Title: Connecting affine Yangians with W-algebras

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Collection: Cohomological Hall Algebras in Mathematics and Physics

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Abstract: Alday-Gaiotto-Tachikawa connect instanton counts in gauge theory with conformal blocks for W-algebras. We realize this mathematically by relating q-deformed W-algebras with the affine q-Yangians that control gauge theory, thus offering an affine, q-deformed generalization of the well-known Brundan-Kleshchev construction in type A

AGT from an algebro-geometric perspective

smooth projective $S \supset H$ ample

<u>Def</u>: let $\mathcal{M}_r$ be the moduli stable sheaves $\mathcal{F}$ on S, of $\mathrm{rank}(\mathcal{F}) = r$ and fixed c_1 but arbitrary $c_2 \in \mathbb{Z}$

↳ bounded below $\geq \left\lceil \frac{r-1}{2r} c_1^2 \right\rceil$

• Assume $K_S \cong \mathcal{O}_S$ or $c_1(K_S)\cdot H < 0$

$\Rightarrow \mathcal{M}_r$ smooth

Assume $\gcd(r, c_1(\mathcal{F})\cdot H) = 1$

↳ $\exists$ a universal sheaf $\mathcal{U} \leftarrow \mathcal{F}$

$$\begin{array}{ccc} \mathcal{U} & \leftarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{M}\times S & \hookleftarrow & \{\mathcal{F}\}\times S \end{array}$$

Thm 1 (N) the K-theory group $K(\mathcal{M})$ is a <u>module</u> for the <u>algebra</u> $W_{q_1,q_2}(\mathfrak{gl}_r)$

↳ $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{\mathrm{sym}}$-algebra given by explicit generators and relation

$\exists$ abelian group homomorphism

$$W_{q_1,q_2}(\mathfrak{gl}_r) \longrightarrow \mathrm{Hom}(K(\mathcal{M}), K(\mathcal{M}\times S))$$
$$a \rightsquigarrow \phi_a$$

sati

• ϕ

$\to \mathcal{F}$

$\to \{\mathcal{F}\}\times S$

$M)$

$W_{q_1,q_2}(\mathfrak{gl}_n)$

$q_2^{\pm 1}]^{Sym}$ - algebra

by explicit

s and relation

satisfying compatibility conditions: $\forall\, a,b \in W_{q_1,q_2}(\mathfrak{gl}_n)$, $c \in \mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]^{Sym}$

- $\phi_{ab} = K(M) \xrightarrow{\phi_b} K(M\times S) \xrightarrow{\phi_a \times Id_S} K(M\times S\times S) \xrightarrow{Id_M \times \Delta^*} K(M\times S)$
- $\phi_{c\cdot a} = K(M) \xrightarrow{\phi_a} K(M\times S) \xrightarrow{Id_M \times \text{mult by } f(c)} K(M\times S)$

where $f: \mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]^{Sym} \to K(S)$ is the ring homomorphism sending $q_1+q_2 \to [\Omega^1_S]$, $q = q_1 q_2 \to [K_S]$

- $[\phi_a, \phi_b] = \Delta_*\left(\phi_{\frac{[a,b]}{(1-q_1)(1-q_2)}}\right)$

where the LHS is the difference of the following two compositions

$K(M) \xrightarrow{\phi_b} K(M\times S) \xrightarrow{\phi_a \times Id_S} K(M\times S\times S)$

$K(M) \xrightarrow{\phi_a} K(M\times S) \xrightarrow{\phi_b \times Id_S} K(M\times S\times S) \xrightarrow{Id_M \times flip^*} K(M\times S\times S)$

Sym

Thm 2 (N) the bifundamental matter contribution to gauge, which is encoded by the Ext operator $A_m : K(\mathcal{M}'_r) \longrightarrow K(\mathcal{M}_r)$

$\overset{\Uparrow}{\mathrm{Pic}(S)}$

"commutes" with the $W_{q_1,q_2}(\mathfrak{gl}_r)$ algebra action of $K(\mathcal{M}_r)$

$$\begin{array}{ccc} E_m & \longleftrightarrow & R\mathrm{Hom}(\mathcal{F}', \mathcal{F} \otimes m) \\ \vdots & & \vdots \\ \mathcal{M}_r \times \mathcal{M}'_r & \ni & (\mathcal{F}, \mathcal{F}') \end{array}$$

$$\mathcal{M}_r \xleftarrow{\pi_1} \mathcal{M}_r \times \mathcal{M}'_r \xrightarrow{\pi_2} \mathcal{M}'_r$$

$$A_m = \pi_{1*}\left(\left[\wedge^{\bullet} E_m\right] \otimes \pi_2^*\right)$$

Def (F

$$E_m \longleftrightarrow R\mathrm{Hom}(\mathcal{F}', \mathcal{F}\otimes m)$$

$$\begin{tikzcd} & \mathcal{M}_r \times \mathcal{M}'_r \ni (\mathcal{F}, \mathcal{F}') \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_r & & \mathcal{M}'_r \end{tikzcd}$$

...amental

... which

...nerator $A_m : K(\mathcal{M}'_r) \longrightarrow K(\mathcal{M}_r)$

$m \in \mathrm{Pic}(S)$

$..._2(\mathfrak{gl}_r)$ algebra action of $K(\mathcal{M}_r)$

$$A_m = \pi_{1*}\left(\left[\Lambda^\bullet E_m\right] \otimes \pi_2^*\right)$$

$$\zeta(x) = \frac{(1-xq_1)(1-xq_2)}{(1-x)(1-xq)}$$

$q = q_1 q_2$

Def (Feigin-Frenkel-Reshetikhin, Awata-Kubo-Odake-Shiraishi)

$\mathcal{W}_{q_1,q_2}(\mathfrak{gl}_r)$ is the $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{sym}$-algebra generated by $\{W_{d,k}\}^{d\in\mathbb{Z}}_{1\le k\le r}$ modulo quadratic relations $\left(\text{write } W_k(x) = \sum_{d\in\mathbb{Z}} \frac{W_{d,k}}{x^d}\right)$

$$W_k(x)\, W_L(y) \prod_{i=\max(0,k-L)}^{k-1} \zeta\left(\frac{y q^i}{x}\right) - W_L(y)\, W_k(x) \prod_{i=\max(0,L-k)}^{L-1} \zeta\left(\frac{x}{y q^i}\right) =$$

$$= (1-q_1)(1-q_2) \sum_{\text{various } i\text{'s}} W_i(x)\, W_{k+L-i}(y)\, \delta\left(\frac{x}{y q^*}\right) \underset{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{sym}}{\mathrm{const}_{i,k,L}}$$

$$E_m \longleftrightarrow R\mathrm{Hom}(\mathcal{F}', \mathcal{F} \otimes m)$$

$$\begin{array}{ccc} & \mathcal{M}_n \times \mathcal{M}_n' \ni (\mathcal{F}, \mathcal{F}') & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{M}_n & & \mathcal{M}_n' \end{array}$$

...amental

... which

...nerator $A_m : K(\mathcal{M}_n') \to K(\mathcal{M}_n)$

$\qquad\qquad \uparrow$

$\qquad\quad \mathrm{Pic}(S)$

$_2(\mathfrak{gl}_n)$ algebra action of $K(\mathcal{M}_n)$

$$A_m = \pi_{1*}\left(\left[\wedge^\bullet E_m\right] \otimes \pi_2^*\right)$$

$$\boxed{\zeta(x) = \frac{(1-xq_1)(1-xq_2)}{(1-x)(1-xq)} \qquad q = q_1 q_2}$$

Def (Feigin - Frenkel - Reshetikhin, Awata - Kubo - Odake - Shiraishi)

$\mathcal{W}_{q_1,q_2}(\mathfrak{gl}_n)$ is the $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{\mathrm{sym}}$ - algebra generated by $\{W_{d,k}\}_{1 \leq k \leq n}^{d \in \mathbb{Z}}$ modulo quadratic relations $\left(\text{write } W_k(x) = \sum_{d \in \mathbb{Z}} \frac{W_{d,k}}{x^d}\right)$

$$W_k(x) W_L(y) \prod_{i=\max(0,k-L)}^{k-1} \zeta\left(\frac{y q^i}{x}\right) - W_L(y) W_k(x) \prod_{i=\max(0,L-k)}^{L-1} \zeta\left(\frac{x}{y q^i}\right) =$$

expand in $|y| \ll |x|$ $\qquad\qquad$ expand in $|x| \ll |y|$

$$= (1-q_1)(1-q_2) \sum_{\text{various } i\text{'s}} W_i(x) W_{k+L-i}(y)\, \delta\left(\frac{x}{y q^*}\right) \mathrm{const}_{i,k,L}$$

$$\mathrm{const}_{i,k,L} \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{\mathrm{sym}}$$

$$\mathcal{M} = \mathcal{M}_r$$

$W_{q_1,q_2}(\mathfrak{gl}_n)$, $c \in \mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]^{sym}$

$\mathrm{Id}_{\mathcal{M}} \times \Delta^*$

$\longrightarrow K(\mathcal{M} \times S)$

$f(c)$

$\longrightarrow K(\mathcal{M} \times S)$

in the ring

$q_2 \to [\Omega_S^1]$

$q_1 q_2 \to [K_S]$

d_S

$\to K(\mathcal{M} \times S \times 3)$

Thm 2 (N) the bifundamental matter contribution to gauge, which is encoded by the Ext operator $A_m : K(\mathcal{M}_r') \to K(\mathcal{M}_r)$, $m \in \mathrm{Pic}(S)$

"commutes" with the $W_{q_1,q_2}(\mathfrak{gl}_n)$ algebra action of $K(\mathcal{M}_r)$

$$A_m \circ \Phi_{W_k(x)}(1-x) = m^k \, \Phi_{W_k(\gamma x)}\left(1 - \frac{x}{q^k}\right) \circ A_m$$

as an identity of operators $K(\mathcal{M}') \to K(\mathcal{M} \times S)$

where $\gamma = \dfrac{m^r}{q^r} \dfrac{\det \mathcal{F}}{\det \mathcal{F}'}$

$$E_m \longleftrightarrow R\mathrm{Hom}(\mathcal{F}', \mathcal{F} \otimes m)$$

$$\mathcal{M}_r \times \mathcal{M}_r' \ni (\mathcal{F}, \mathcal{F}')$$

$\pi_1 : \mathcal{M}_r \times \mathcal{M}_r' \to \mathcal{M}_r$, $\pi_2 : \mathcal{M}_r \times \mathcal{M}_r' \to \mathcal{M}_r'$

$$A_m = \pi_{1*}\left([\wedge^{\bullet} E_m] \otimes \pi_2^*\right)$$

$$\zeta(x) = \frac{(1-xq_1)(1-xq_2)}{(1-x)(1-xq)}$$

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W

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Def (Feigin-Frenkel-Reshetikhin, Awata-Kubo-Odake-Shiraishi)

$\mathcal{W}_{q_1,q_2}(\mathfrak{gl}_n)$ is the $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{sym}$-algebra generated by $\{W_{d,k}\}^{d\in\mathbb{Z}}_{1\le k\le n}$ modulo quadratic relations (write $W_k(x) = \sum_{d\in\mathbb{Z}} \frac{W_{d,k}}{x^d}$)

expand in $|x| \ll |y|$

$$W_k(x)\,W_L(y) \prod_{i=\max(0,k-L)}^{k-1} \varphi\left(\frac{yq^i}{x}\right) - W_L(y)\,W_k(x) \prod_{i=\max(0,L-k)}^{L-1} \varphi\left(\frac{x}{yq^i}\right) =$$

expand in $|y| \ll |x|$

$$= (1-q_1)(1-q_2) \sum_{\text{various } i\text{'s}} W_i(x)\, W_{k+L-i}(y)\, \delta\left(\frac{x}{yq^*}\right) \mathrm{const}_{i,k,L}$$

$\mathrm{const}_{i,k,L} \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{sym}$

Corollary

a $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-

...a-Kubo-(Odake-Shiraishi)

generated

...ons (write $W_k(x)=\sum_{d\in\mathbb{Z}} \frac{W_{d,k}}{x^d}$)

$\dots(y)\,\dots x) \prod_{i=\max(0,L-k)}^{L-1} \varphi\left(\frac{x}{y q_2^{i}}\right) =$

$\dots_{L-i}(y)\,\delta\left(\frac{x}{y q_2^{*}}\right)\,\mathrm{const}_{i,k,L}$

$\mathrm{const}_{i,k,L} \in \mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]^{\mathrm{sym}}$

<u>Corollary</u>: $\overbrace{W_{q_1,q_2}(\mathfrak{gl}_n)}^{\leadsto \text{ allow infinite sums}}$ has a $\mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]$-basis given by

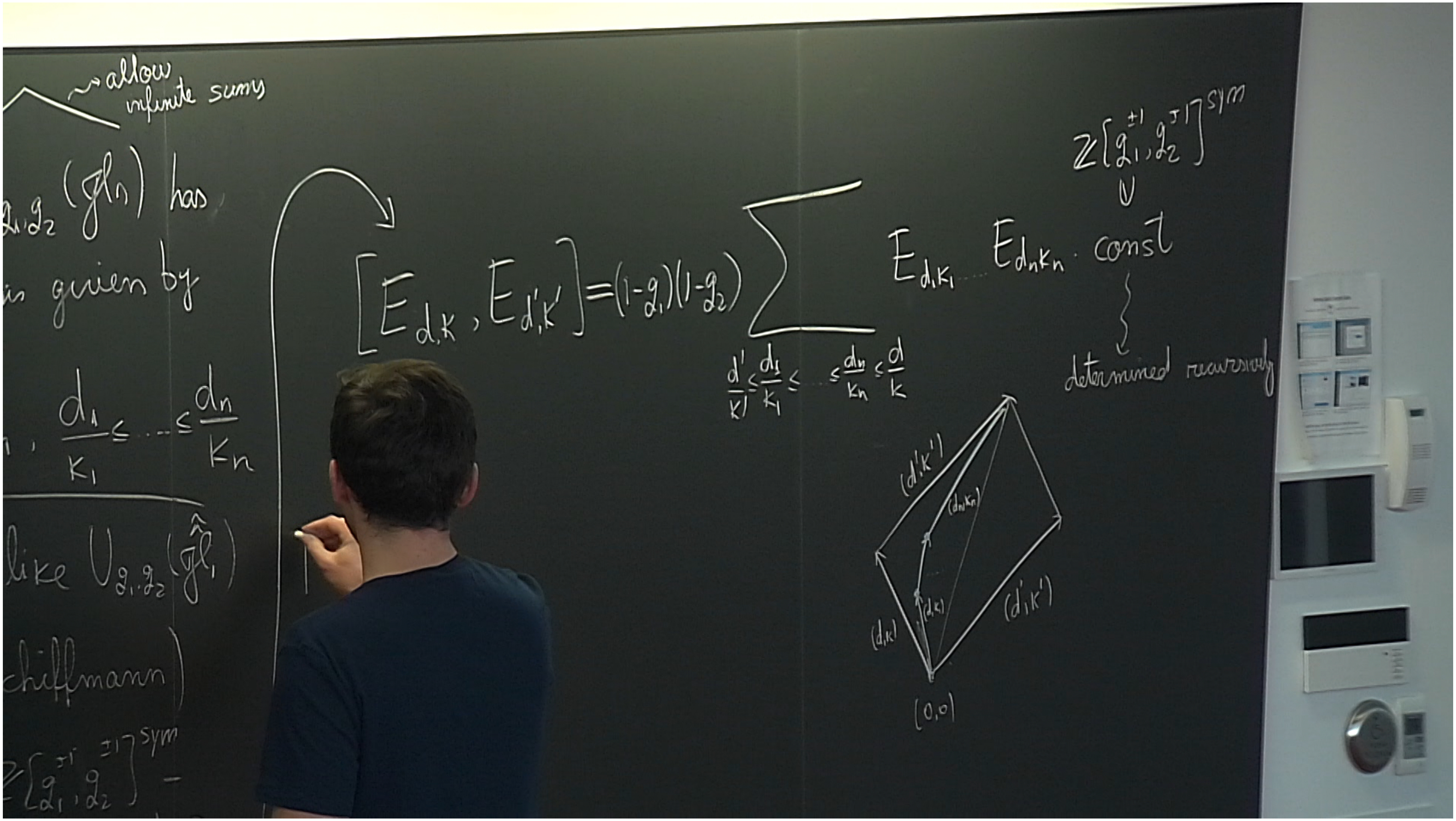
$W_{d_1k_1}\dots W_{d_nk_n}$, $\frac{d_1}{k_1}\le \dots \le \frac{d_n}{k_n}$

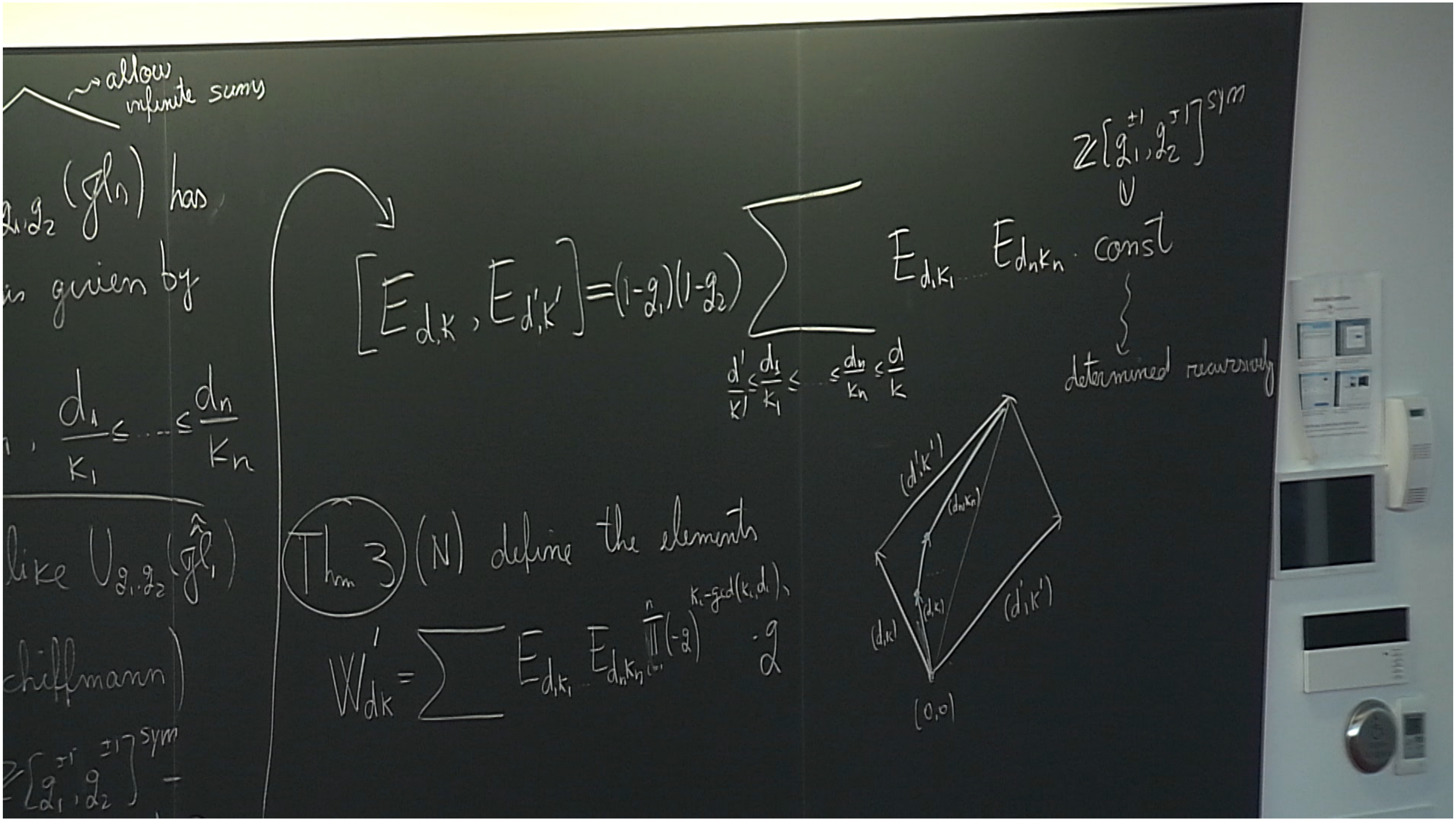
this looks a lot like $U_{q_1,q_2}(\hat{\mathfrak{gl}}_1)$

<u>Def</u> (Burban-Schiffmann) $U_{q_1,q_2}(\hat{\mathfrak{gl}}_1)^+$ is the $\mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]^{\mathrm{sym}}$-algebra generated by $\{E_{d,k}\}_{k>0}^{d\in\mathbb{Z}}$ modulo relations

$$[E_{d,k}, E_{d',k'}] = (1-q_1)(1-q_2)\sum_{\frac{d'}{k'}\le\frac{d_1}{k_1}\le\dots\le\frac{d_n}{k_n}\le\frac{d}{k}} E_{d_1k_1}\dots E_{d_nk_n}\cdot \underbrace{\mathrm{const}}_{\text{determined recursively}}$$

$\mathrm{const} \in \mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]^{\mathrm{sym}}$


~> allow infinite sums
$g_1, g_2(\mathfrak{gl}_n)$ has
is given by
$\frac{d_1}{k_1} \le \dots \le \frac{d_n}{k_n}$
like $U_{g_1,g_2}(\hat{\mathfrak{gl}}_1)$
chiffmann)
$\mathbb{Z}[g_1^{\pm1}, g_2^{\pm1}]^{sym}$
$[E_{d,k}, E_{d',k'}] = (1-g_1)(1-g_2) \sum_{\frac{d'}{k'} \le \frac{d_1}{k_1} \le \dots \le \frac{d_n}{k_n} \le \frac{d}{k}} E_{d_1k_1} \dots E_{d_nk_n} \cdot \text{const}$
determined recursively
(d',k')
(d_n,k_n)
(d,k)
(d',k')
(d,k)
$(0,0)$


allow infinite sums
$\mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]^{sym}$
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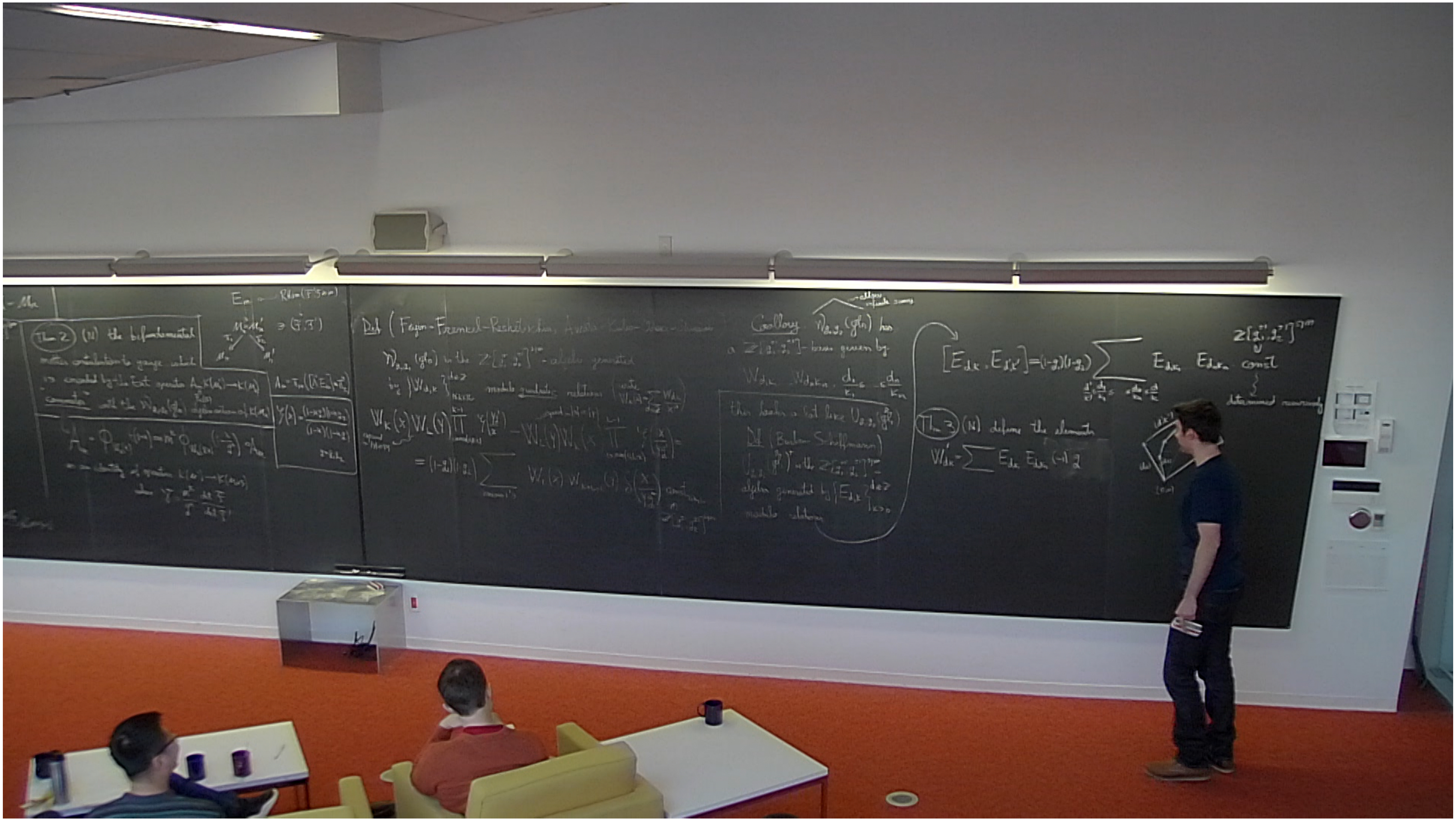
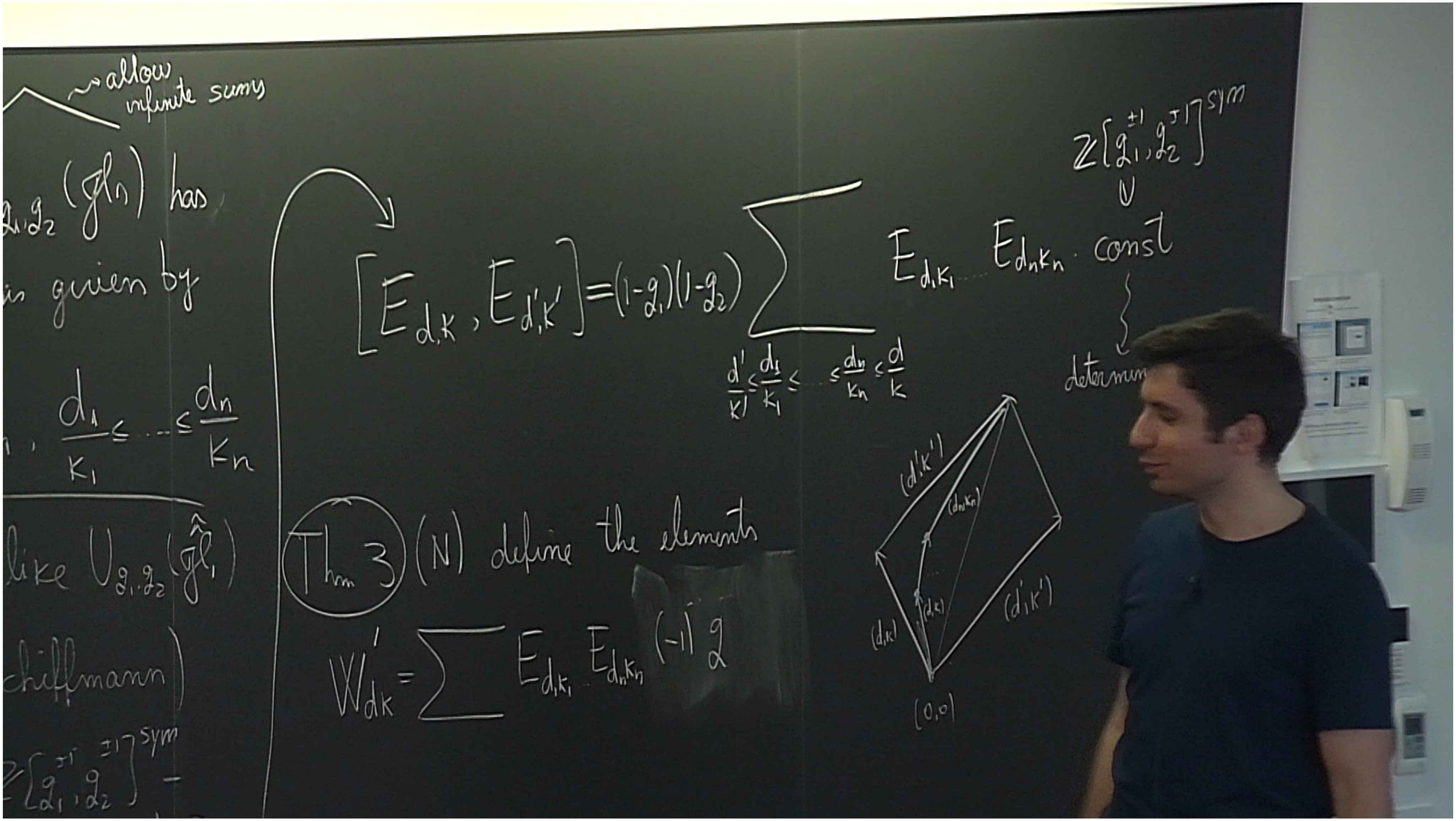
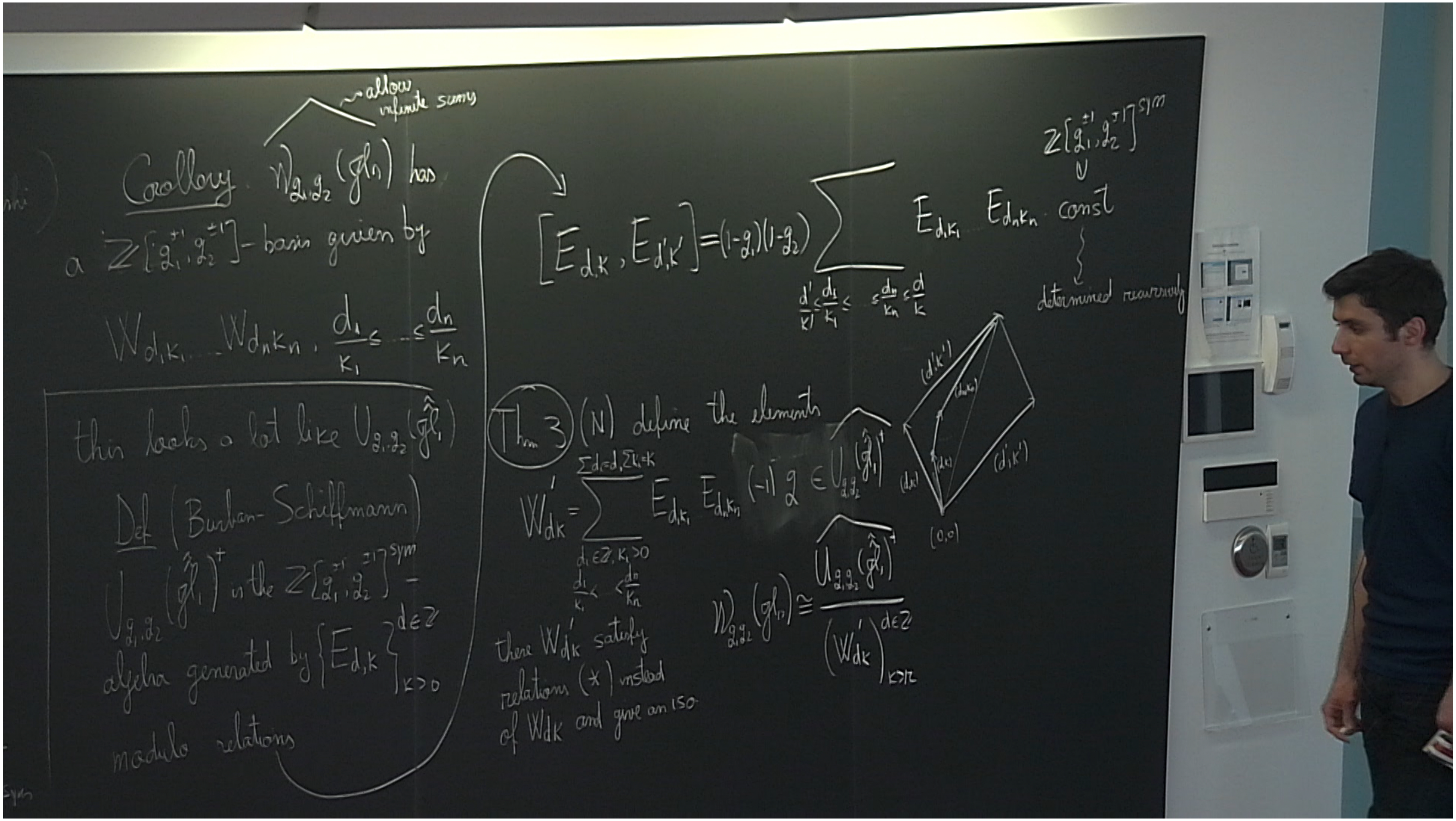
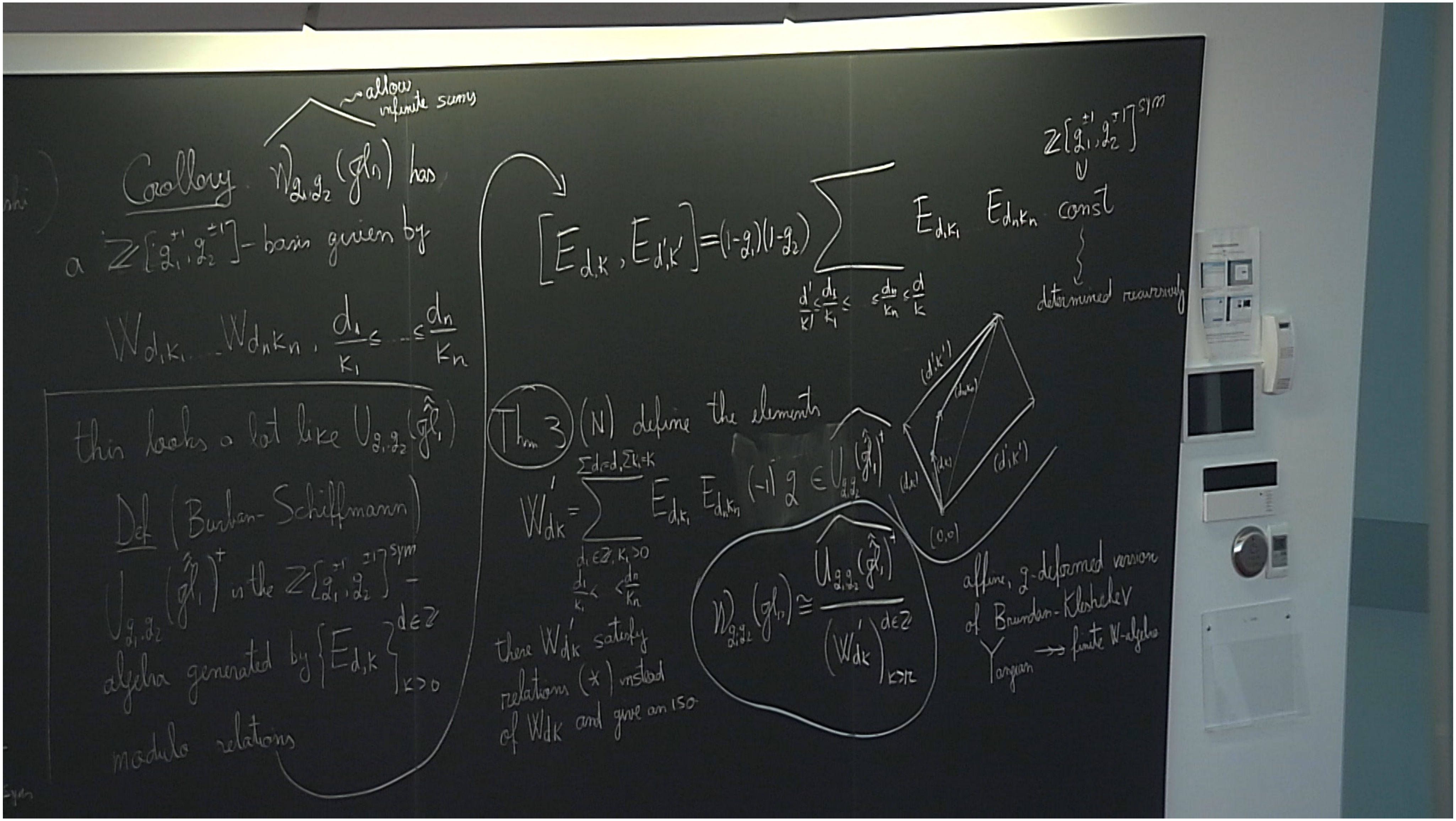
$[E_{d,k}, E_{d',k'}] = (1-q_1)(1-q_2) \sum_{\frac{d'}{k'} \le \frac{d_1}{k_1} \le \dots \le \frac{d_n}{k_n} \le \frac{d}{k}} E_{d_1 k_1} \dots E_{d_n k_n} \cdot const$
determined recursively
Thm 3 (N) define the elements
$W'_{dk} = \sum E_{d_1 k_1} \dots E_{d_n k_n} \prod_{i=1}^{n} (-q)^{k_i - \gcd(k_i, d_i)} \cdot q$
(d',k')
(d_n,k_n)
(d,k)
(d,k)
(d',k')
$(0,0)$




allow infinite sums
has
is given by
like $U_{q_1,q_2}(\hat{gl}_1)$
Schiffmann
$\mathbb{Z}[q_1^{\pm1}, q_2^{\pm1}]^{sym}$
$[E_{d,k}, E_{d',k'}] = (1-q_1)(1-q_2) \sum_{\frac{d'}{k'} \le \frac{d_1}{k_1} \le \dots \le \frac{d_n}{k_n} \le \frac{d}{k}} E_{d_1k_1} \dots E_{d_nk_n} \cdot \text{const}$
determin
Thm 3 (N) define the elements
$W'_{dk} = \sum E_{d_1k_1} \dots E_{d_nk_n} (-1) q$
(d',k')
(d_n,k_n)
(d,k)
$(0,0)$



Corollary $W_{g_1,g_2}(gl_n)$ has
~ allow infinite sums
a $\mathbb{Z}[g_1^{\pm1}, g_2^{\pm1}]$ - basis given by
$W_{d_1k_1} \dots W_{d_nk_n}$, $\frac{d_1}{k_1} \le \dots \le \frac{d_n}{k_n}$
this looks a lot like $U_{g_1,g_2}(\hat{gl}_1)$
Def (Burban-Schiffmann)
$U_{g_1,g_2}(\hat{gl}_1)^+$ is the $\mathbb{Z}[g_1^{\pm1}, g_2^{\pm1}]^{sym}$ - algebra generated by $\{E_{d,k}\}^{d\in\mathbb{Z}}_{k>0}$
modulo relations
$[E_{d,k}, E_{d',k'}] = (1-g_1)(1-g_2) \sum_{\frac{d'}{k'} \le \frac{d_1}{k_1} \le \dots \le \frac{d_n}{k_n} \le \frac{d}{k}} E_{d_1k_1} \dots E_{d_nk_n}$ const
$\mathbb{Z}[g_1^{\pm1}, g_2^{\pm1}]^{sym} \ni$ const
determined recursively
Thm 3 (N) define the elements
$W'_{dk} = \sum_{\substack{\sum d_i = d, \sum k_i = k \\ d_i \in \mathbb{Z}, k_i > 0 \\ \frac{d_1}{k_1} < \dots < \frac{d_n}{k_n}}} E_{d_1k_1} E_{d_nk_n} (-1) g \in \widehat{U_{g_1g_2}(\hat{gl}_1)^+}$
these W'_{dk} satisfy relations (*) instead of W_{dk} and give an iso
$W_{g_1g_2}(gl_n) \cong \widehat{U_{g_1g_2}(\hat{gl}_1)^+} / (W'_{dk})^{d\in\mathbb{Z}}_{k>n}$
(d',k')
(dn,kn)
(d,k)
(d1,k1)
(d',k')
(0,0)



Corollary $\widehat{W_{q_1,q_2}(gl_n)}$ has
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a $\mathbb{Z}[q_1^{\pm1},q_2^{\pm1}]$-basis given by
$W_{d_1k_1}\dots W_{d_nk_n}$, $\frac{d_1}{k_1}\le\dots\le\frac{d_n}{k_n}$
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$U_{q_1,q_2}(\hat{\hat{gl}}_1)^+$ is the $\mathbb{Z}[q_1^{\pm1},q_2^{\pm1}]^{sym}$-algebra generated by $\{E_{d,k}\}_{k>0}^{d\in\mathbb{Z}}$ modulo relations
$[E_{d,k},E_{d',k'}]=(1-q_1)(1-q_2)\sum_{\frac{d'}{k'}\le\frac{d_1}{k_1}\le\dots\le\frac{d_n}{k_n}\le\frac{d}{k}} E_{d_1k_1}\dots E_{d_nk_n}$ const
$\mathbb{Z}[q_1^{\pm1},q_2^{\pm1}]^{sym}$
determined recursively
Thm 3 (N) define the elements
$W'_{dk}=\sum_{\substack{\Sigma d_i=d,\,\Sigma k_i=k \\ d_i\in\mathbb{Z},\,k_i>0 \\ \frac{d_1}{k_1}<\dots<\frac{d_n}{k_n}}} E_{d_1k_1}\dots E_{d_nk_n}(-1)^{?}q^{?}\in\widehat{U_{q_1,q_2}(\hat{\hat{gl}}_1)}^+$
these W'_{dk} satisfy relations (*) instead of W_{dk} and give an iso
$W_{q_1,q_2}(gl_n)\cong\dfrac{\widehat{U_{q_1,q_2}(\hat{\hat{gl}}_1)}^+}{(W'_{dk})^{d\in\mathbb{Z}}_{k>n}}$
(d',k')
(d_n,k_n)
(d_1,k_1)
(0,0)
affine, q-deformed version of Brundan-Kleshchev
Yangian → finite W-algebra

with a little bit of work you can show that

$\mathcal{F} \subseteq_x \mathcal{F}'$ is that $\mathcal{F} \subset \mathcal{F}'$ and $\mathcal{F}'/\mathcal{F} \cong \mathbb{C}_x$

$W'_{d,k}$ acts on $K(\mathcal{M})$ as the following

$$\sum_{\substack{\Sigma k_i = k,\ d_2 - d_1 = d \\ k_0, k_1, k_2 > 0 \\ d_0, d_2 \geq 0}} (p_1 \times p_S)_* \left(\Lambda^{k_0}(\tilde{U}) \mathcal{L}_1^{k_1} \mathcal{L}_2^{k_2} \left(\tfrac{\pm \text{line}}{\text{bundle}}\right)_{d_1 d_2} p_2^* \right)$$

$$\left\{ \left(\mathcal{F} \subseteq_x \cdots \subseteq_x \mathcal{F}_1 \subseteq_x \tilde{\mathcal{F}} \supseteq_x \mathcal{F}_2 \supseteq_x \cdots \supseteq_x \mathcal{F}' \right) \right\}$$

$p_1: \to \mathcal{F} \in \mathcal{M}$, $p_S: \to x \in S$, $p_2: \to \mathcal{M} \ni \mathcal{F}'$

Thm 2 (Schiffmann-Vasserot, Feigin-Tsymbaliuk for $\mathbb{A}^2$) (N for general S)

$U_{q_1, q_2}(\ddot{\mathfrak{gl}}_1) \curvearrowright K(\mathcal{M})$

To deduce Thm 1 you must show that

- $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ acts correctly on $K(\mathcal{M})$ (follows from c_2 bounded below)
- $W'_{d,k}$ act as 0 on $K(\mathcal{M})$

sumo

gcd(r,

$\Rightarrow \exists$ a un

Thm 1 (N): th

is a module

$\exists$ abelian group hom

$W_{q_1,q_2}(\mathfrak{gl}_r) \xrightarrow{\phi} H$

$a \rightsquigarrow$

$\mathcal{F} \subseteq_x \mathcal{F}'$
so that $\mathcal{F} \subset \mathcal{F}'$
and $\mathcal{F}'/\mathcal{F} \cong \mathbb{C}_x$

with a little bit of work you can show that

W'_{dk} acts on $K(M)$ as the following

$$\sum_{\substack{\sum k_i = k,\, d_2 - d_1 = d \\ k_0, k_1, k_2 \geq 0 \\ d_0, d_2 \geq 0}} (p_1 \times p_S)_* \left(\Lambda^{k_0}(\tilde{\mathcal{U}})(-1)^{k_0} \mathcal{L}_1^{k_1} \mathcal{L}_2^{k_2} \left(\pm \text{line bundle}\right)_{d_1 d_2} p_2^* \right) = 0$$

$$D^b(\mathrm{Coh}(M))$$

$$\left\{ \left(\mathcal{F} \subseteq_x \cdots \subseteq_x \mathcal{F}_1 \subseteq_x \tilde{\mathcal{F}} \supseteq_x \mathcal{F}_2 \supseteq_x \cdots \supseteq_x \mathcal{F}' \right) \right\}$$

$\xrightarrow{p_1}$ $\mathcal{F} \in M$, $\xrightarrow{p_S}$ $x \in S$, $\xrightarrow{p_2}$ $M \ni \mathcal{F}'$

Thm 4 (Schiffmann-Vasserot, Feigin-Tsymbaliuk for $\mathbb{A}^2$)
(N for general S)

$$U_{q_1, q_2}(\hat{\hat{\mathfrak{gl}}}_1) \curvearrowright K(M)$$

To deduce Thm 1 you must show that

- $U_{q_1 q_2}(\hat{\hat{\mathfrak{gl}}}_1)$ acts correctly on $K(M)$
(follows from ς_2 bounded below)
- $W'_{d,k}$ act as 0 on $K(M)$

$\mathcal{L}_1 = \Gamma(\tilde{\mathcal{F}}/\mathcal{F}_1)$ $\quad$ $\tilde{\mathcal{U}}$ univ sheaf
$\mathcal{L}_2 = \Gamma(\tilde{\mathcal{F}}/\mathcal{F}_2)$ $\quad$ parametrizes $\tilde{\mathcal{F}}$

assume
$\gcd(r, \ldots$
$\Rightarrow \exists$ a un…

Thm 1 (N): th…
is a module…

$\exists$ abelian group hom…
$W_{q_1, q_2}(\mathfrak{gl}_r) \xrightarrow{\Phi} H\ldots$
$a \mapsto \ldots$

$(\mathcal{F}', \mathcal{F} \otimes m)$

$, \mathcal{F}')$

$([\Lambda^i E_m] \otimes \Pi_2^{\times})$

$$\frac{(1-xq_1)(1-xq_2)}{(1-x)(1-xq)}$$

$q = q_1 q_2$

fix smooth $D \subset S$

$\pi_1, \ldots, \pi_n \in \mathbb{N}$

$\mathcal{F}_\bullet = \{ \mathcal{F}_n(-D) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \quad c_1(\mathcal{F}_i) = c_1(\mathcal{F}_{i-1}) + \pi_i [D] \}$

$\mathcal{M}_{\pi_1 \ldots \pi_n} = \{ \text{parabolic sheaves } \mathcal{F}_\bullet \}$

To Be Defined

Expectation: $K(\mathcal{M}_{\pi_1 \ldots \pi_n})$ should be a module for $\mathcal{U}_{q_1, q_2}(\mathfrak{gl}_{\pi_1 + \cdots + \pi_n}, \text{nilp of Jordan decomposition } \pi_1 \ldots \pi_n)$

Thm 5 (Finkelberg–Tsymbaliuk for $\mathbb{A}^2$)

$$U_{q_1, q_2}(\hat{\hat{\mathfrak{gl}}}_n)_{\underbrace{\pi_1 \ldots \pi_n}_{\text{shift}}} \curvearrowright K(\mathcal{M}_{\pi_1 \ldots \pi_n})$$

$\to RHom(\mathcal{F}', \mathcal{F} \otimes m)$

$\ni (\mathcal{F}, \mathcal{F}')$

M'_π

$A_m = \pi_{1*}\left([\wedge^i E_m] \otimes \pi_2^*\right)$

$$\boxed{\varsigma(x) = \frac{(1-xq_1)(1-xq_2)}{(1-x)(1-xq)} \qquad q = q_1 q_2}$$

fix smooth $D \subset S$

$\pi_1, \ldots, \pi_n \in \mathbb{N}$

$\mathcal{M}_{\pi_1 \ldots \pi_n} = \left\{ \begin{array}{l} \text{parabolic} \\ \text{sheaves} \end{array} \right.$

$\mathcal{F}_\bullet = \{ \mathcal{F}_n(-D) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_i \subset \mathcal{F}_n \}$

$\mathcal{F}'_\bullet = \{ \mathcal{F}'_n(-D) = \mathcal{F}'_0 \subset \mathcal{F}'_1 \subset \mathcal{F}'_i \subset \mathcal{F}'_n \}$

$\mathcal{F}_i / \mathcal{F}'_i \cong \mathbb{C}_x$

To Be Defined

Expectation: $K(\mathcal{M}_{\pi_1 \ldots \pi_n})$ should be a module for $N_{q_1, q_2}(\mathfrak{gl}_{\pi_1 \ldots \pi_n})$ nilpot Jordan decomposition $\pi_1 \ldots \pi_n$

Thm 5 (Feigelberg–Tsymbaliuk for $\mathbb{A}^2$)

$U_{q_1, q_2}(\hat{\hat{\mathfrak{gl}}}_n)_{\underbrace{\pi_1 \ldots \pi_n}_{\text{shift}}} \curvearrowright K(\mathcal{M}_{\pi_1 \ldots \pi_n})$

find elements $f_1 \ldots \in U_{q_1 q_2}(\hat{\mathfrak{gl}}_n)_{\pi_1 \ldots \pi_n}$ which act by 0 on $K(\mathcal{M}_{\pi_1 \ldots \pi_n})$

Thm (N)

the following operators act by 0 on $K(\mathcal{M}_{\pi_1 \ldots \pi_n})$

fix $(i, j) \in \mathbb{Z}^2 / (n,n)\mathbb{Z}$

fix $k > \max(\pi_1, \ldots \pi_n)$

$$\sum_{\substack{k_0 + k_1 + k_2 = k,\ k > 0 \\ S \leq i,j}} (p_1 \times p_3)_* \left(\begin{array}{l} \text{line bundle on} \\ \text{on left board} \end{array} \ p_2^* \right) = 0$$

$$\left\{ \mathcal{F}_\bullet \overset{i-1}{\underset{x}{\subset}} \ldots \overset{s+1}{\underset{x}{\subset}} \tilde{\mathcal{F}}_\bullet \overset{s}{\underset{x}{\subset}} \tilde{\mathcal{F}}_\bullet \overset{s}{\underset{x}{\supset}} \tilde{\mathcal{F}}_\bullet \overset{s+1}{\underset{x}{\supset}} \ldots \overset{j-1}{\underset{x}{\supset}} \mathcal{F}'_\bullet \right\}$$

$$\mathcal{F}_\bullet \xleftarrow{p_1} \cdot \xrightarrow{p_3} x \in S, \qquad \cdot \xrightarrow{p_2} \mathcal{F}'_\bullet$$

...ffmann-Vasserot, Feigin-Tsymbaliuk for $\mathbb{A}^2$)

...for general S)

$\to K(\mathcal{M})$

you must show that

...orrectly on $K(\mathcal{M})$

from $\leq$ bounded below)

0 in $K(\mathcal{M})$

$(\mathcal{F}_1)$ $\widetilde{\mathcal{U}}$ univ sheaf

parametrizing $\widetilde{\mathcal{F}}$

$$\sum P_k = 0$$

$$P_k = A_k - B_k, \quad A_{k+1} = B_k, \quad A_0 = 0$$

sums

$\gcd(r, c_1(\mathcal{F}) \cdot H) = 1$

$\Rightarrow \exists$ a universal sheaf $\mathcal{U} \leftrightarrow \mathcal{F}$

$\mathcal{M} \times S \leftarrow \{\mathcal{F}\} \times S$

Thm 1 (N) the K-theory group $K(\mathcal{M})$ is a module for the algebra $\mathcal{W}_{q_1, q_2}(\mathfrak{gl}_r)$

$\exists$ abelian group homomorphism

$\mathcal{W}_{q_1, q_2}(\mathfrak{gl}_r) \xrightarrow{\Phi} \mathrm{Hom}(K(\mathcal{M}), K(\mathcal{M} \times S))$

$\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{\mathrm{Sym}}$-algebra given by explicit generators...

satisfying compatibility co...

- $\phi_{ab} = K(\mathcal{M}) \xrightarrow{\phi_b} K(\mathcal{M} \times S) \ldots$
- $\phi_{c \cdot a} = K(\mathcal{M}) \xrightarrow{\phi_a} K(\ldots$

where $f \cdot \mathbb{Z}[q_1^{\pm}, \ldots$ homomor...

- $[\phi_a, \phi_b] = \Delta_*\left(\phi_{\frac{[a,b]}{(1-q_1)(1-q_2)}} \ldots\right.$

where the LHS is the difference of the follo...