyl} + \frac{R}{6} \mathbb{1} \right) P_+$

$C := P^- \,\text{Riemann}\, P^- = P_- \left(\text{Weyl} + \frac{R}{6} \mathbb{1} \right) P_-$

$\underline{\text{Riemann}}$: $\Lambda^2 \to \Lambda^2$ $\qquad R_{\mu\nu\rho\sigma}$

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$B := P^+ \,\text{Riemann}\, P^-$ $\qquad B = 0 \iff \text{Ricci}|_{tf} = 0$

In other words, Riemann can be thought of as consisting of 3×3 blocks

$$\text{Riemann} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

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In other words, Riemann can be thought of as consisting of 3×3 blocks

$$\text{Riemann} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

A: $W^+ + R\cdot\mathbb{1}$

B: Ricci$|_{tf}$

C: $W^- + R\cdot\mathbb{1}$

With A, C having equal traces = multiples of R

With A, C having equal traces – multiples of R and tracefree parts encoding Weyl$^{\pm}$, and B encoding the tracefree part of Ricci

Lemma: $B = P^+ \,\text{Riemann}\, P^-$ vanishes if and only if Riemann commutes with the Hodge star

$$* \,\text{Riemann} = \text{Riemann} \, *$$

Corollary: Since B encodes the tracefree part of Ricci, a metric is Einstein if and only if Riemann tensor viewed as an endomorphism of Λ^2 commutes with the Hodge star.

4 / 30

220 %

5 / 30

Proof of $\quad P^+ \text{Riemann}\, P^- = 0 \iff * \text{Riemann} = \text{Riemann} *$

$$P^+ \text{Rieman}\, P^- = \frac{1}{2}\left(\mathbb{1} + \frac{1}{\sqrt{\sigma}} *\right) \text{Riemann}\, \frac{1}{2}\left(\mathbb{1} - \frac{1}{\sqrt{\sigma}} *\right)$$

$$= \frac{1}{4}\left(\text{Riemann} + \frac{1}{\sqrt{\sigma}}\left(* \text{Riemann} - \text{Riemann} * \right) - \frac{1}{\sigma} * \text{Riemann} * \right)$$

Easiest to see when $\sqrt{\sigma} = i$

Then equating to zero real and imaginary pa

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$$P^+ \, \text{Rieman} \, P^- = \underbrace{\frac{1}{2}\left(\mathbb{1} + \frac{1}{\sqrt{\sigma}} *\right)}_{P^+} \text{Riemann} \, \underbrace{\frac{1}{2}\left(\mathbb{1} - \frac{1}{\sqrt{\sigma}} *\right)}_{P^-}$$

$$= \frac{1}{4}\left(\text{Riemann} + \frac{1}{\sqrt{\sigma}}\left(*\text{Riemann} - \text{Riemann} * \right) - \frac{1}{\sigma} \underline{*\,\text{Riemann}\,*} \right)$$

Easiest to see when $\underline{\sqrt{\sigma} = i}$

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σ

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Then equating to zero real and imaginary parts

$$* \,\text{Riemann} = \text{Riemann}\, *$$

$$\text{Riemann} = \frac{1}{\sigma} * \text{Riemann}\, *$$

One implies the other because $*^2 = \sigma \cdot \mathbb{1}$

Hodge star.

Proof: $* \text{Riemann} = \text{Riemann} *$ is equivalent to

$$R\, \delta^{\alpha}_{[\mu} \delta^{\beta}_{\nu]} - 2\, \delta^{\alpha}_{[\mu} R^{\beta}_{\nu]} + 2\, \delta^{\beta}_{[\mu} R^{\alpha}_{\nu]} = 0$$

which is equivalent to $R^{\alpha}{}_{\beta} = (R/4)\, \delta^{\alpha}_{\beta}$

On the other hand $P^{+} \text{Rieman}\, P^{-} = 0 \iff$

$* \text{Riemann} = \text{Riemann} *$

Corollary: To impose Einstein condition, only need access to 'just one of the two rows of the matrix

$$\text{Riemann} = \begin{pmatrix} A & B \\ B^{T} & C \end{pmatrix}$$

Riemann commutes with the Hodge star

$$* \text{Riemann} = \text{Riemann} *$$

Corollary: Since B encodes the tracefree part of Ricci, a metric is Einstein if and only if Riemann tensor viewed as an endomorphism of Λ^2 commutes with the Hodge star.

$\text{Rieman} = \sigma * \text{Riemann} *$

Proof: $* \text{Riemann} = \text{Riemann} *$ is equivalent to

$$R\, \delta^{\alpha}_{[\mu} \delta^{\beta}_{\nu]} - 2\, \delta^{\alpha}_{[\mu} R^{\beta}_{\nu]} + 2\, \delta^{\beta}_{[\mu} R^{\alpha}_{\nu]} = 0$$

which is equivalent to $R^{\alpha}_{\beta} = (R/4)\, \delta^{\alpha}_{\beta}$

On the other hand $P^{+} \text{Rieman}\, P^{-} = 0 \iff B = 0$

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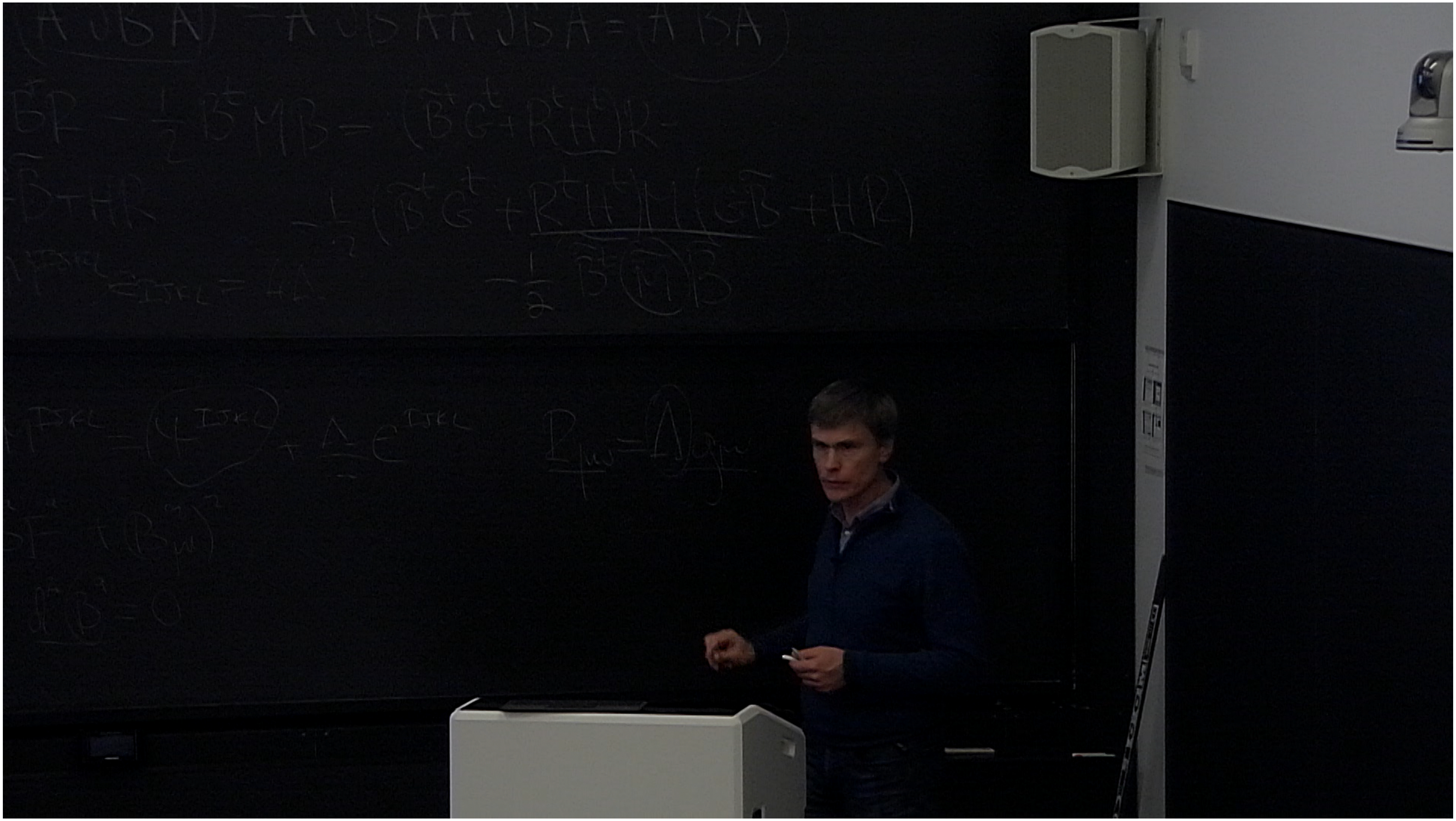

$$R\,\delta_{[\mu}{}^{\sigma}\delta_{\nu]} - 2\,\delta_{[\mu}R_{\nu]} + 2\,\delta_{[\mu}R_{\nu]} = 0$$

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$\Updownarrow$ $R_{\mu\nu} \sim g_{\mu\nu}$

<u>Corollary</u>: To impose Einstein condition, only need access to just one of the two rows of the matrix

$$\underline{\text{Riemann}} = \begin{pmatrix} A & B \\ B^{T} & C \end{pmatrix}$$

I.e. only need the self-dual projection of Riemann with respect to one of the pair of indices

To put this to use we recall that the curvature of the spin connection (metric torsion-free) encodes all of Riemann.

This means that we only need

$$\overset{+}{R}{}^{IJ}_{\mu\nu}(\omega) = \frac{1}{2}\left(\delta^{I}_{[K}\delta^{J}_{L]} + \frac{1}{2\sqrt{\sigma}}\epsilon^{IJ}{}_{KL}\right) R^{KL}_{\mu\nu}(\omega)$$

But because Lie algebra of Lorentz group splits into two commuting SD/ASD parts we have

$$R^{+\,IJ}(\omega) = R^{IJ}(\omega^{+})$$

Self-dual part of the curvature of the spin

Spin connection (metric torsion-free) encodes all of Riem

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But because Lie algebra of Lorentz group splits into two commuting SD/ASD parts we have

$$R^{+\,IJ}(\omega) = R^{IJ}(\omega^{+})$$

Self-dual part of the curvature of the spin connection equals the curvature of the self-dual part of the spin connection spin connection to impose the Einstein condition

Chiral Einstein-Cartan action

Because $*^2$ is a multiple of identity can always add to the Einstein-Cartan kinetic term

$$\int (ee) * R$$

a multiple of $\int (ee) *^2 R = \sigma \int e^I e^J R_{IJ} \sim \sigma \int \overset{\omega}{d} e^I \overset{\omega}{d} e_I$

vanishes (modulo surface term) when torsion is zero

Can adjust the coefficient in the added term so as to obtain the self-dual projector

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Thus, modulo a surface term, get Einstein-Cartan equal to its chiral version

6 / 30

220 %

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Thus, modulo a surface term, get Einstein-Cartan equal to its chiral version

$$S_{chiral}[e,\omega] = \frac{\sqrt{\sigma}}{8\pi G}\int e_I e_J \underline{P^{+IJ}{}_{KL}}\left(R^{KL}(\omega) - \frac{\Lambda}{6} e^K e^L\right)$$

$$= \frac{\sqrt{\sigma}}{8\pi G}\int e_I e_J R^{IJ}(\omega^+) - \frac{\Lambda}{24\sqrt{\sigma}} \epsilon_{IJKL} e^I e^J e^K e^L$$

We already know the conceptual explanation of why it is possible to only keep ω^+ in the action.

7 / 30

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ω^+ has just 12 components - half of components of ω

Thus, modulo a surface term, get Einstein-Cartan equation to its chiral version

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The mismatch 16 components of e^I_μ

vs. 12 components of ω^{+IJ}_μ

220 %

$$= \frac{\sqrt{\sigma}}{8\pi G}\int e_I e_J R^{IJ}(\omega^+) - \frac{\Lambda}{24\sqrt{\sigma}}\epsilon_{IJKL}\, e^I \ldots$$

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ω^+ has just 12 components - half of components of ω

The mismatch 16 components of e^I_μ

vs. 12 components of ω^{+IJ}_μ

is now just due to gauge.

No second class constraints in this chiral version of the Einstein - Cartan theory

Price to pay for this economy - the action is not

7/30 220 %

Comparison with chiral version of YM theory

$$\mathcal{L}_{YM} = \frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2$$

However, the term $\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$ is a total derivative - Pontryagin class

Can add any multiple of it to the Lagrangian without changing the theory

In particular, can choose this multiple to get

$$\frac{1}{2g^2} F^a_{\mu\nu} P^{+\mu\nu}{}_{\rho\sigma} F^{a\rho\sigma} = \frac{1}{2g^2}\left(F^{+a}_{\mu\nu}\right)^2$$

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where $P^{+\mu\nu}{}_{\rho\sigma} = \frac{1}{2}\left(g^{\mu}{}_{\rho}\, g^{\nu}{}_{\sigma} + \frac{1}{2\sqrt{\sigma}} \epsilon^{\mu\nu}{}_{\rho\sigma}\right)$

A chiral Lagrangian, in particular not manifestly real in Lorentzian signature

8 / 30

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A chiral Lagrangian, in particular not manifestly real in Lorentzian signature

Can also give a first-order version

$$S_{YM}[B,A] = \int B^{+a}_{\mu\nu} F^{a\mu\nu} - \frac{g^2}{2}\left(B^{+a}_{\mu\nu}\right)^2$$

self dual 2-form with values in the Lie algebra

This gives a very useful cubic formalism for YM, with nice perturbation theory

The chiral Einstein-Cartan is gravity analog of this formal

204 %

A chiral Lagrangian, in particular not manifestly real in Lorentzian signature

Can also give a first-order version F^{+}.

$$S_{YM}[B, A] = \int B^{+a}_{\mu\nu} F^{a\mu\nu} - \frac{g^2}{2} (B^{+a}_{\mu\nu})^2$$

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This gives a very useful cubic formalism for YM, with nice perturbation theory

The chiral Einstein-Cartan is gravity analog of this formalism

Spinorial form of chiral Einstein-Cartan

SD/ASD projections in 4D are most clearly performed