Title: Partition functions and the McKay correspondence

Speakers: Balazs Szendroi

Collection: Cohomological Hall Algebras in Mathematics and Physics

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Abstract: I will explain some results on certain sheaf-theoretic partition functions defined on Calabi-Yau orbifolds, and their connection to the McKay correspondence, the representation theory of affine Lie algebras, and cohomological Hall algebras. Based on joint work with Gyenge and Nemethi, respectively Davison and Ongaro

joint w. Gyenge, Némethi
Davison, Ongaro

I Intro

$\Gamma \circlearrowright \mathbb{C}^d \quad d \leq 3$

finite

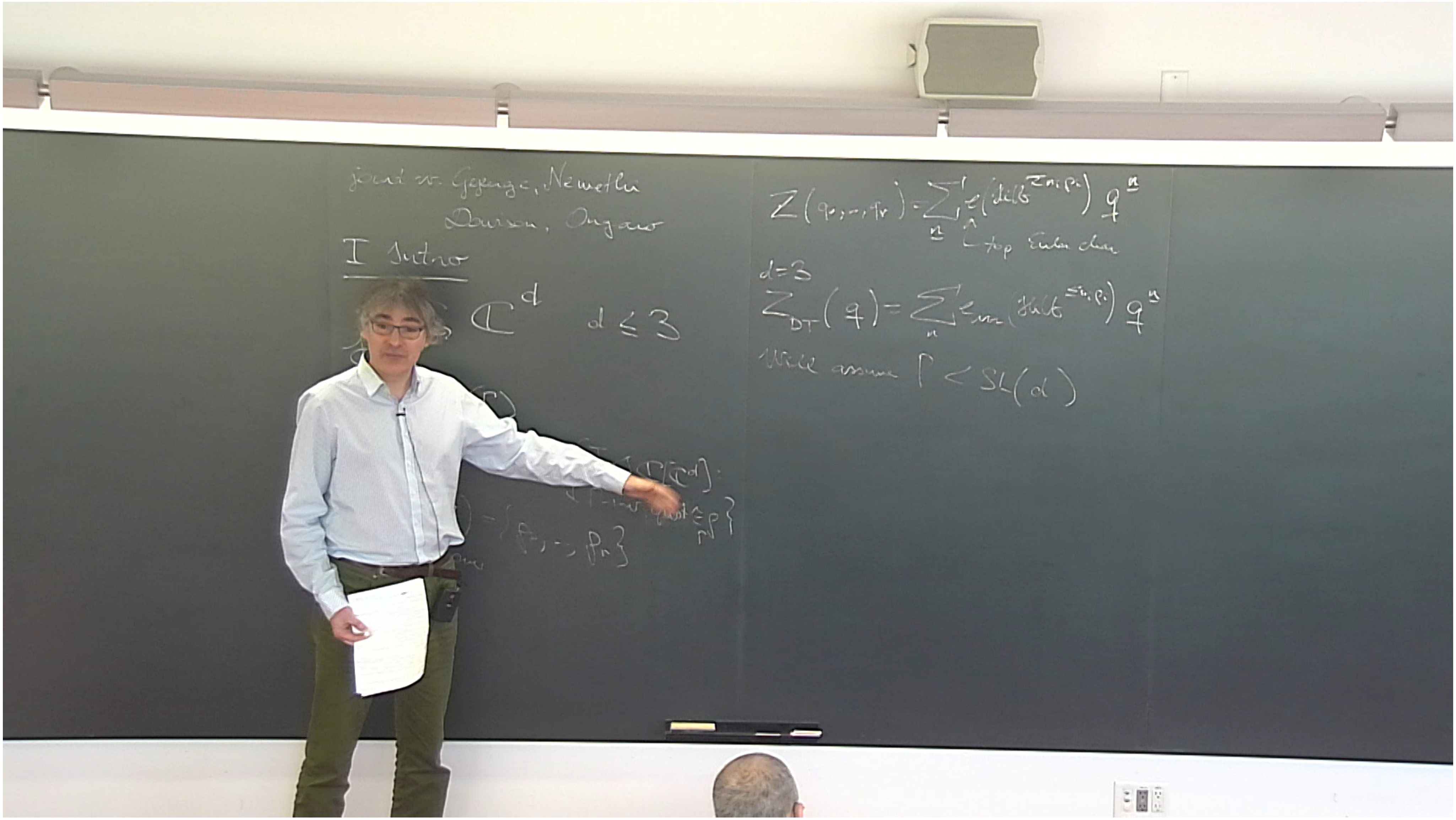
$\rho \in \mathrm{Rep}(\Gamma)$

$\rightsquigarrow \mathrm{Hilb}^{\rho}(\mathbb{C}^d) = \{ I \lhd \mathbb{C}[\mathbb{C}^d] : I \ \Gamma\text{-inv}, \text{quot} \underset{\Gamma}{\cong} \rho \}$

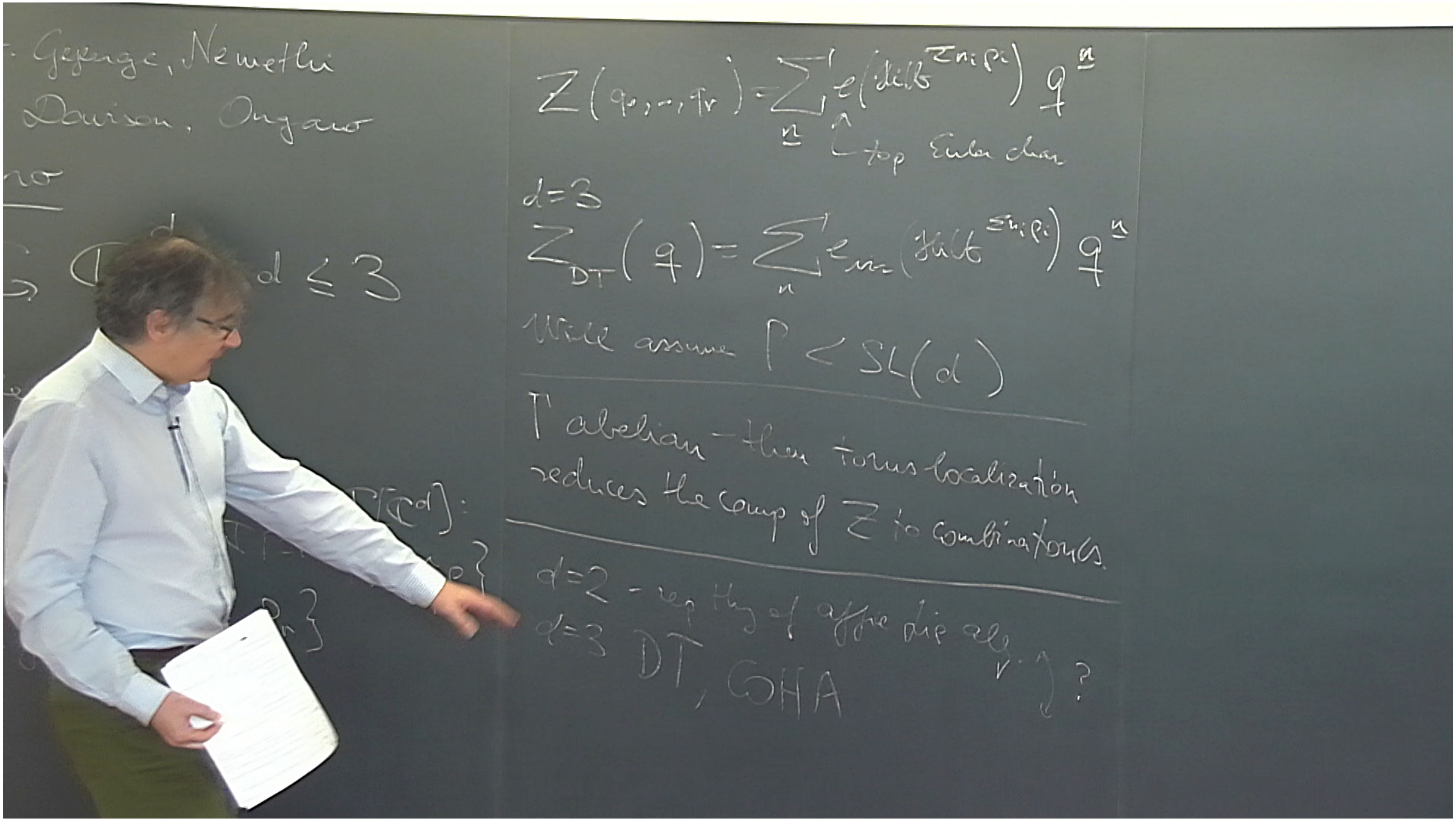
$\mathrm{Irrep}(\Gamma) = \{\rho_0, \dots, \rho_n\}$

then can form

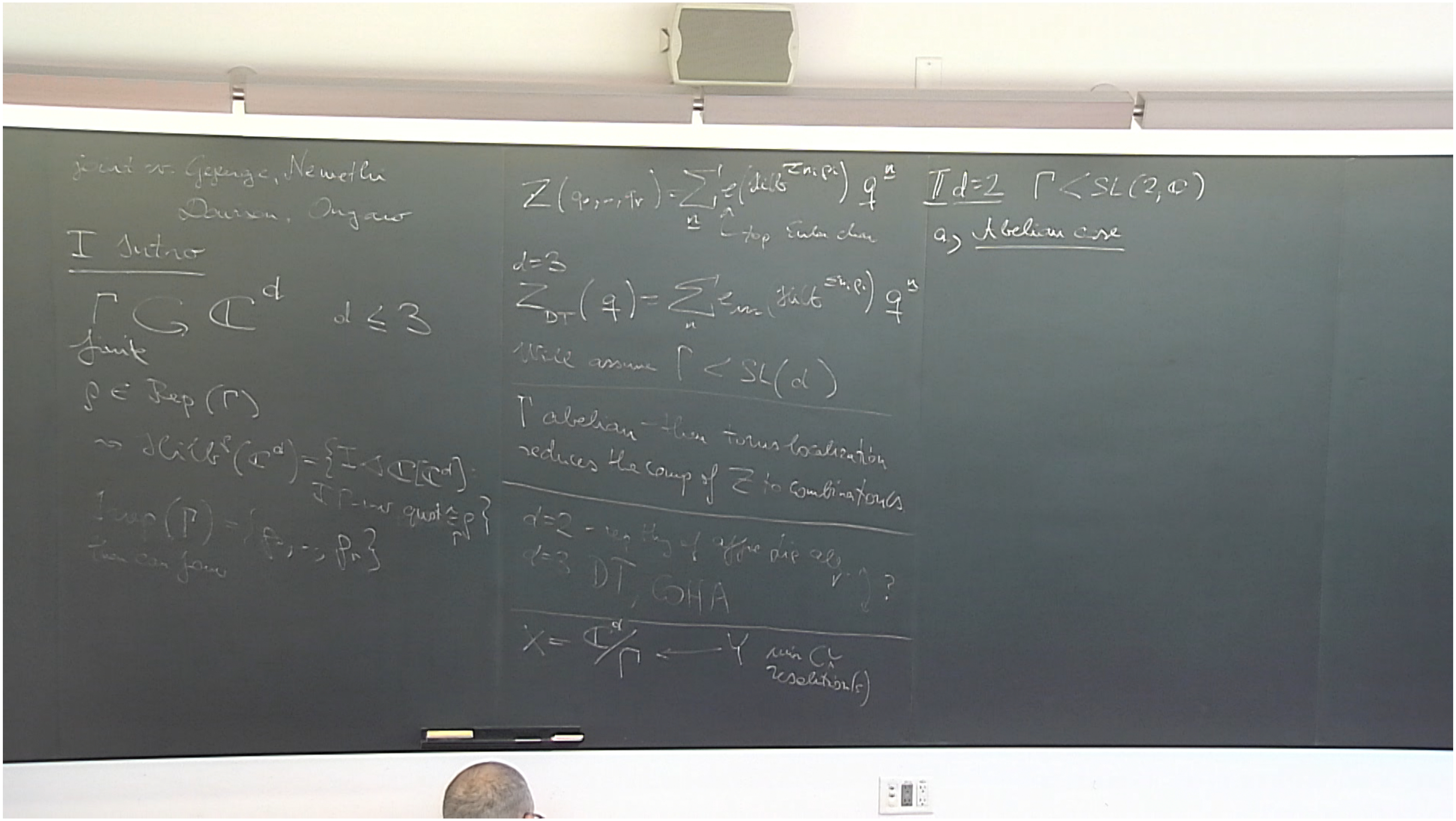
$Z(q_0, \dots, q_r) = \sum_{\underline{n}}' \dots$

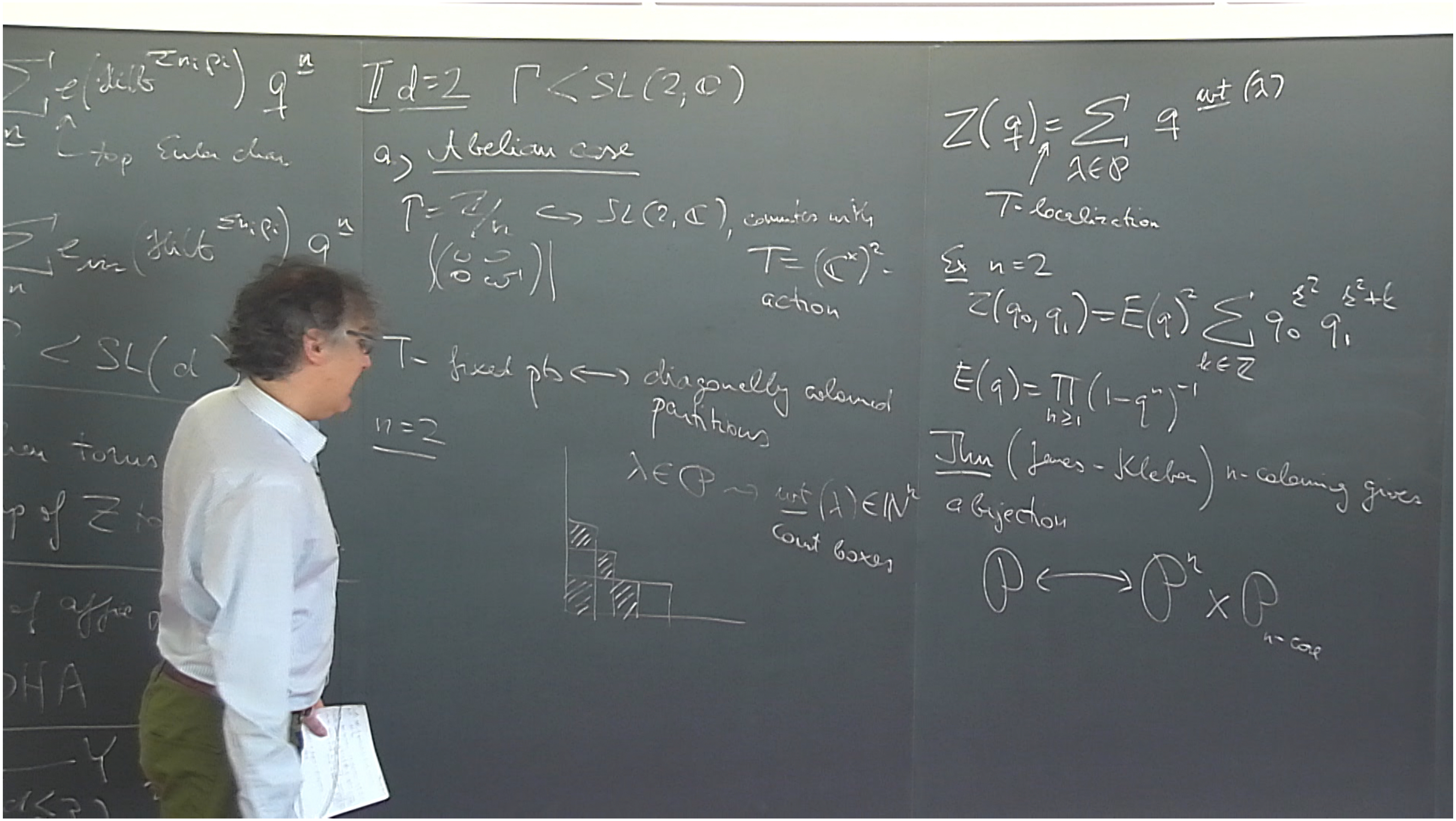


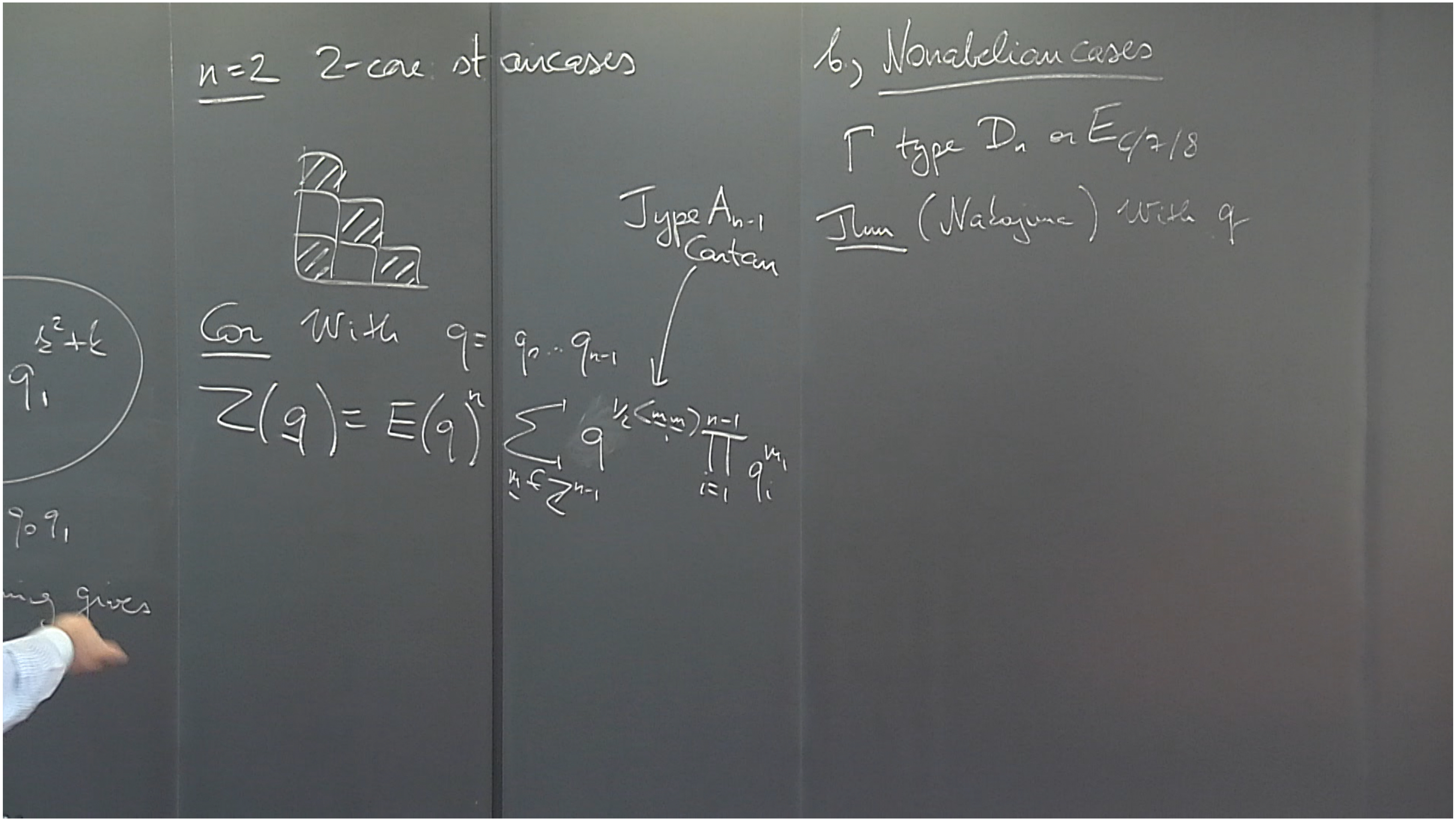
I Intro
$\mathbb{C}^d$ $d \leq 3$
$Z(q_1, \dots, q_r) = \sum_n e(\mathrm{Hilb}^{\sum n_i \rho_i}) q^n$
top Euler char
$d=3$
$Z_{DT}(q) = \sum_n e_{vir}(\mathrm{Hilb}^{\sum n_i \rho_i}) q^n$
Will assume $\Gamma < SL(d)$



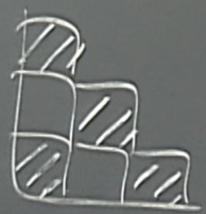
Gyenge, Nemethi
Davison, Ongaro
top Euler char
d=3
Will assume Γ < SL(d)
Γ abelian - then torus localization
reduces the comp of Z to combinatorics
d=3 DT, COHA


joint w. Gyenge, Nemethi
Davison, Ongaro
I Intro
Γ ↷ ℂ^d d ≤ 3
finite
ρ ∈ Rep(Γ)
Irrep(Γ) = {ρ_0, ..., ρ_n}
Z(q_0, ..., q_n) = Σ' e(Hilb^{Σ n_i ρ_i}) q^n
top Euler char
d=3
Z_DT(q) = Σ' e_vir(Hilb^{Σ n_i ρ_i}) q^n
Will assume Γ < SL(d)
Γ abelian - then torus localization reduces the comp of Z to combinatorics
d=2 - rep thy of affine Lie alg ?
d=3 DT, COHA
X = ℂ^d/Γ ← Y min CL resolution(s)
II d=2 Γ < SL(2, ℂ)
a) Abelian case


top Euler char
torus
DHA
II d=2 Γ < SL(2,C)
a) Abelian case
Γ = Z/n ↪ SL(2,C), commutes with T = (C*)²-action
T-fixed pts ⟷ diagonally colored partitions
n=2
λ ∈ P ⇝ wt(λ) ∈ Nⁿ
count boxes
Z(q) = Σ_{λ∈P} q^{wt(λ)}
T-localization
Ex n=2
Z(q₀, q₁) = E(q)² Σ_{k∈Z} q₀^{k²} q₁^{k²+k}
E(q) = Π_{n≥1} (1−qⁿ)⁻¹
Thm (James-Kleber) n-coloring gives a bijection
P ⟷ Pⁿ × P_{n-core}


n=2 2-core staircases
Cor With q = q_0 ... q_{n-1}
Z(q) = E(q)^n Σ' q^{½<m,m>} ∏_{i=1}^{n-1} q_i^{m_i}
m ∈ Z^{n-1}
Type A_{n-1} Cartan
b) Nonabelian cases
Γ type D_n or E_{6/7/8}
Thm (Nakajima) With q
q_0 q_1
gives

$n=2$ 2-core staircases



Cor With $q = q_0 \cdots q_{n-1}$

$$Z(\underline{q}) = E(q)^n \sum_{m \in \mathbb{Z}^{n-1}} q^{\frac{1}{2}\langle m, m\rangle} \prod_{i=1}^{n-1} q_i^{m_i}$$

Type A_{n-1} Cartan

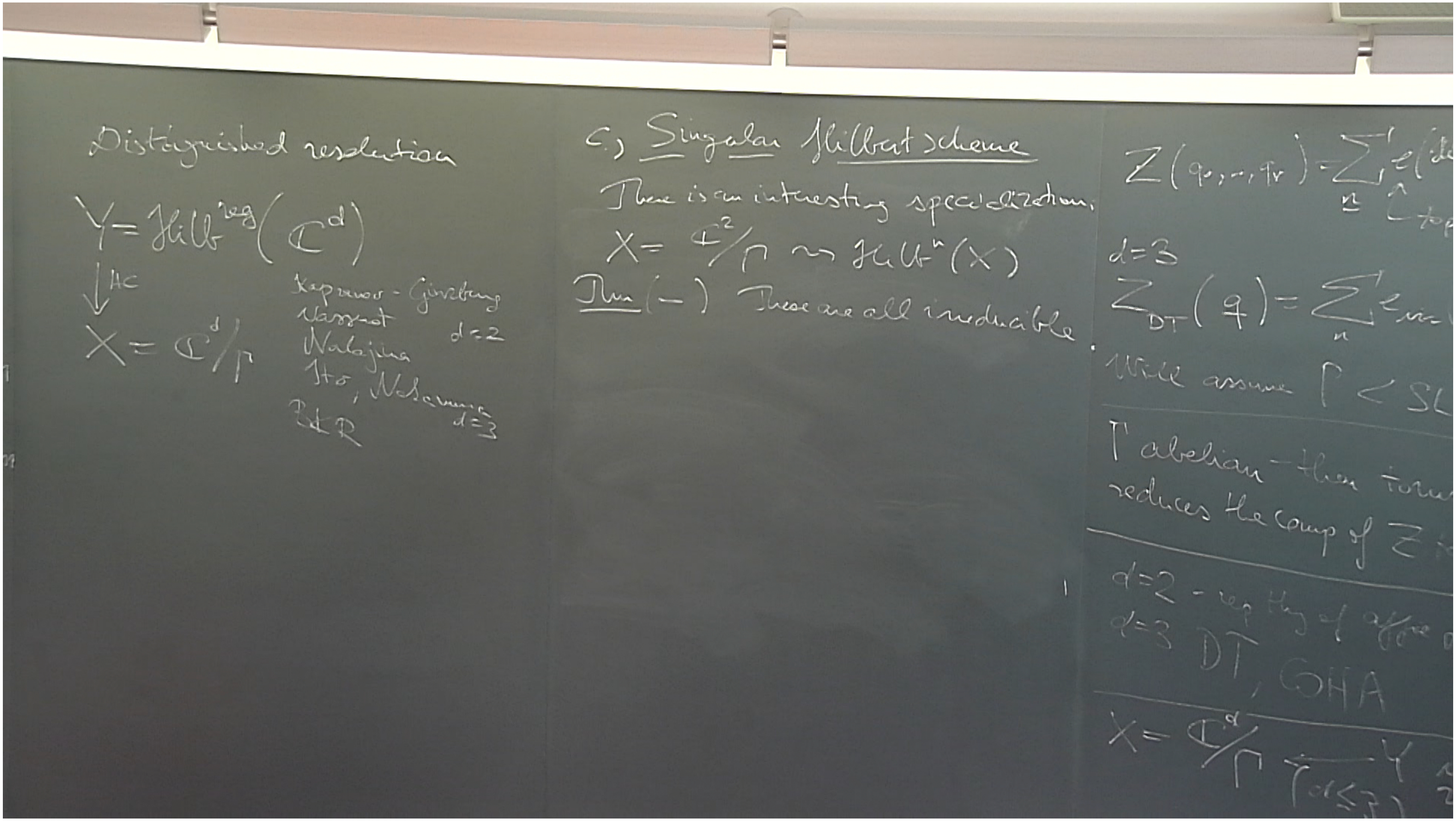
b) Nonabelian cases

Γ type D_n or $E_{6/7/8}$

Thm (Nakajima) With $q = \underline{q}^{\delta}$, $\delta = (\dim(\rho_i))$, we have

$$Z(\underline{q}) = E(q)^{r+1} \sum_{m \in \Lambda_R} q^{\frac{1}{2}\langle m, m\rangle} \prod_{i=1}^{r} q_i^{m_i}$$

Proof The CYs are certain Nakajima q. var's on the affine quiver.


Distinguished resolution
Y = Hilb^reg (C^d)
HC
X = C^d/Γ
Kapranov - Ginzburg
Vasserot d=2
Nakajima
Ito, Nakamura d=3
BKR
c) Singular Hilbert scheme
There is an interesting specialization:
X = C^2/Γ ⇝ Hilb^n(X)
Thm (–) These are all irreducible.
d=3
Will assume Γ < SL
Γ abelian - then toric
reduces the comp of Z
d=2 - rep thy of affine
d=3 DT, CoHA
X = C^d/Γ ← Y

Distinguished resolution

$Y = \mathrm{Hilb}^{\mathrm{reg}}(\mathbb{C}^d)$

$\downarrow$ HC

$X = \mathbb{C}^d/\Gamma$

Kapranov - Ginzburg
Vasserot $\quad d=2$
Nakajima
Ito, Nakamura $\quad d=3$
BKR

c) Singular Hilbert scheme

There is an interesting specialization:

$X = \mathbb{C}^2/\Gamma \rightsquigarrow \mathrm{Hilb}^n(X)$

Thm (–) These are all irreducible.

Thm (Gyenge/Némethi/–) + Conj

$$\sum_{n\geq 0}' e(\mathrm{Hilb}^n(X))\, q^n = Z_\Gamma(q)\Big|_{q_1=\dots=q_r= \exp\left(\frac{2\pi i}{1+h}\right)}, \quad q = q^{\sigma}$$

$Z(q_0, \dots, q_r) = \sum_{n}' e(\dots) \dots$

$d=3$

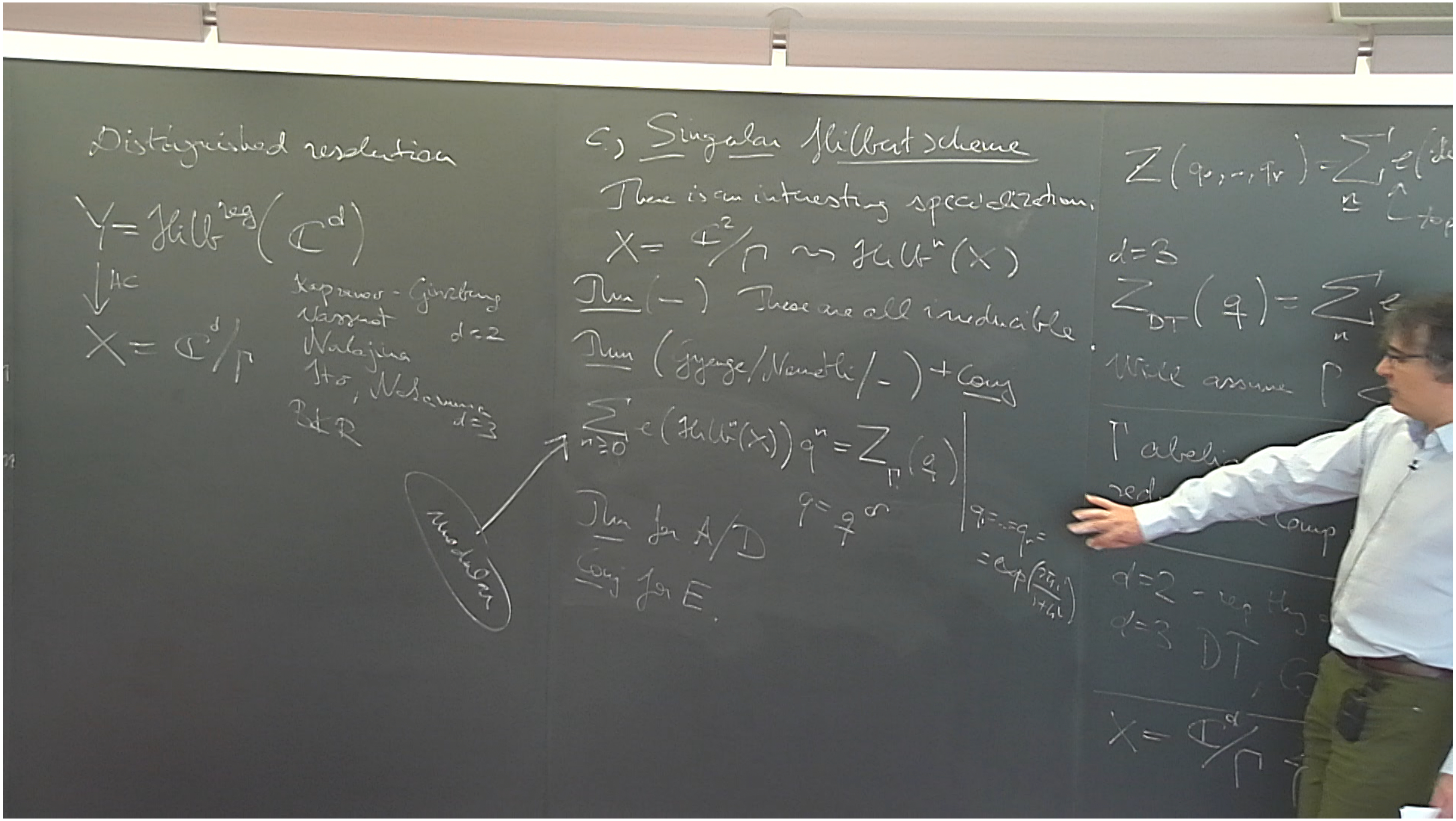
$Z_{DT}(q) = \sum_n' e_{\dots}$

Will assume $\Gamma < SL$

Γ abelian – then torus
reduces the comp of Z …

$d=2$ – rep thy of affine …
$d=3$ DT, CoHA

$X = \mathbb{C}^d/\Gamma \longleftarrow Y$ ($d\leq 3$)


Distinguished resolution
Y = Hilb^reg(ℂ^d)
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Kapranov - Ginzburg
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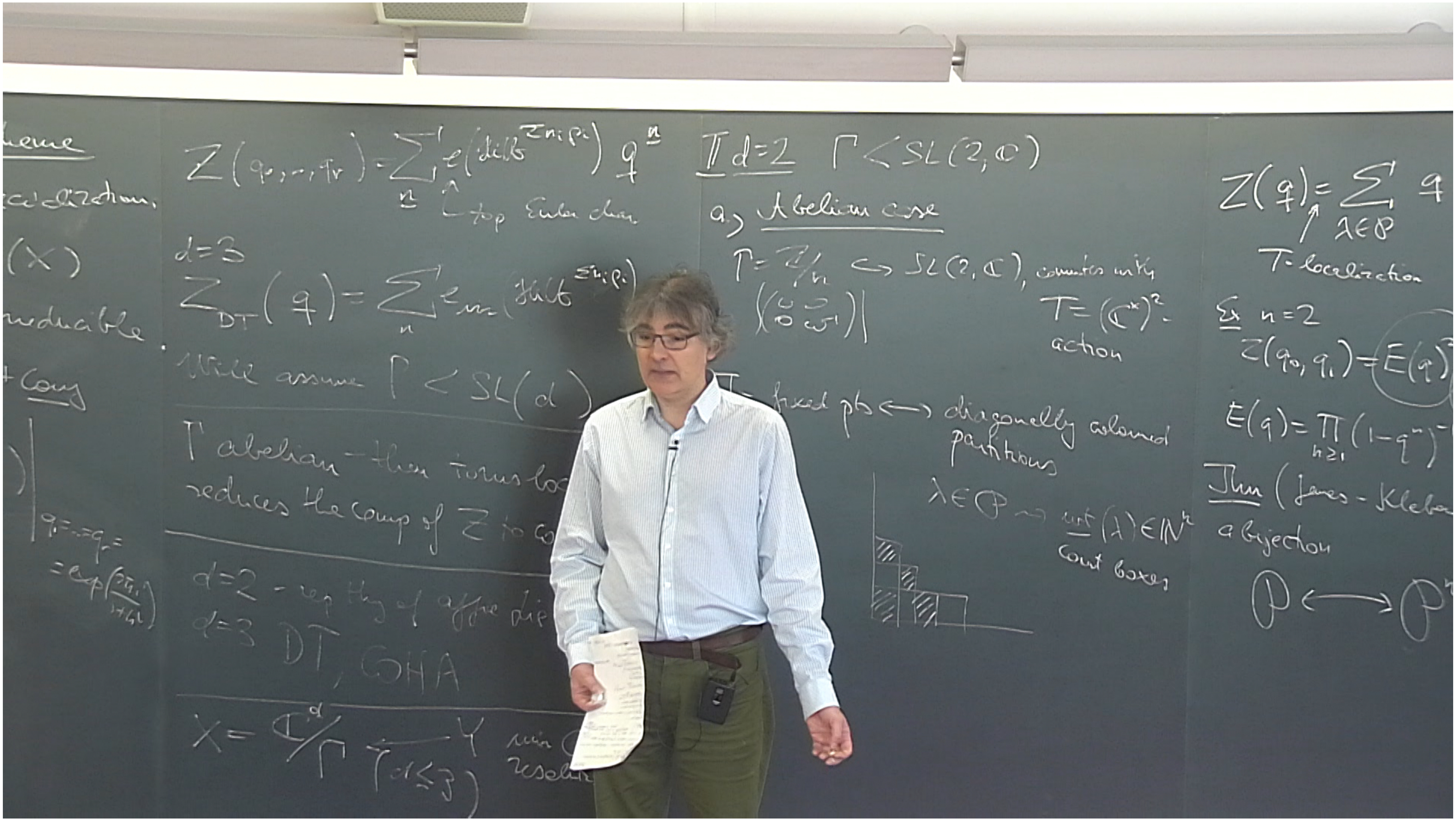
c, Singular Hilbert scheme
There is an interesting specialization:
X = ℂ²/Γ ⇝ Hilb^n(X)
Thm (–) These are all irreducible.
Thm (Gyenge/Némethi/–) + Conj
Σ_{n≥0} e(Hilb^n(X)) q^n = Z_Γ(q)
Thm for A/D
Conj for E.
d=3
Will assume
Γ abelian
d=2
d=3 DT,
X = ℂ^d/Γ


II d=2 Γ < SL(2,ℂ)
a) Abelian case
Will assume Γ < SL(d)
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T-localization
T = (ℂ*)² - action
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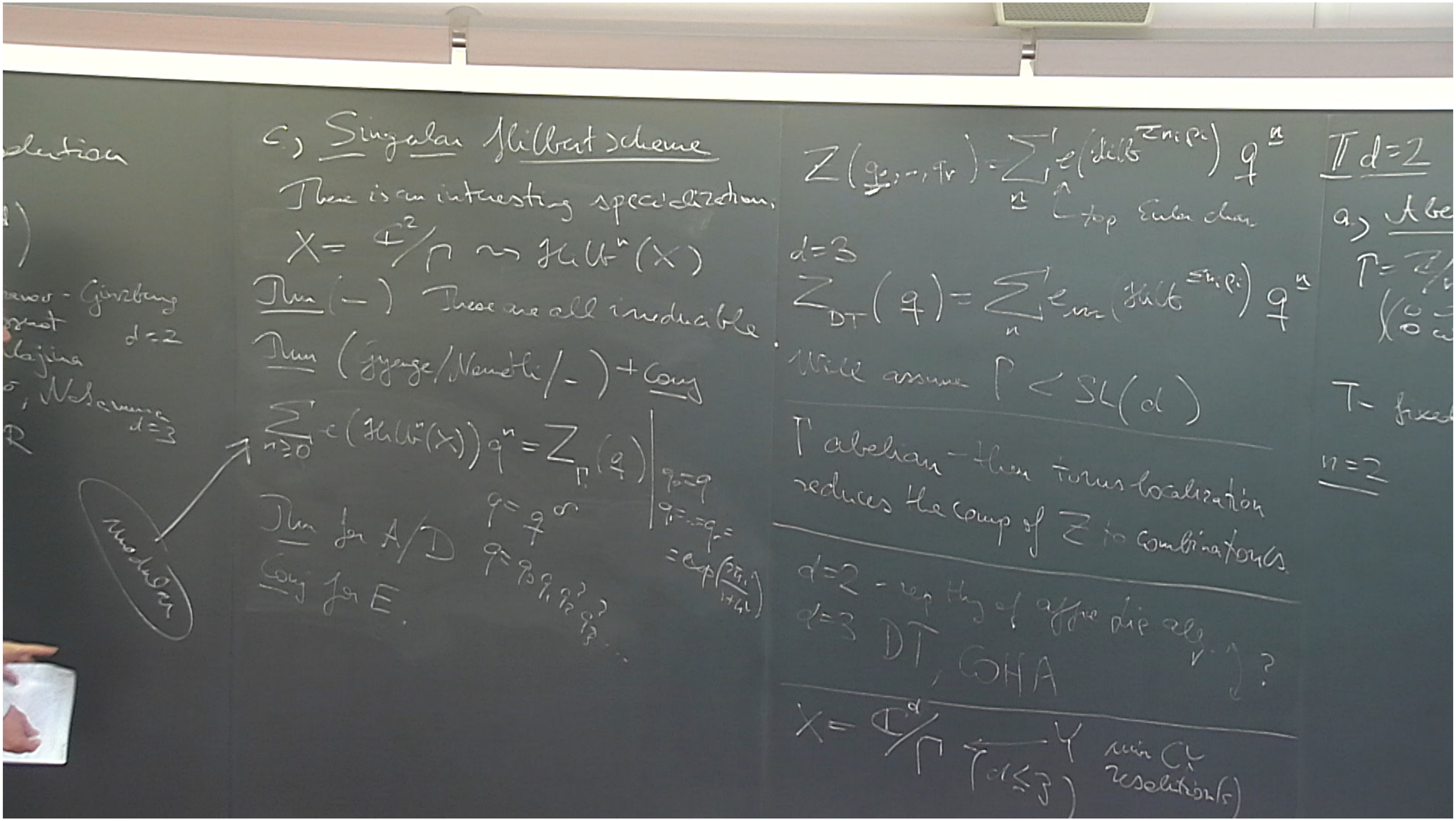
c) Singular Hilbert scheme
There is an interesting specialization:
X = C²/Γ ⇝ Hilb^n(X)
Thm (–) These are all irreducible.
Thm (Gyenge/Némethi/–) + Conj
Σ' e(Hilb^n(X)) q^n = Z_Γ(q) | q_0=q, q_1=...=q_r = exp(2πi/(1+h))
n≥0
q = q^σ
q = q_0 q_1 q_2² q_3² ...
Thm for A/D
Conj for E.
Z(q_0,...,q_r) = Σ' e(Hilb^{Σ n_i ρ_i}) q^n
top Euler char
d=3
Z_DT(q) = Σ' e_vir(Hilb^{Σ n_i ρ_i}) q^n
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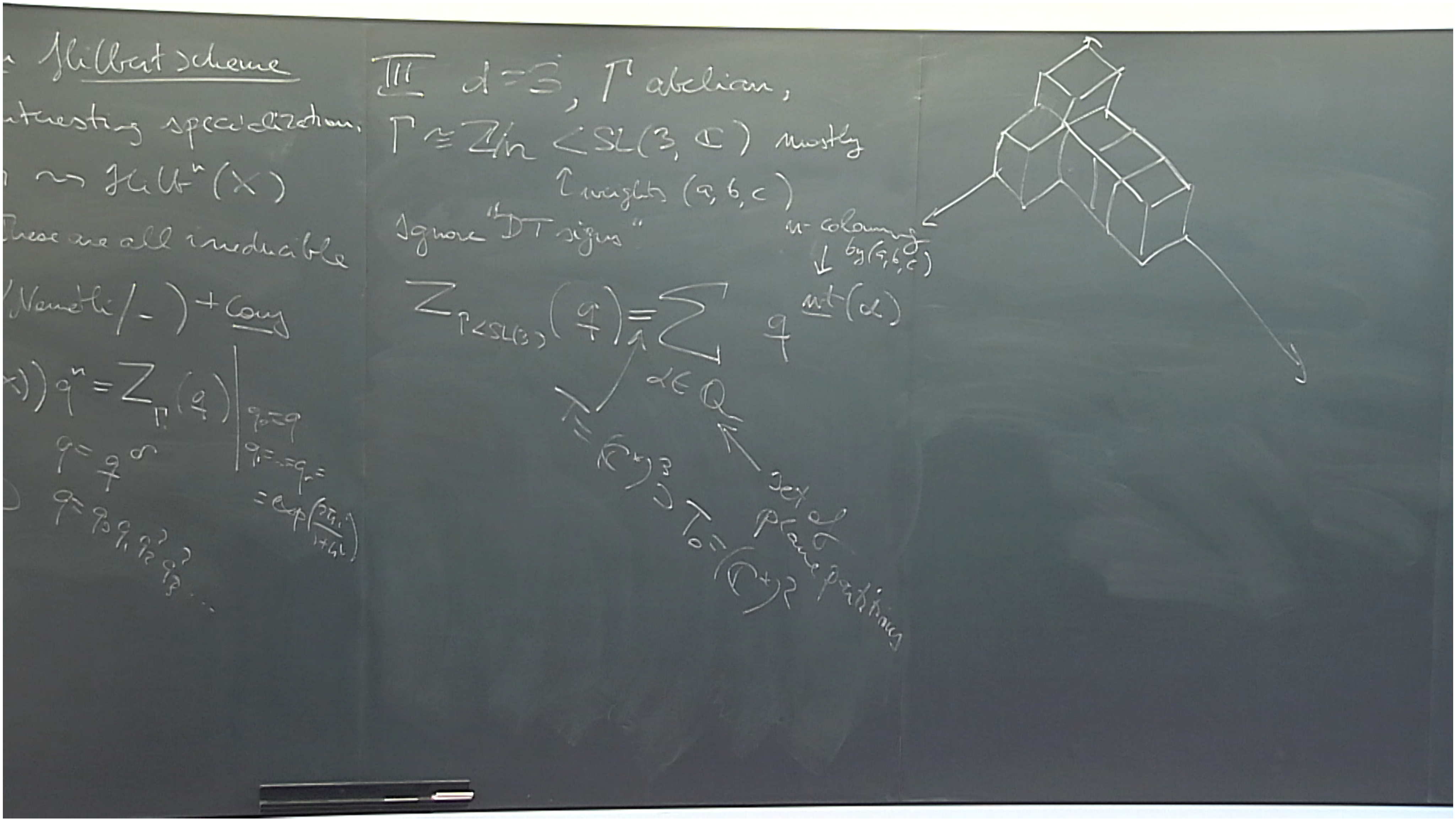
(d≤3)
II d=2
a) Abe
n=2
d=2
d=3


Hilbert scheme
interesting specialization:
Hilb^n(X)
These are all irreducible
(Nemethi/...) + Conj
q^n = Z_Γ(q)
q_0 = q
q = q^σ
q_1 = ... = q_r =
= exp(2πi/1+n)
q = q_0 q_1 q_2^2 q_3^2 ...
III d = 3, Γ abelian,
Γ ≅ Z/n < SL(3, C) mostly
weights (a, b, c)
Ignore "DT signs"
n-colouring by (a, b, c)
Z_{Γ<SL(3)}(q) = Σ_{α∈Q} q^{wt(α)}
T ≅ (C*)^3 ⊃ T_0 ≅ (C*)^2
set of plane partitions

... Hilbert scheme

...teresting specialization:

... ⇝ $\mathrm{Hilb}^n(X)$

These are all irreducible

... $(\ldots/-)$ + Conj

$\ldots = Z_{\Gamma}(q)\Big|_{\substack{q_0 = q \\ q_1 = \ldots = q_r = \\ = \exp\left(\frac{2\pi i}{1+r}\right)}}$

$\ldots = q^{n}$

$q_0 q_1 q_2^{?} q_3^{?} \ldots$

III $d = 3$, Γ abelian,

$\Gamma \cong \mathbb{Z}/n < SL(3, \mathbb{C})$ mostly

↑ weights (a, b, c)

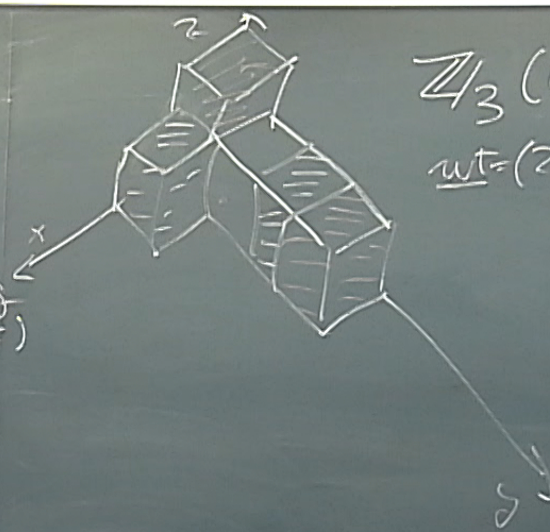
Ignore "DT signs"

$$Z_{\Gamma < SL(3)}(q) = \sum_{\alpha \in Q} q^{\underline{wt}(\alpha)}$$

n-colouring by (a, b, c) ↓ $\underline{wt}(\alpha)$

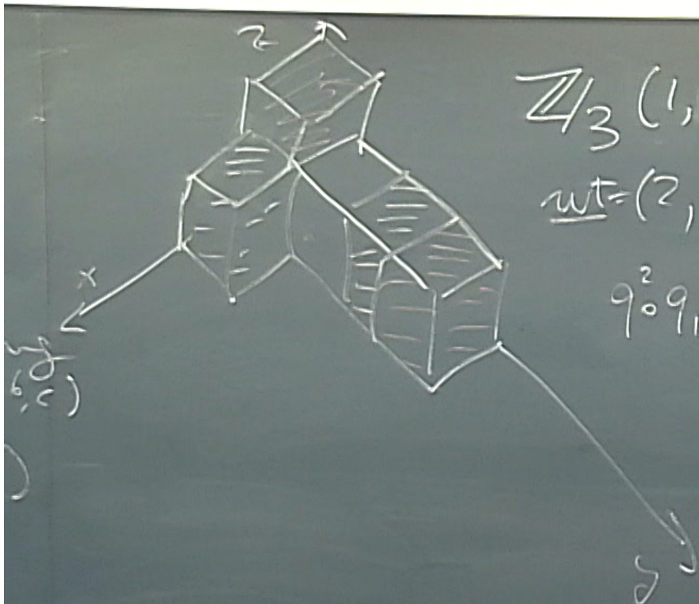
$T \cong (\mathbb{C}^*)^3 \supset T_0 \cong (\mathbb{C}^*)^2$

set of plane partitions



$\mathbb{Z}/3\,(1, -1, 0)$

$\underline{wt} = (2, 1, 3)$



$\mathbb{Z}/3\ (1,-1,0)$

wt $= (2,1,3)$

$q_0^2 q_1 q_2^3$

$$Z(q) = \sum_{\alpha} q^{wt(\alpha)} \qquad q = (q_0, \dots, q_{n-1})$$

T-localization

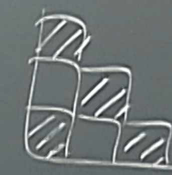
Conjecture With $q = q_0 \cdots q_{n-1}$

$$Z_{\Gamma \subset SL(3)}(q) = M(q)^n Z_\Gamma^{red}(q)$$

with $M(q) = \prod_{n \geq 1} (1-q^n)^{-n} = \sum_{\alpha \in Q} q^{|\alpha|}$

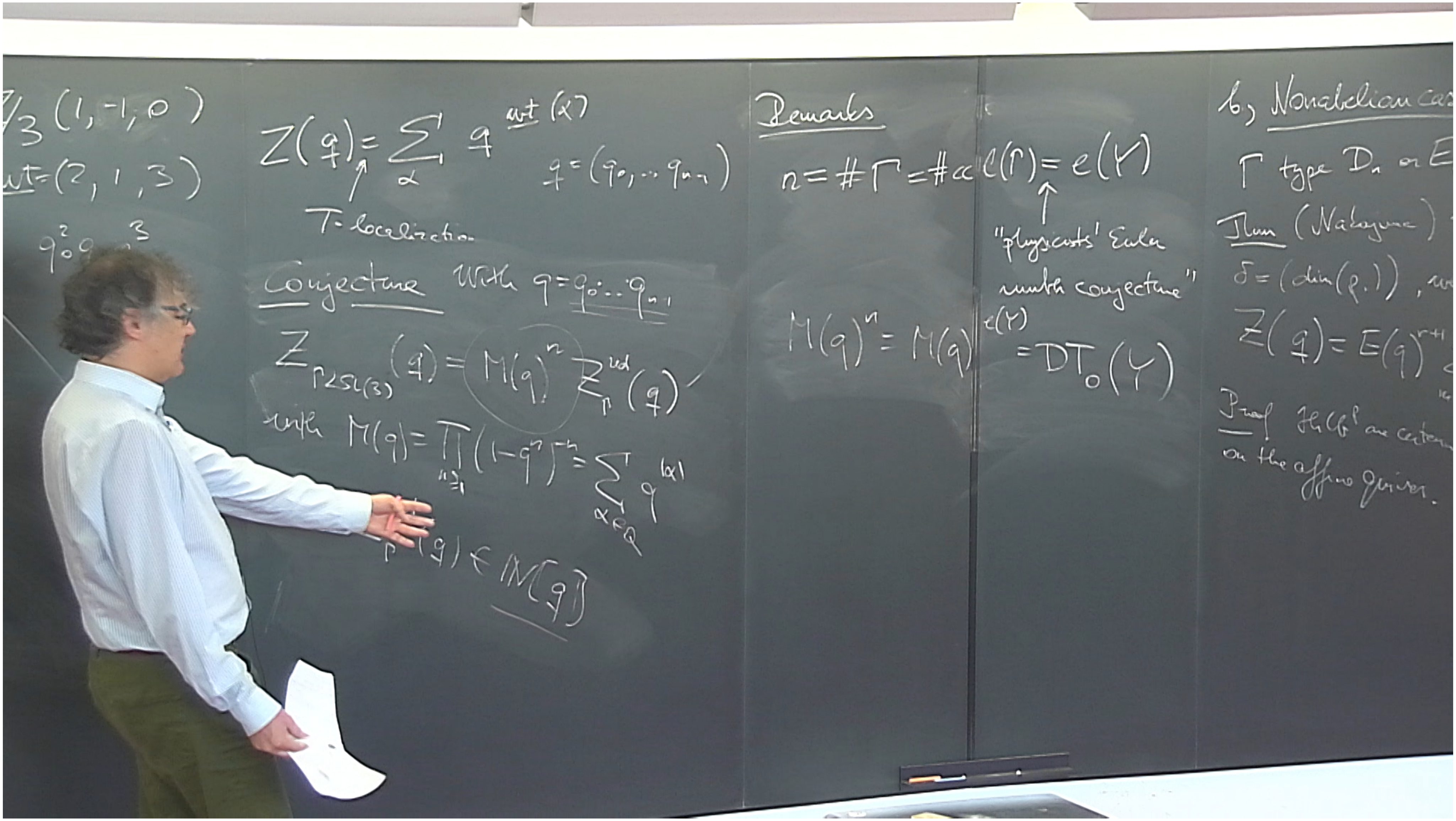
and $Z_\Gamma^{red}(q) \in \mathbb{N}[q]$

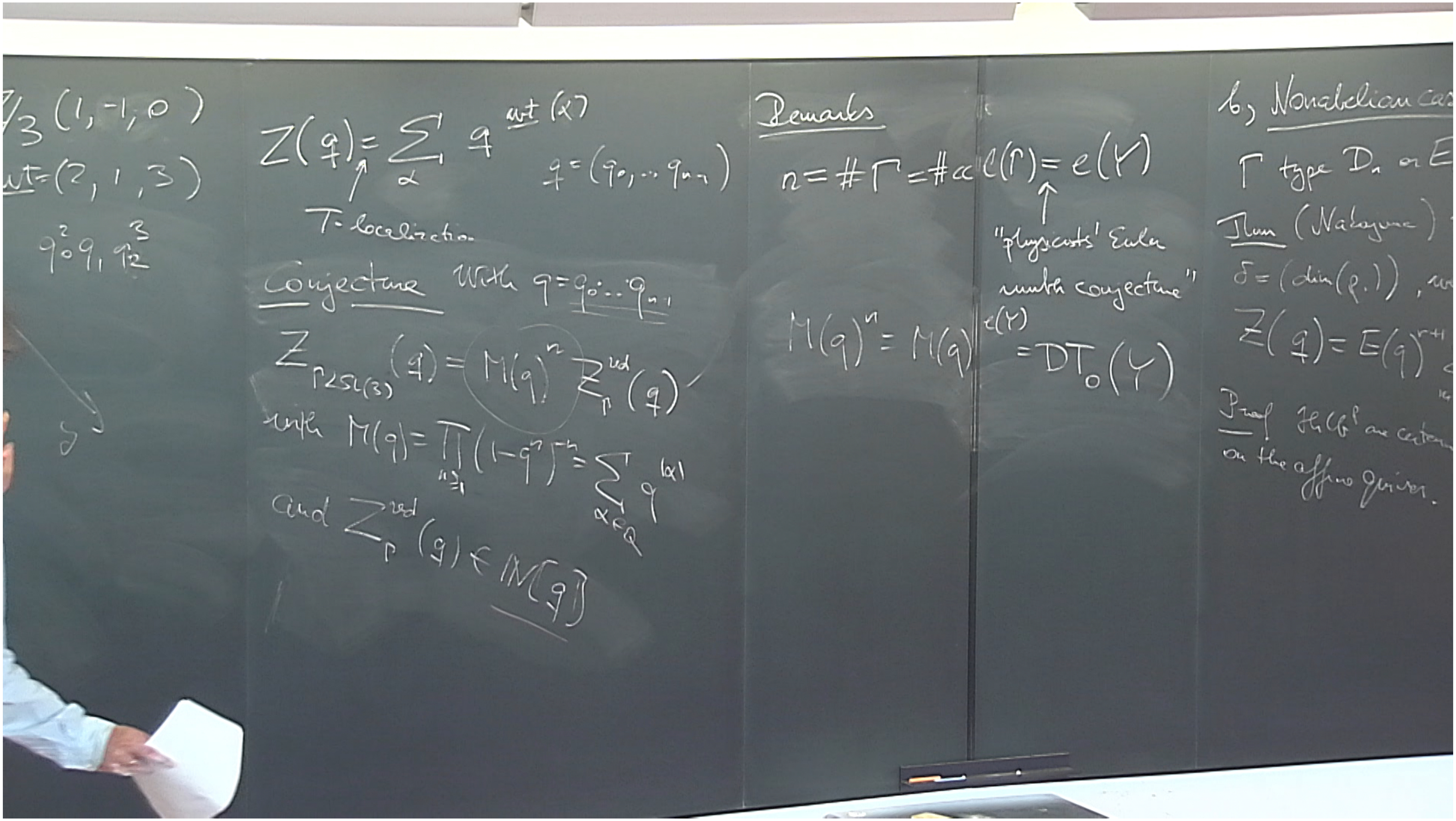
$n=2$ 2-core st



Cor With $q =$

$$Z(q) = E(q)^n$$


Z(q) = Σ q^wt(α)
T-localization
q = (q0, … qn-1)
Conjecture With q = q0 ⋯ qn-1
Remarks
"physicists' Euler number conjecture"
= DT0(Y)
b) Nonabelian case
Γ type Dn or E
Proof
on the affine quiver.


/3 (1, -1, 0)
wt = (2, 1, 3)
$q_0^2 q_1 q_2^3$
$Z(q) = \sum_\alpha q^{wt(\alpha)}$
$q = (q_0, \dots q_{n-1})$
T-localization
Conjecture With $q = q_0 \cdots q_{n-1}$
$Z_{\Gamma < SL(3)}(q) = M(q)^n Z_\Gamma^{red}(q)$
with $M(q) = \prod_{n \geq 1} (1-q^n)^{-n} = \sum_{\alpha \in Q} q^{|\alpha|}$
and $Z_\Gamma^{red}(q) \in \mathbb{N}[q]$
Remarks
$n = \#\Gamma = \#cc$
$e(\Gamma) = e(Y)$
"physicists' Euler number conjecture"
$M(q)^n = M(q)^{e(Y)} = DT_0(Y)$
b) Nonabelian cas
Γ type D_n or E
Thm (Nakajima)
$\delta = (\dim(\rho_i))$
$Z(q) = E(q)^{r+1}$
Proof
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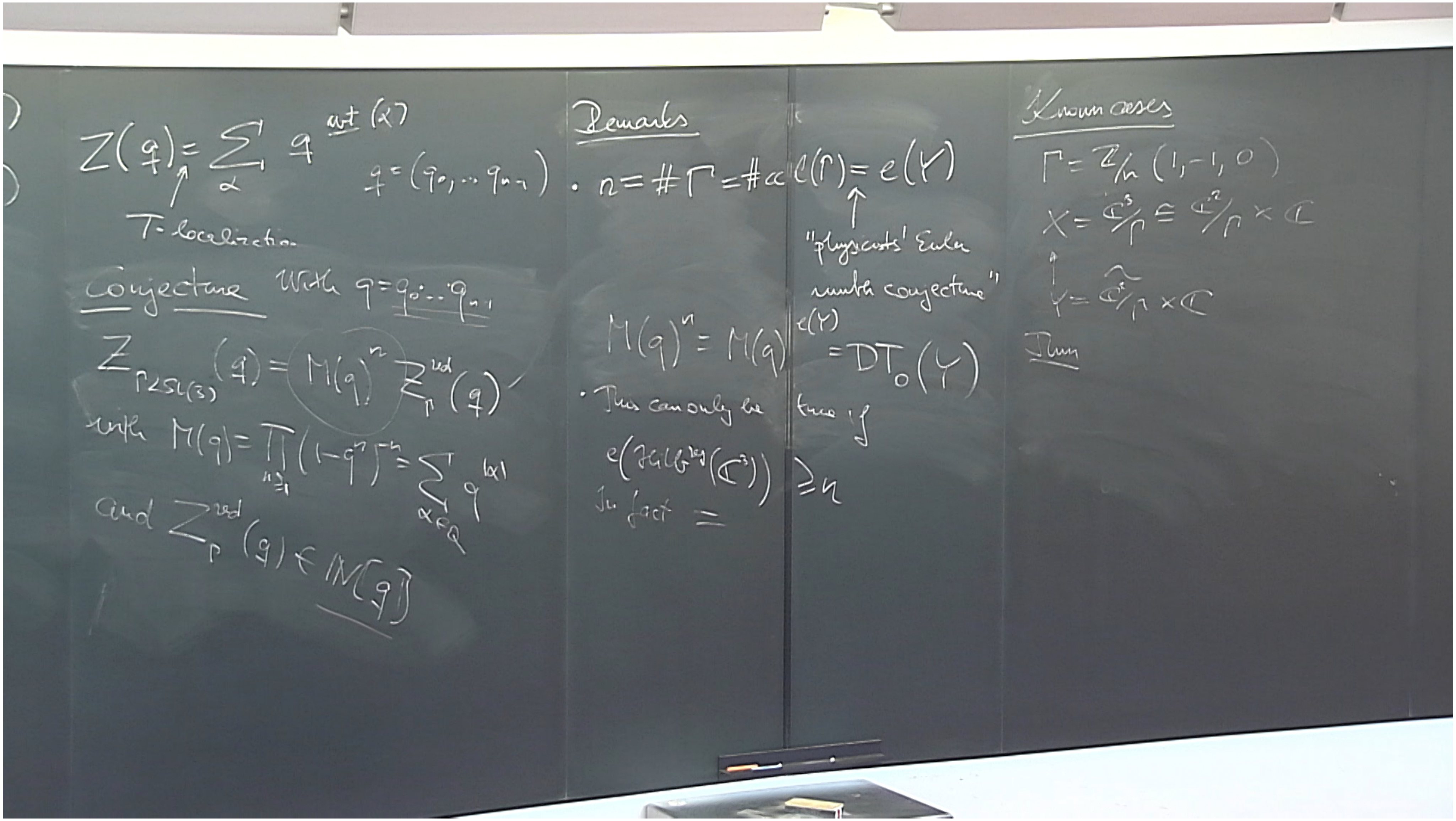

Z(q) = Σ_α q^{wt(α)}
q = (q_0, ... q_{n-1})
T-localization
Conjecture With q = q_0 ... q_{n-1}
Z_{Γ<SL(3)}(q) = M(q)^n Z_Γ^{red}(q)
with M(q) = Π_{n≥1} (1-q^n)^{-n} = Σ_{α∈Q} q^{|α|}
and Z_Γ^{red}(q) ∈ ℕ[q]
Remarks
• n = #Γ = #cc e(Γ) = e(Y)
"physicists' Euler number conjecture"
M(q)^n = M(q)^{e(Y)} = DT_0(Y)
• This can only be true if
e(Hilb^{reg}(ℂ^3)) ≥ n
In fact =
Known cases
Γ = ℤ/n (1,-1,0)
X = ℂ^3/Γ ≅ ℂ^2/Γ × ℂ
Y = ℂ^2/Γ × ℂ
Thm

$(q_0, \dots q_{n-1})$

$\dots q_{n-1}$

$Z^{2nd}_{\Gamma}(q)$

$\sum_{\alpha \in Q} q^{|\alpha|}$

$\mathrm{Exp}\left(\dots \right)$

Remarks

• $n = \#\Gamma = \# cc\ell(\Gamma) = e(Y)$

↑ "physicists' Euler number conjecture"

$M(q)^n = M(q)^{e(Y)} = DT_0(Y)$

• This can only be true if

$e(\mathrm{Hilb}^{reg}(\mathbb{C}^3)) \geq n$

In fact $=$

Known cases

$\Gamma = \mathbb{Z}/n\,(1,-1,0)$

$X = \mathbb{C}^3/\Gamma \cong \mathbb{C}^2/\Gamma \times \mathbb{C}$

↑

$Y = \widetilde{\mathbb{C}^2/\Gamma} \times \mathbb{C}$

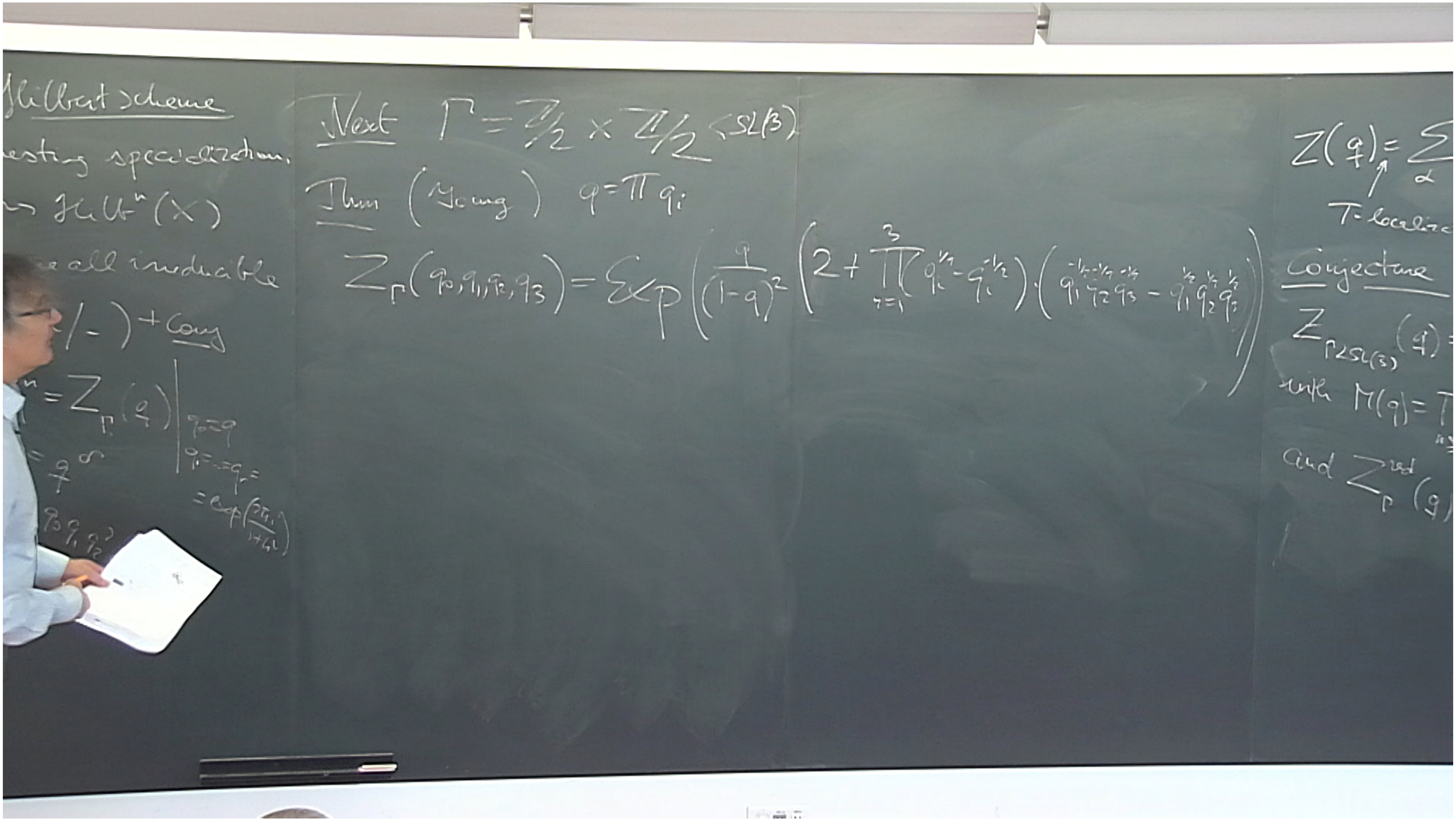
Thm (Young)

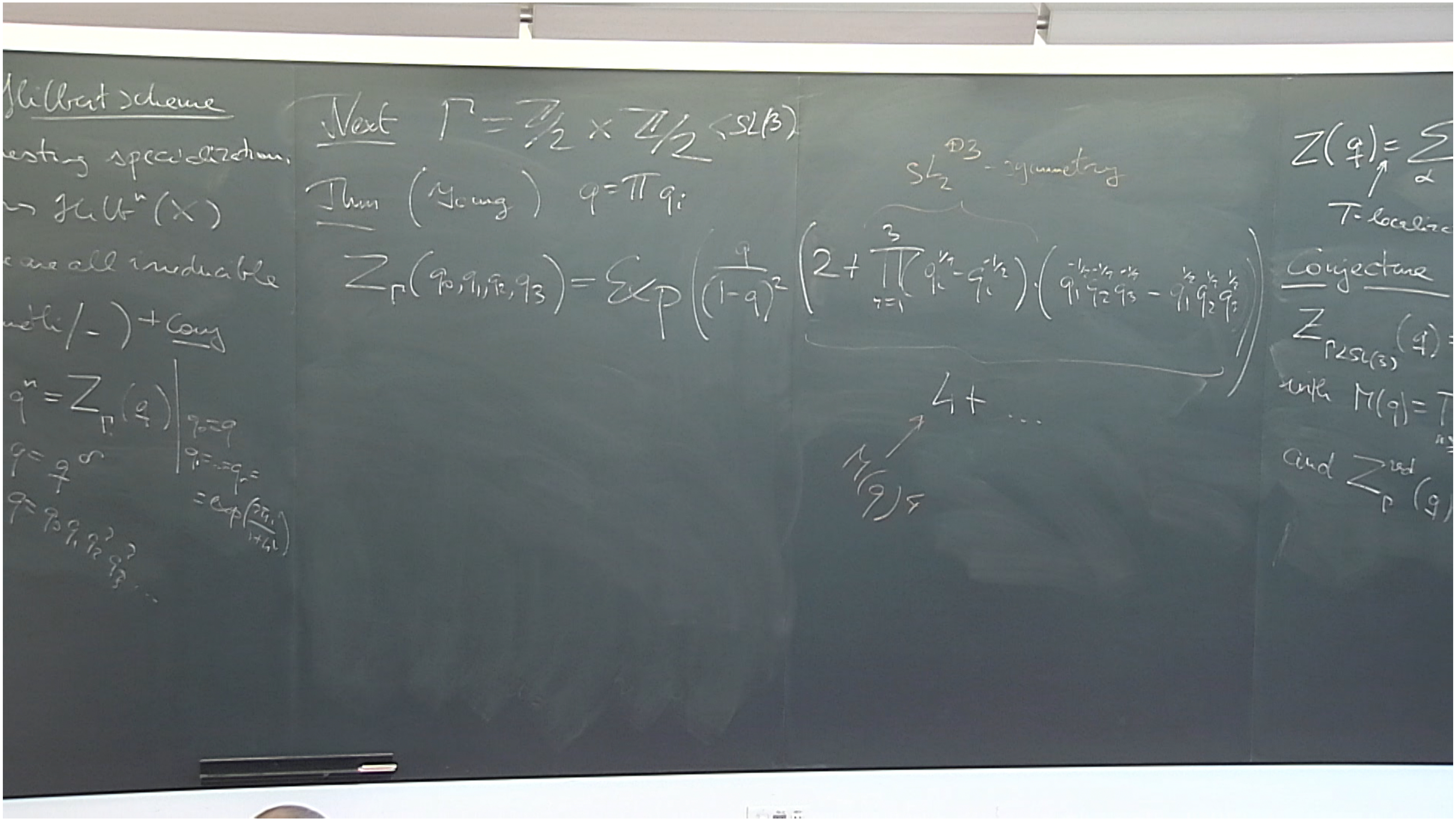
$Z(q) = \mathrm{Exp}\left(\frac{q}{(1-q)^2} \left(n + \sum_{\alpha \in \Delta_R} q^{\alpha} \right) \right)$

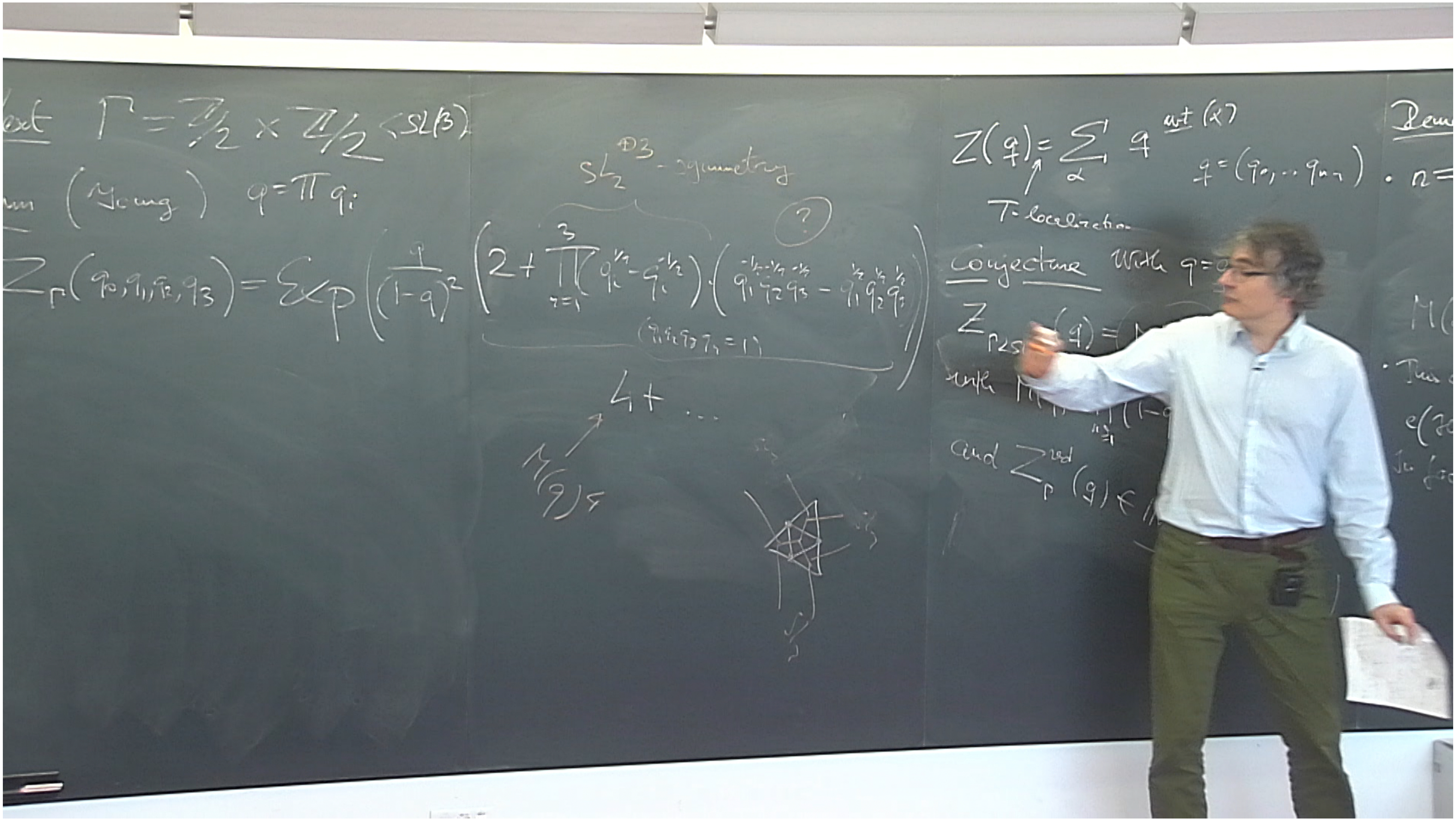
Conj. $M(q)^n$

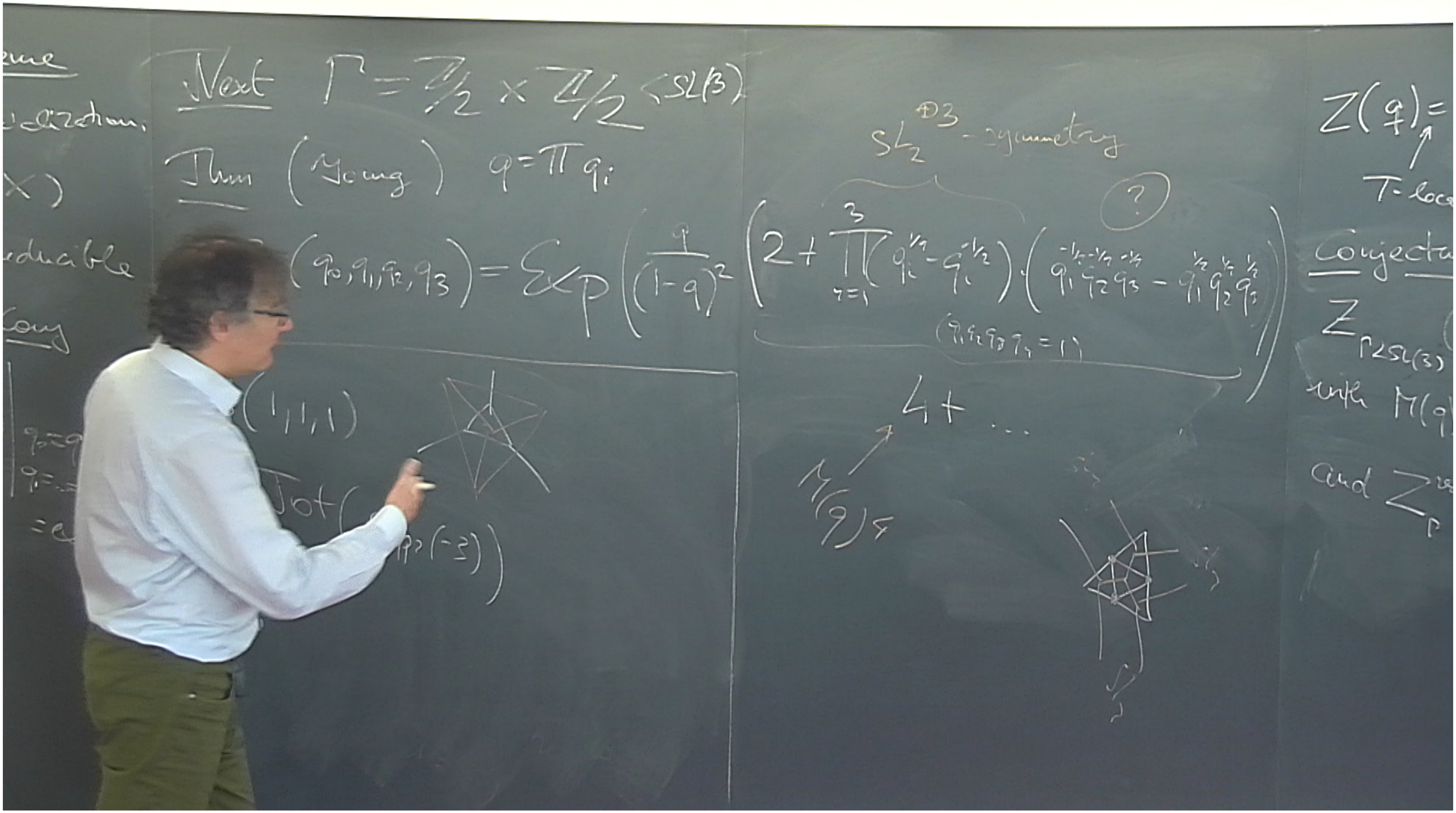
sum of adjoint of Sl_{n-1}

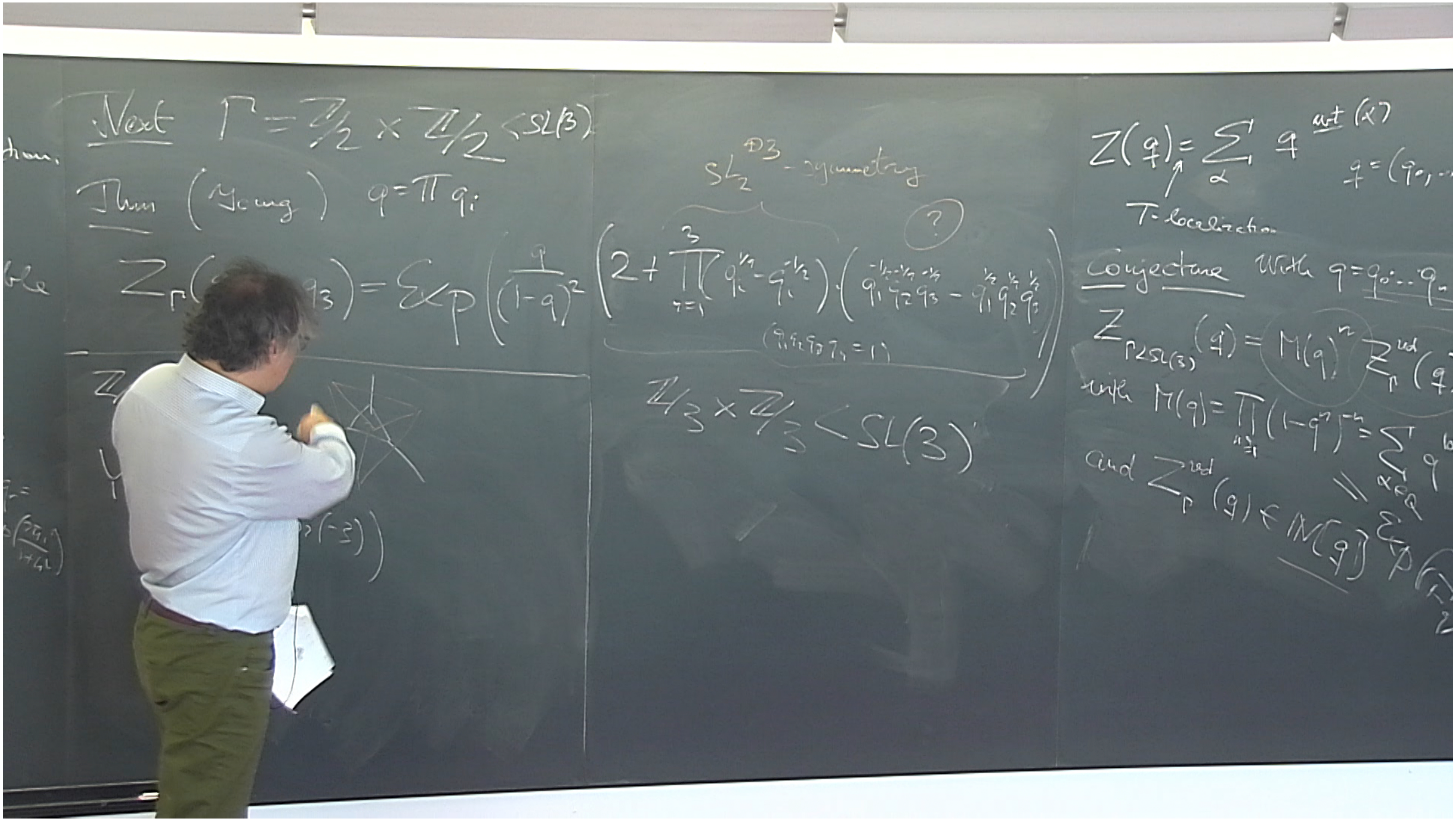
root lattice of A_{n-1}

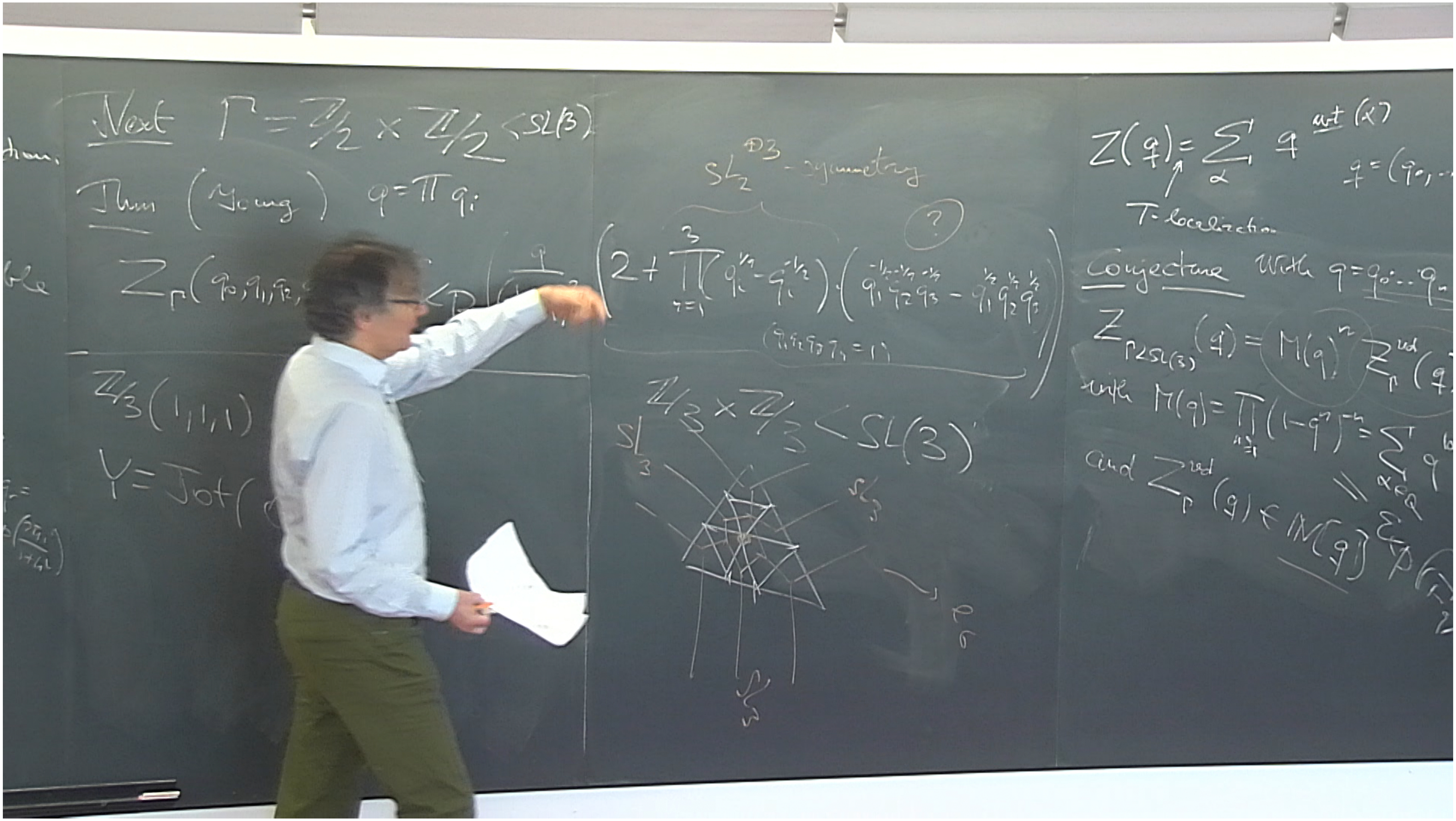

Hilbert scheme
Next Γ = Z/2 × Z/2 < SL(3)
Thm (Young) q = Π q_i
Z_Γ(q_0, q_1, q_2, q_3) = Exp
Z(q) =
Conjecture
with M(q) =
and


Hilbert scheme
Next Γ = Z/2 × Z/2 < SL(3)
Thm (Young) q = Π q_i
Z_Γ(q_0, q_1, q_2, q_3) = Exp
SL_2^{⊕3} - symmetry
4 + ...
Conjecture


symmetry
Young
T-localization
Conjecture
with
and


Next Γ = Z/2 × Z/2 < SL(3)
Thm (Young) q = Π q_i
(q_0, q_1, q_2, q_3) = Exp(q/(1-q)^2 (2 + Π_{i=1}^{3} (q_i^{1/2} - q_i^{-1/2}) · (q_1^{-1/2} q_2^{-1/2} q_3^{-1/2} - q_1^{1/2} q_2^{1/2} q_3^{1/2})))
SL_2^{⊕3}-symmetry
(q_0 q_1 q_2 q_3 = 1)
4 + ...
(1,1,1)
Z(q) =
T-loc
Conjectu
Z_{Γ<SL(3)}
with M(q)
and Z_Γ


Next Γ = Z/2 × Z/2 < SL(3)
Thm (Young) q = Π q_i
SL_2^{D3} - symmetry
?
T-localization
Z(q) = Σ_α q^{wt(α)}
Conjecture With q = q_0 ⋯ q_n
with M(q) = Π_{n≥1} (1-q^n)^{-n} = Σ_α q
and Z_Γ^{red}(q) ∈ M[q] Exp
Z/3 × Z/3 < SL(3)


Next Γ = Z/2 × Z/2 < SL(3)
Thm (Young) q = Π q_i
Z/3 (1,1,1)
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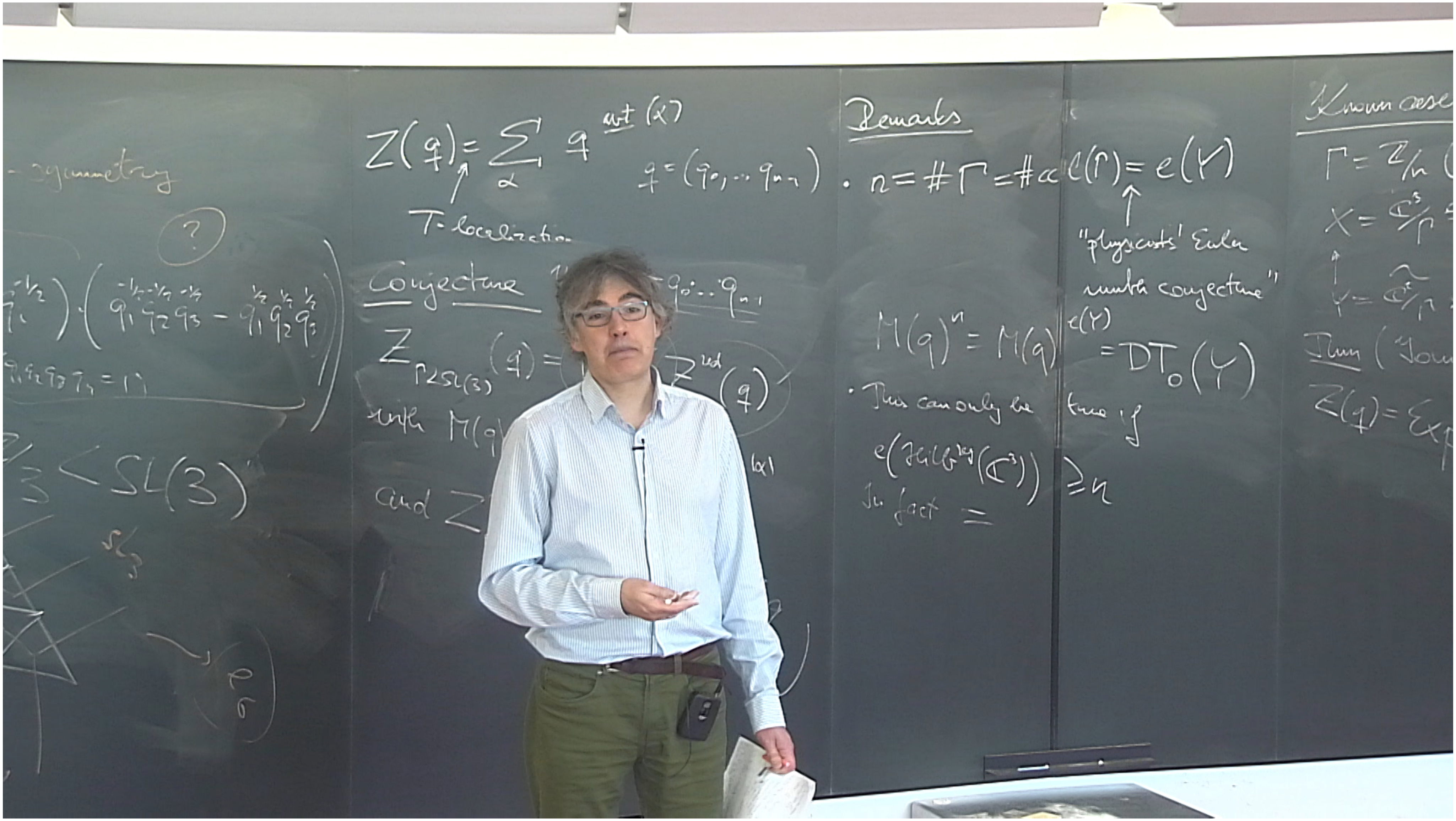
SL_2 - symmetry
Z/3 × Z/3 < SL(3)
Z(q) = Σ q
T-localization
Conjecture With q = q_0 ... q_n
with M(q) = Π (1-q^n)^-n
and


Remarks
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Conjecture
"physicists' Euler number conjecture"
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In fact