Title: Formulations of General Relativity (Part 1 of 4)

Speakers: Kirill Krasnov

Series: Quantum Gravity

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Abstract: The goal of this series is to collect various different formulations of General Relativity, with emphasis on four spacetime dimensions and formulations that use differential forms. The (non-exhaustive) list of formulations to be covered is per this plan:

Lecture 1): Motivations, followed by the usual Einstein-Hilbert to start with, first order Palatini, second order pure affine connection Eddington-Schroedinger.

Lecture 2) Cartan's geometry of soldering. First order Einstein-Cartan tetrad formulation, second order pure spin connection formulation, MacDowell-Mansouri formulation.

Lecture 3) Non-chiral BF-type formulations. Explicit pure spin connection Lagrangian, field redefinitions, BF plus potential for the 2-form field formulation.

Lecture 4) Chiral formulations in four dimensions. Chiral Einstein-Cartan, Plebanski formulation, pure SU(2) connection formulation, SU(2) BF plus potential for the 2-form field. Self-dual gravity. Concluding remarks.

Formulations of General Relativity

Kirill Krasnov
(Nottingham)

4 Lectures one hour each Monday and Tuesday

The goal is to collect all known Lagrangian formulations of General Relativity, with emphasis on four spacetime dimensions

Why bother?

GR person

1) Physics is independent of any choice of variables one uses to describe it. We have the formulation that is most developed (metric). Why develop unnecessary alternatives?

Field (String) theorist

2) GR is the unique low energy Lorentz-invariant interacting theory of massless spin two particles. This statement is independent of any Lagrangian formulation of it. Lagrangian is irrelevant, everything follows from amplitudes

Answers to GR person objection

Practical: some of the formulations exhibit much less non-linearity than the Einstein-Hilbert formulation. It may be (is) easier to compute in one of the alternatives

Conceptual:

> "*There is always another way to say the same thing that doesn't look at all like the way you said it before. I don't know what the reason for this is. I think it is somehow a representation of the simplicity of nature? Perhaps a thing is simple if you can describe it fully in several different ways without immediately knowing that you are describing the same thing.*"
>
> Richard Feynman, *Nobel Lecture*, 1965

One will never fully appreciate the beauty of GR without absorbing all of its equivalent but not obviously so reformulations

Another conceptual point:

> “ *Theories of the known, which are described by different physical ideas may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made and hence are not equivalent in the hypotheses one generates from them in ones attempt to understand what is not yet understood. I, therefore, think that a good theoretical physicist today might find it useful to have a wide range of physical viewpoints and mathematical expressions of the same theory available to him.* ”

Richard Feynman, *Nobel Lecture*, 1965

We all believe that GR is just an effective theory, and as such just an approximation to some more fundamental description

But it may well be that it contains hints as to where to go in the search for this more fundamental theory. Such a hint may only be apparent in one of its formulations

GR is a dynamical theory of geometry, but there are many different types of geometry apart from the metric one. If the sought fundamental theory is geometric (?), which of the types of geometry we can use to rephrase GR is holding the key?

Which geometry is to be generalised?

The fundamental question that is my personal motivation

Einstein, one year before death, speaking
to a group of Wheeler's students:

"There is much reason to be attracted to a theory with no space and no time. But nobody has any idea how to build it."

There is a metric apparently filling all of the Universe. GR describes dynamics of this metric if it is assumed to be there.

But it does not answer the question why it is there in the first place. My hope is that by thinking about geometry(ies) of GR one can get closer to answering "Why non-zero metric?"

These lectures will cover:

I) Metric and related formulations:

- Einstein-Hilbert
- First-order Palatini
- Pure connection Eddington-Schroedinger

II) Tetrad and related formulations:

- First-order Einstein-Cartan
- MacDowell-Mansouri, Stelle-West
- Pure spin connection

III) BF and related:

- BF plus constraints
- Pure spin connection in closed form
- BF plus potential

IV) Chiral formulations of 4D GR

Part I : Metric and related formulations

Historical remark:

Einstein's GR is formulated in the language of Riemannian geometry, the only type of geometry sufficiently developed in 1912 when Einstein returned to Zurich and was learning geometry with the help of his friend and ex-classmate M. Grossmann

Geometry is far reacher now as compared to what it was in 1912, thanks in particular to fundamental contributions by Cartan. Part of the motivation here is to learn to think about GR using the 20th century geometry of Cartan rather than 19th century tensor calculus of Ricci and Levi-Civita

Geometry of fibre bundles, differential forms and connections

Einstein's GR

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(Geometry of fibre bundles, differential forms and connections

Einstein's GR

$$\text{Geometry} \rightleftarrows \text{Matter}$$

G - Newton's constant
$g = \det(g_{\mu\nu})$
Λ - Cosmological constant

$$S_{EH}[g] = \frac{1}{16\pi G}\int\sqrt{-g}\,(R-2\Lambda)$$

sign in front is signature dependent plus sign for the mostly plus signature

It is interesting to remark that GR is non-Machian - geometry exists even without matter
Related to "Why non-zero metric?"

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Conventions: $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\alpha\mu} V^\alpha$ $\quad [\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\alpha\mu\nu} V^\alpha$

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} + \Gamma^\alpha_{\rho\nu}\Gamma^\sigma_{\alpha\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\alpha\nu}$$

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Einstein equations (in vacuum)

Einstein metrics

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = 0 \implies R_{\mu\nu} = \Lambda g_{\mu\nu}$$

(in 4 dimensions)

Expansion around flat space (Minkowski) - schematically

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G} h_{\mu\nu}$$

$$\mathcal{L} = (\partial h)^2 + \sqrt{G}\, h (\partial h)^2 + \dots$$

negative mass dimension coupling, power counting non-renormalizable

derivative interactions

Propagating DOF

4 components of $g_{\mu\nu}$ are Lagrange multipliers

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for 4 constraints

$10 - 4 - 4 = 2$ DOF

2 propagating polarizations of the graviton

Metric Affine formulation - first order Palatini

$$S_{Palatini}[g, \Gamma] = \frac{1}{16\pi G} \int \sqrt{-g} \left(g^{\mu\nu} R_{\mu\nu}(\Gamma) - 2\Lambda \right)$$

Variation wrt $\Gamma^{\rho}_{\mu\nu}$ gives $\nabla_{\rho} g^{\mu\nu} = 0 \Rightarrow \Gamma = \Gamma(g)$

- Only first derivatives in the action - like Hamiltonian formulation
- Note $R_{\mu\nu}(\Gamma)$ is not automatically symmetric even when $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{(\mu\nu)}$ torsion free

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- Only first derivatives in the action - like Hamiltonian formulation
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- When $\Lambda = 0$ introducing $\sigma^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}$ gives a cubic Lagrangian for GR!

a cubic Lagrangian for GR!

- Too many fields 10+40 to be of practical use

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Second order pure affine connection – Eddington formulation

When $\Lambda \neq 0$ can "integrate out" the metric instead of $\Gamma^{\rho}_{\mu\nu}$

Field equation for the metric $R_{\mu\nu}(\Gamma) = \Lambda g_{\mu\nu}$

and so $g_{\mu\nu}(\Gamma) = \frac{1}{\Lambda} R_{\mu\nu}(\Gamma)$

Metric algebraically constructed from the curvature of the connection

only possible when $\Lambda \neq 0$

Substituting back into the action gives

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Substituting back into the action gives

$$S_{\text{Eddington}}[\Gamma] = \frac{1}{8\pi G\Lambda} \int \sqrt{-\det R_{\mu\nu}(\Gamma)}$$

Field equation for Γ

$$\nabla_\mu R^{(\alpha\beta)}(\Gamma) = 0$$

Second order PDE for the affine connection

The metric constructed from solutions of this PDE

Field equation for Γ

$$\nabla_\Gamma R^{(\alpha\beta)}(\Gamma) = 0$$

Second order PDE for the affine connection

The metric constructed from solutions of this PDE is automatically Einstein

- Too many fields to be useful
- Can't expand around flat space ($\Lambda \neq 0$)
- Matter can be added without problems, the metric can be always "integrated out"
- The Lagrangian is no longer unique – can construct other invariants using the $R_{[\mu\nu]}$ part of the curvature. This is a rather general feature – other formulations typically have more ambiguity than Einstein – H

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Less familiar second order formulations - $\Gamma\Gamma$

$$\int \sqrt{-g}R = \int \sqrt{-g}\,\Gamma\Gamma + \int \partial_\mu \left(\sqrt{-g}\,\omega^\mu\right)$$

Modulo surface term can rewrite the GR action in the form $(\partial g)^2$. There is no covariant way to do this, and this is why the only covariant action is of the form $g\partial^2 g$

Explicitly

$$S_{\Gamma\Gamma}[g] = \frac{1}{16\pi G}\int \sqrt{-g}\left(g^{\rho\sigma}\left(\Gamma^\mu{}_{\nu\rho}\Gamma^\nu{}_{\mu\sigma} - \Gamma^\mu{}_{\rho\sigma}\Gamma^\nu{}_{\nu\mu}\right) - 2\Lambda\right)$$

Action explicitly in terms of the metric is of the form $g\partial^2 g$

Explicitly

$$S_{\Gamma\Gamma}[g] = \frac{1}{16\pi G}\int\sqrt{-g}\left(g^{\rho\sigma}\left(\Gamma^{\mu}_{\nu\rho}{}^{(g)}\Gamma^{\nu}_{\mu\sigma} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\nu}_{\nu\mu}\right) - 2\Lambda\right)$$

Action explicitly in terms of the metric

$$S[g] = \frac{1}{16\pi G}\int\sqrt{-g}\Big(\partial_\mu g^{\rho\alpha}\partial_\nu g_{\sigma\alpha}\left(\frac{1}{4}g^{\mu\nu}\delta^\sigma_\rho - \frac{1}{2}g^{\mu\sigma}\delta^\nu_\rho\right) - g^{\mu\nu}\partial_\mu\partial_\nu(\ln\sqrt{-g}) - 2\Lambda\Big)$$

Convenient starting point for linearization around Minkowski, but introducing $\mathfrak{G}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ can al

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Action explicitly in terms of the metric

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}\left(\partial_\mu g^{\rho\alpha}\,\partial_\nu g_{\sigma\alpha}\left(\frac{1}{4}g^{\mu\nu}\delta^\sigma_\rho - \frac{1}{2}g^{\mu\sigma}\delta^\nu_\rho\right) - g^{\mu\nu}\partial_\mu\partial_\nu\left(\ln\sqrt{-g}\right) - 2\Lambda\right)$$

Convenient starting point for linearization around Minkowski, but introducing $\mathfrak{G}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}$ can also be used for getting a very compact perturbative expansion to any order. The number of terms at every order does not proliferate, unlike on the usual $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ expansion.

Cheung, Remmen 2017

Part II: Tetrad and related formulations

Historical remarks: In 1920's Cartan, motivated partly by Einstein's GR that came to prominence in 1919, partly by his previous work on Lie algebras and differential forms, came up with a completely new way of thinking about Riemannian geometry. In fact what Cartan discovered was the most natural generalization of Riemannian geometry, sought by many during the same period, in particular by Weyl.

In Cartan's description fibre bundles and connections as Lie algebra valued 1-forms play key role. In fact, Cartan discovered connections (gauge fields) much before they appeared in physics. Mathematicians now learn about Riemannian geometry. In fact what Cartan discovered was the most natural generalization of Riemannian geometry, sought by many during the same period, in particular by Weyl.

In Cartan's description fibre bundles and connections as Lie algebra valued 1-forms play key role. In fact, Cartan discovered connections (gauge fields) much before they appeared in physics. Mathematicians now learn geometry a la Cartan. GR can be formulated using this powerful language. Differential forms and exterior derivative simplify things greatly, even technically. It is time grav. physicists take this language on board. Things are much more deep than the non-coordinate bases typically mentioned in this context in GR books.

Soldering form

technically. It is time grav. physicists take this language on board. Things are much more deep than the non-coordinate bases typically mentioned ~~in this context~~ in GR books.

Soldering form

The natural object that lives in fibre bundles (principal bundles, vector bundles) is a connection

But connection does not tie (at least not in general) the geometry of the fibre to geometry of the base in any way.

Cartan's idea is to introduce precisely such an object. This is the general idea of soldering

As we shall see, soldering has many different incarnations
We start with the most familiar one

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We start with the most familiar one

Let $V \hookrightarrow E \to M$ be a vector bundle over our manifold M, with fibres $V \sim \mathbb{R}^n$ $\quad \dim M = n$

We will also require that E is of the same topological type as TM

preparing for the fact that there will be a relation to geometry of M

Can put many different connections on E

Definition: Co-frame or a soldering form at $x \in M$ is an isomorphism

$$e : T_x M \to V$$

same topological type as TM

Can put many different connections on E

Definition: Co-frame or a soldering form at $x \in M$ is an isomorphism

$$e : T_x M \to V$$

Locally, it is a 1-form on M with values in V

$$e^I = e^I_\mu \, dx^\mu$$

Extending to all points of M get an isomorphism

$$e : TM \to E$$

This is an object that ties geometry of the fib

preparing for the fa[ct] that there will be a relation to geometry of M.

This is an object that ties geometry of the fibre to that of the base

Note that $GL(n,\mathbb{R})$ acts (transitively) on the space of coframes at a point

$$GL(n,\mathbb{R}) \ni g : e \to g^{-1} e$$

coframe map followed by g^{-1} on V

This makes the space of coframes on M into the principal $GL(n,\mathbb{R})$ bundle over M

We did not yet put any geometric structure either on V or on M. If we do this we are led to the notion of G-structures.

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Geometric structures

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Geometric structures

We now want to put some geometric structure on M.
This can be e.g. metric, but it can also be a non-degenerate 2-form, or an almost complex structure
$I : TM \to TM \quad I^2 = -\mathbb{1}$

It will be very convenient to fix some model object of this type in V. Thus, we take either

Q – non-degenerate symmetric tensor in $V^* \otimes V^*$ (metric)

ω – non-degenerate anti-symmetric tensor in $V^* \otimes V^*$

I – map $I : V \to V$ s.t. $I^2 = -\mathbb{1}$

In each case the group $GL(n, \mathbb{R})$ acting on V reduces to a subgroup of transformations that p

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ω – non-degenerate anti-symmetric tensor in $V^* \otimes V^*$

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In each case the group $GL(n, \mathbb{R})$ acting on V reduces to a subgroup of transformations that preserve the model object

$O(V, Q)$ orthogonal group

$Sp(V, \omega)$ symplectic group

$GL(m, \mathbb{C})$ general linear complex $2m = n$

Let us now put on M a geometric structure of one of the above types – metric, non-degenerate 2-form, ACS

Definition: A co-frame (soldering) is called adapted to a given geometric structure if this structure arises

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I.e. $e: TM \to E$ is adapted to a metric $g \in S^2 T^*M$ if g is the pull back via e from Q in E

$$g_{\mu\nu} = e^I_\mu e^J_\nu Q_{IJ}$$

metric on M — the model metric

Similarly, $e: TM \to E$ is adapted to $B \in \Lambda^2 M$ if

$$B = e^*(\omega) \qquad B_{\mu\nu} = e^I_\mu e^J_\nu \ldots$$

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$e : TM \to E$ is adapted to $J \in TM \otimes TM^*$ if

$$J_\mu{}^\nu = e^I_\mu e^\nu_J I_I{}^J$$

inverse soldering (or frame)

The space of coframes adapted to a given geometric structure becomes a principal G bundle over M, where

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The space of coframes adapted to a given geometric structure becomes a principal G bundle over M, where G is one of the $GL(n,\mathbb{R})$ subgroups, i.e. $O(V,G)$, $Sp(V,\omega)$ or $GL(m,\mathbb{C})$

We can now put a connection on this principal G bundle. This gives rise to a connection in E that preserves the model geometric structure in V. Concretely, a 1-form

inverse soldering
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$GL(m,\mathbb{C})$

The pull-back of this connection to TM via the soldering map gives an affine connection on TM model geometric structure in V. Concretely, a 1-form on M with values in the Lie algebra of $O(V,Q)$, $Sp(V,\omega)$, $GL(m,\mathbb{C})$

The pull-back of this connection to TM via the soldering map gives an affine connection on TM

This is how soldering relates the vector bundle with some geometric structure and a connection that preserves this structure to a geometric structure on the manifold, and an affine connection.

Concretely $\quad u^I = e_\mu^I u^\mu$

$$e_\mu^I \nabla u^\mu := d^\omega u^I = du^I + \omega^I{}_J u^J$$

or

total cov. derivative

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or

$$e^I{}_\sigma \Gamma^\sigma_{\nu\mu} = \partial_\mu e_\nu^I + \omega_\mu{}^I{}_J e^J{}_\nu$$

total cov. derivative is zero

Can rewrite as $\boxed{\nabla^\omega_\mu e_\nu^I = \partial_\mu e_\nu^I + \omega_\mu{}^I{}_J e^J{}_\nu - \Gamma^\sigma_{\nu\mu} e_\sigma^I = 0}$

The above story is completely general, and allows to treat metric, symplectic and complex geometry in parallel.

Can rewrite as $\nabla^{\omega}_{\mu} e^{I}_{\nu} = \partial_{\mu} e^{I}_{\nu} + \omega_{\mu}{}^{I}{}_{J}\, e^{J}_{\nu} - \Gamma^{\sigma}_{\nu\mu}\, e^{I}_{\sigma} = 0$

The above story is completely general, and allows to treat metric, symplectic and complex geometry in parallel.

We now specialize to the metric case

Torsion: $T^{I} = d^{\omega} e^{I}$ 2-form on M with values in V

torsion only exists because of soldering.
There is no such thing in a general bundle

Fundamental lemma:

There exists a unique torsion free and metric connection in E. It is called the spin connection. Explicitly

$$\omega_{\mu}{}^{I}{}_{J} = e^{\rho I} e^{\sigma}_{J} \left(- C_{\mu\rho\sigma} + C_{\rho\sigma\mu} + C_{\sigma\mu\sigma} \right)$$

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where $C_{\mu\rho\sigma} = e_{\mu I}\, \partial_{[\rho} e^I_{\sigma]}$

The objects $e_{\mu I} = e^J_\mu Q_{IJ}$ $\quad e^{\rho I} = e^\rho_J Q^{IJ}$

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There is no unique (torsion-free) connection in the symplectic and complex cases, and this is why metric geometry is somewhat exceptional

It is then easy to check that for the metric $g = e^*(Q)$ the metric and torsion free affine connection Γ is the

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It is then easy to check that for the metric $g = e^*(Q)$ the metric and torsion free affine connection Γ is the pull-back of the metric and torsion free connection on E

Indeed $\nabla g_{\mu\nu} = \nabla^\omega g_{\mu\nu} = \nabla^\omega (e^I_\mu e^J_\nu Q_{IJ}) = 0$

$$e^I_\sigma \Gamma^\sigma_{[\mu\nu]} = \partial_{[\mu} e^I_{\nu]} + \omega_{[\mu}{}^I{}_J e^J_{\nu]} = T^I = 0$$

Riemann curvature is curvature of the spin connection

$$0 = 2\nabla^\omega_{[\mu} \nabla^\omega_{\nu]} e^I_\rho = R_{\mu\nu}{}^I{}_J e^J_\rho - R^\alpha{}_{\rho\mu\nu} e^I_\alpha$$

where $R^I{}_J = d\omega^I{}_J + \omega^I{}_K \omega^K{}_J$ curvature 2-form

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Riemann curvature is curvature of the spin connection

$$0 = 2\nabla^\omega_{[\mu} D^\omega_{\nu]} e^I_\rho = R_{\mu\nu}{}^I{}_J(\omega) e^J_\rho - R^\alpha{}_{\rho\mu\nu}(\Gamma) e^I_\alpha$$

where $R^I{}_J = d\omega^I{}_J + \omega^I{}_K \omega^K{}_J$ curvature 2-form

This translates all operations in computing connection and curvature to working with differential forms $e^I, \omega^I{}_J$ and computing the curvature $R^I{}_J(\omega)$

This is much more efficient for explicit computations

$$0 = 2\nabla^{\omega}_{[\mu} D^{\omega}_{\nu]} e^I_\rho = R_{\mu\nu}{}^I{}_J(\omega) e^J_\rho - R^{\alpha}{}_{\rho\mu\nu}(\Gamma)\, e^I_\alpha$$

where $R^I{}_J = d\omega^I{}_J + \omega^I{}_K \omega^K{}_J$ curvature 2-form

This translates all operations in computing connection and curvature to working with differential forms e^I, $\omega^I{}_J$ and computing the curvature $R^I{}_J(\omega)$

This is much more efficient for explicit computations than the usual $\Gamma^\rho_{\mu\nu}$ way. 24 components of $\omega^I_{\mu}{}_J$ as compared to 40 components of $\Gamma^\rho_{\mu\nu}$

Torsion-free affine connection $\Gamma^\rho_{\mu\nu}$ is not a principal connection, and is not a differential form. Soldering maps it to a principal connection and a differential form. Working with differential forms is easy

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Einstein-Cartan first order formulation

Can write the Lagrangian in terms of the wedge product of

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Einstein-Cartan first order formulation

Can write the Lagrangian in terms of the wedge product of differential forms

4D theory (other dimensions analogously)

$$S_{EC}[e,\omega] = \frac{1}{32\pi G}\int \epsilon_{IJKL}\, e^I e^J \left(R^{KL}(\omega) - \frac{\Lambda}{6} e^K e^L \right)$$

need to choose orientation of both M and V to make sense of this integral

The second index on $R^K{}_L$ is raised using the metric in V

Varying wrt spin connection gives the torsion-free condition

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differential forms

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The second index on $R^K{}_L$ is raised using the metric in V

Varying wrt spin connection gives the torsion-free condition

$$d^\omega e^I = 0$$

Varying wrt the coframe gives Einstein equations

$$\epsilon_{IJKL}\, e^J R^{KL} = \frac{\Lambda}{3} \epsilon_{IJKL}\, e^J e^K e^L$$

First order formulation with 16+24 field components

- Polynomial (quartic) even when $\Lambda \neq 0$
- This formulation is unavoidable if wants to couple spinors to gravity, as spinors couple directly to the spin connection (hence the name)
- A drawback of this formulation is mismatch in #'s components of e^I_μ and $\omega_\mu{}^I{}_J$. There are too many "momenta" variables. Some of them are redundant and eliminated by second class constraints
- In contrast with metric GR, this formulation does not require $e^I_\mu \neq 0$. Indeed, all field being zero is still a solution of all the equations. This is a step closer to addressing "why non-zero metric" question. However, this question is of course not answered by this formalism, as it does not explain why e should be non-zero rather than zero. Also, there is no kinetic term around $e = 0$

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Teleparallel formulation

As in the metric case it was possible to rewrite th

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