Title: Cohomological hall algebras from string and M theory

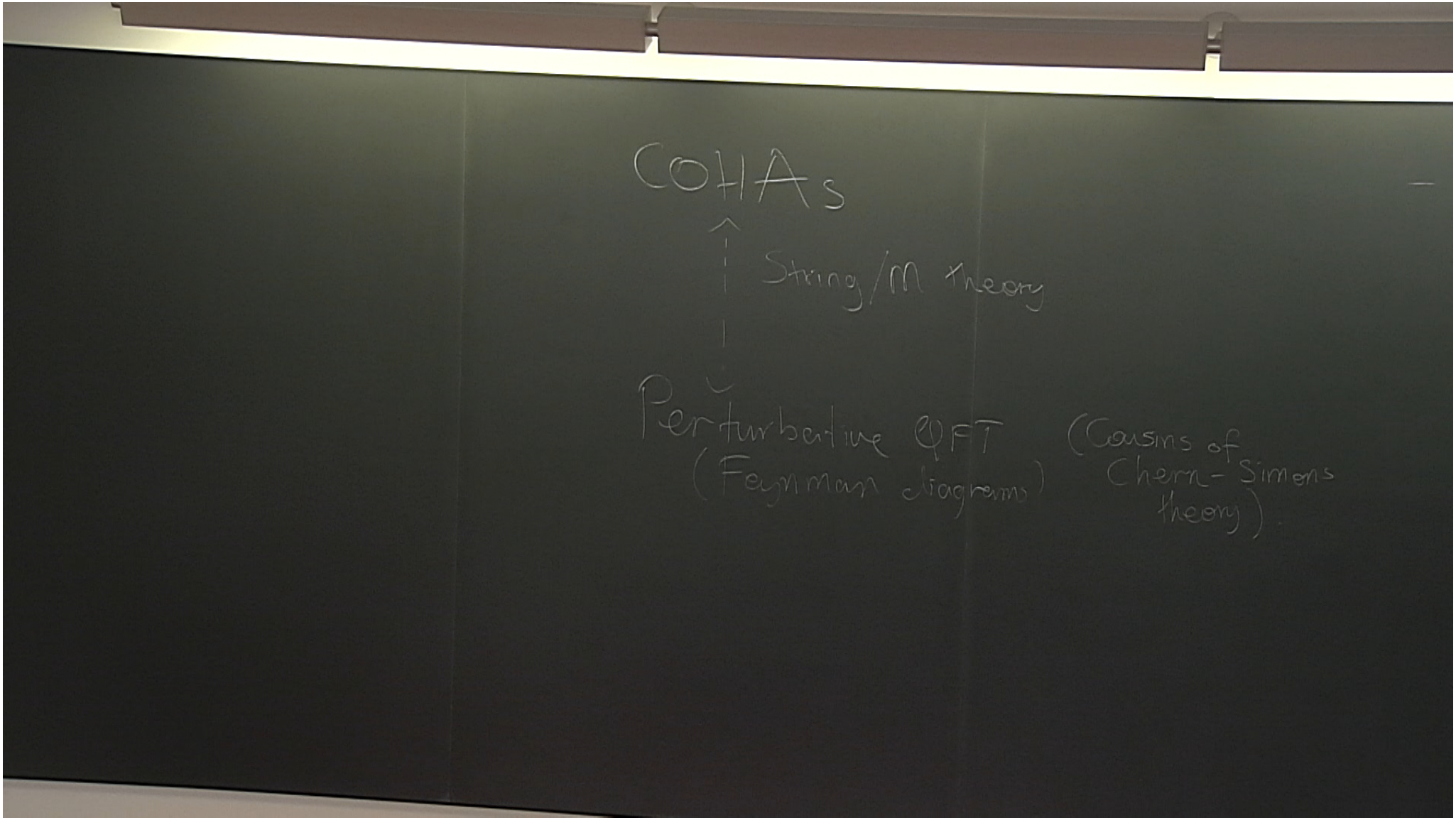
Speakers: Kevin Costello

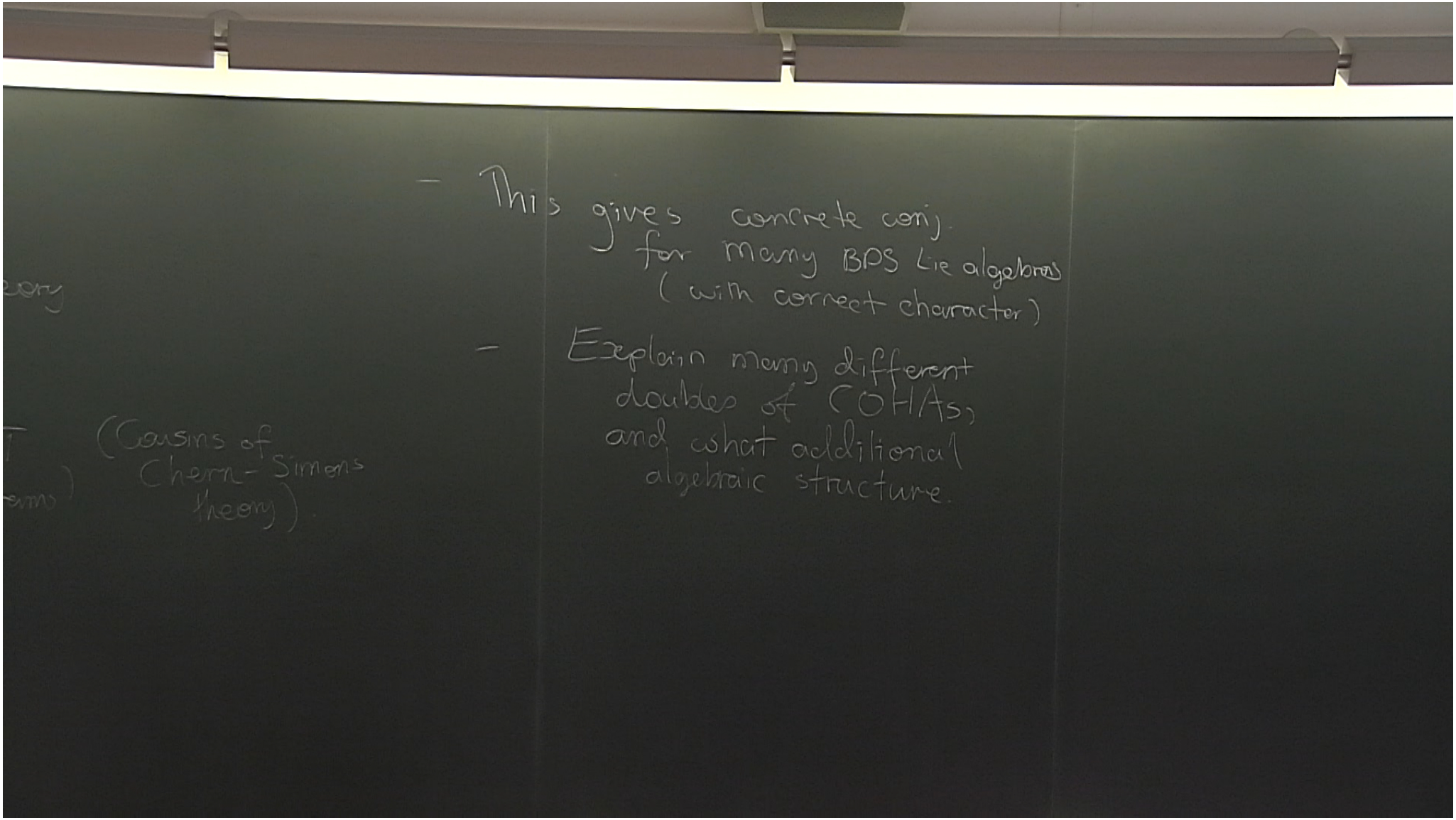
Collection: Cohomological Hall Algebras in Mathematics and Physics

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Abstract: I will outline a framework to understand certain COHAS from a mathematical incarnation of string theory and M theory. As an application, I will give a conjectural description of the COHA of the resolved conifold.


CoHAs
String/M theory
Perturbative QFT (Cousins of Chern-Simons theory)
(Feynman diagrams)


- This gives concrete conj.
for many BPS Lie algebras
(with correct character)
- Explain many different
doubles of COHAs,
and what additional
algebraic structure.
(Cousins of
Chern-Simons
theory)

Chern-Simons theory in dim's 3,4,5,6

3d $\int_{\mathbb{R}^3} CS(A)$ $\quad A \in \Omega^1(\mathbb{R}^3) \otimes g \rightsquigarrow U_q g$

4d $A \in (\Omega^1(\mathbb{R}^3 \times \mathbb{C})/dz) \otimes g \rightsquigarrow Y(g)$ C. Witten, Yamazaki

$\int_{\mathbb{R}^2 \times \mathbb{C}} dz\, CS(A)$ $\quad U_q(\hat{g})$

$\mathbb{R}^2 \times \mathbb{C}^\times$ $\quad E_{q,t}(g)$

$\mathbb{R}^2 \times E$

conj.
S Lie algebras
- character)
fferent
HAs,
ional
ture.

5d $\mathbb{R} \times \mathbb{C}^2$

$$A \in \left(\Omega^1(\mathbb{R} \times \mathbb{C}^2)/\langle dz_1, dz_2 \rangle\right) \otimes g_{N}$$

$$\int dz_1 dz_2 \left(\frac{1}{2}\left(A dA + \frac{i}{3} A \overset{*}{\varepsilon} A \overset{*}{\varepsilon} A\right)\right.$$

$\overset{*}{\varepsilon}$ = Moyal product on $\mathbb{C}^2$

$$A \wedge A \wedge A + \varepsilon A \partial_{z_1} A \partial_{z_2} A + \cdots$$

6d

$A \in \Omega$

$\int dz_1 dz_2$

5d $\mathbb{R} \times \mathbb{C}^2$

$$A \in \left(\Omega^1(\mathbb{R}\times\mathbb{C}^2)/\langle dz_1, dz_2\rangle\right) \otimes g_N$$

$$\frac{1}{g}\int dz_1 dz_2 \left(\frac{1}{2}\left(A dA + \frac{1}{3} A \overset{*}{\varepsilon} A \overset{*}{\varepsilon} A\right)\right.$$

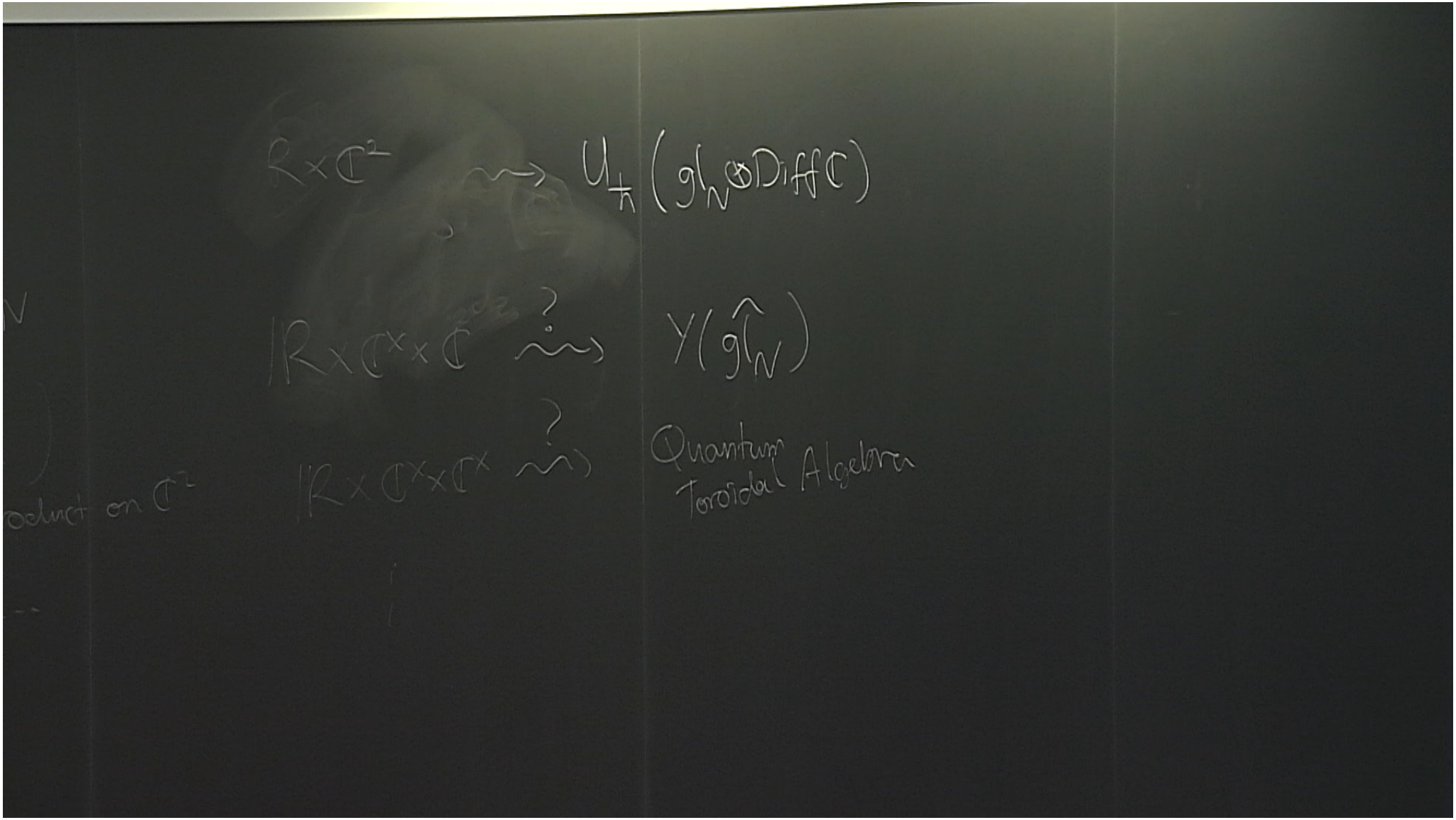
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6d

$A \in \Omega$

$\int dz_1 dz_2$


$\mathbb{R}\times\mathbb{C}^2 \rightsquigarrow U_\hbar(gl_N \otimes Diff\mathbb{C})$
$\mathbb{R}\times\mathbb{C}^\times\times\mathbb{C} \overset{?}{\rightsquigarrow} Y(\widehat{gl_N})$
$\mathbb{R}\times\mathbb{C}^\times\times\mathbb{C}^\times \overset{?}{\rightsquigarrow}$ Quantum Toroidal Algebra

6d

$A \in \Omega^{0,1}(\mathbb{C}^3) \otimes \mathfrak{g}_{N|N}$

$\int dz_1 dz_2 dz_3 \, CS(A)$

+ Kodaira Spenser action

$\mathbb{C}^\times \times \mathbb{C}^2$ ⇝ classically $U(\mathfrak{g}_{N|N}) \otimes \mathcal{O}(\mathbb{C}^\times \times \mathbb{C}^2)$

$\mathcal{X}\mathrm{Vect}_0(\mathbb{C}^\times \times \mathbb{C}^2)$

Div. free

- This

-

Field Theory $\rightsquigarrow$ Algebra

e.g. $\mathbb{R} \times \mathbb{C}^2$

Claim There is a category of line defects along $\mathbb{R} \times (0,0)$

This has a fibre functor $\rightarrow$ Vect

$U_\hbar(gl_N \otimes \mathrm{Diff}\,\mathbb{C})$ is End of fibre

Field Theory $\rightsquigarrow$ Algebra

e.g. $\mathbb{R} \times \mathbb{C}^2$ $\qquad$ $\mathbb{R} \times \mathbb{CP}^2 \longrightarrow \mathbb{R}$

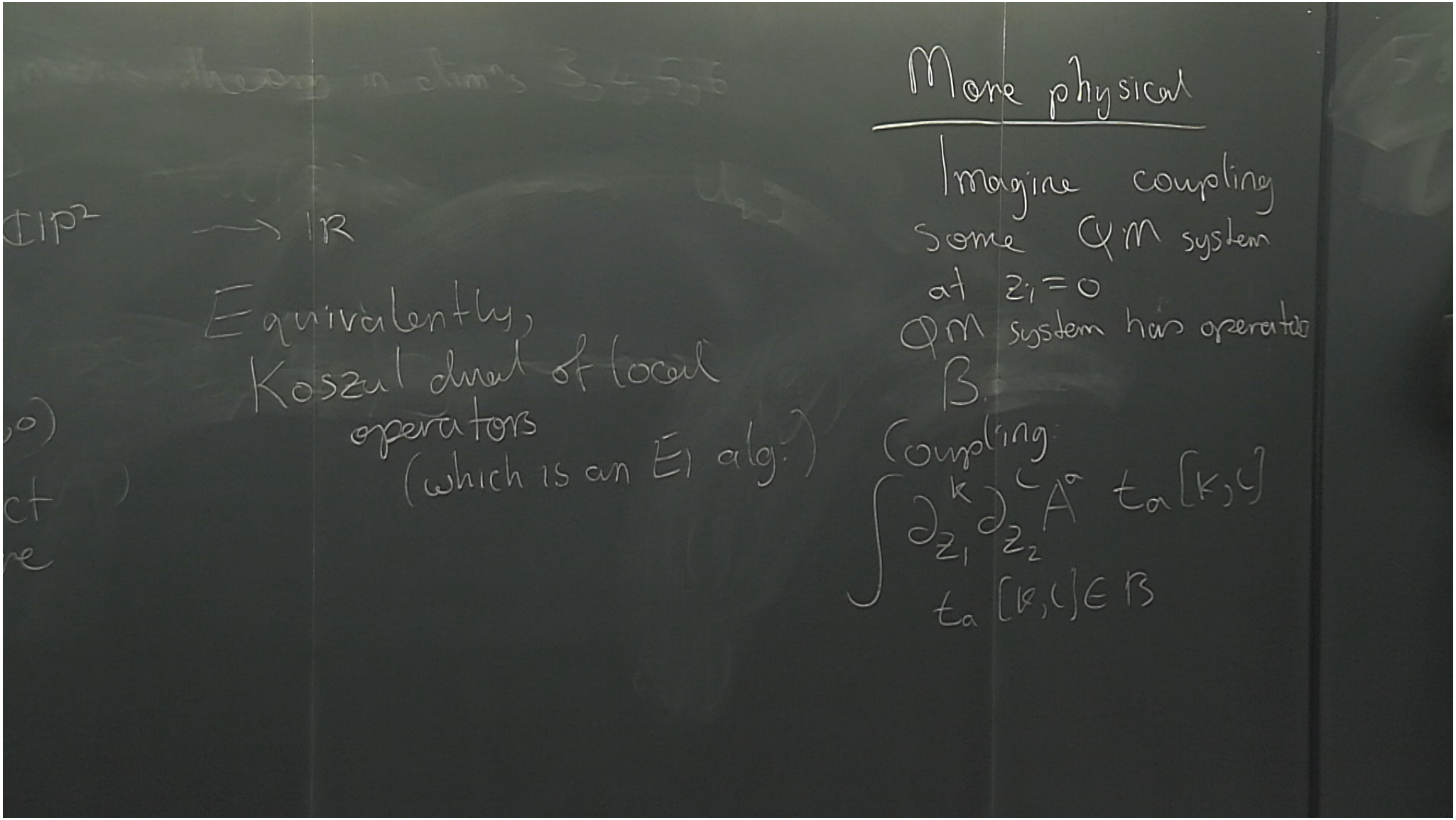
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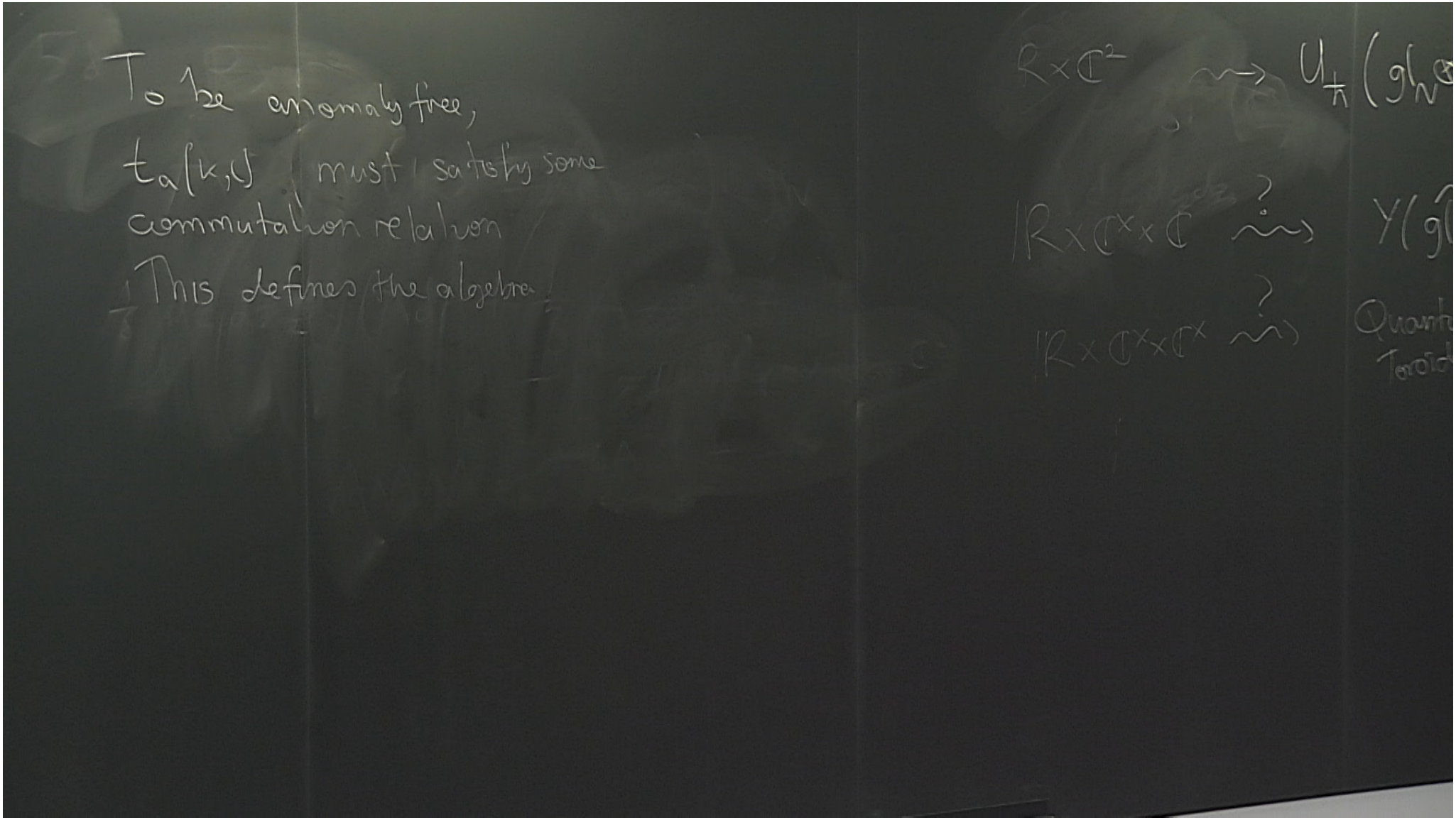
This has a fibre functor $\longrightarrow$ Vect

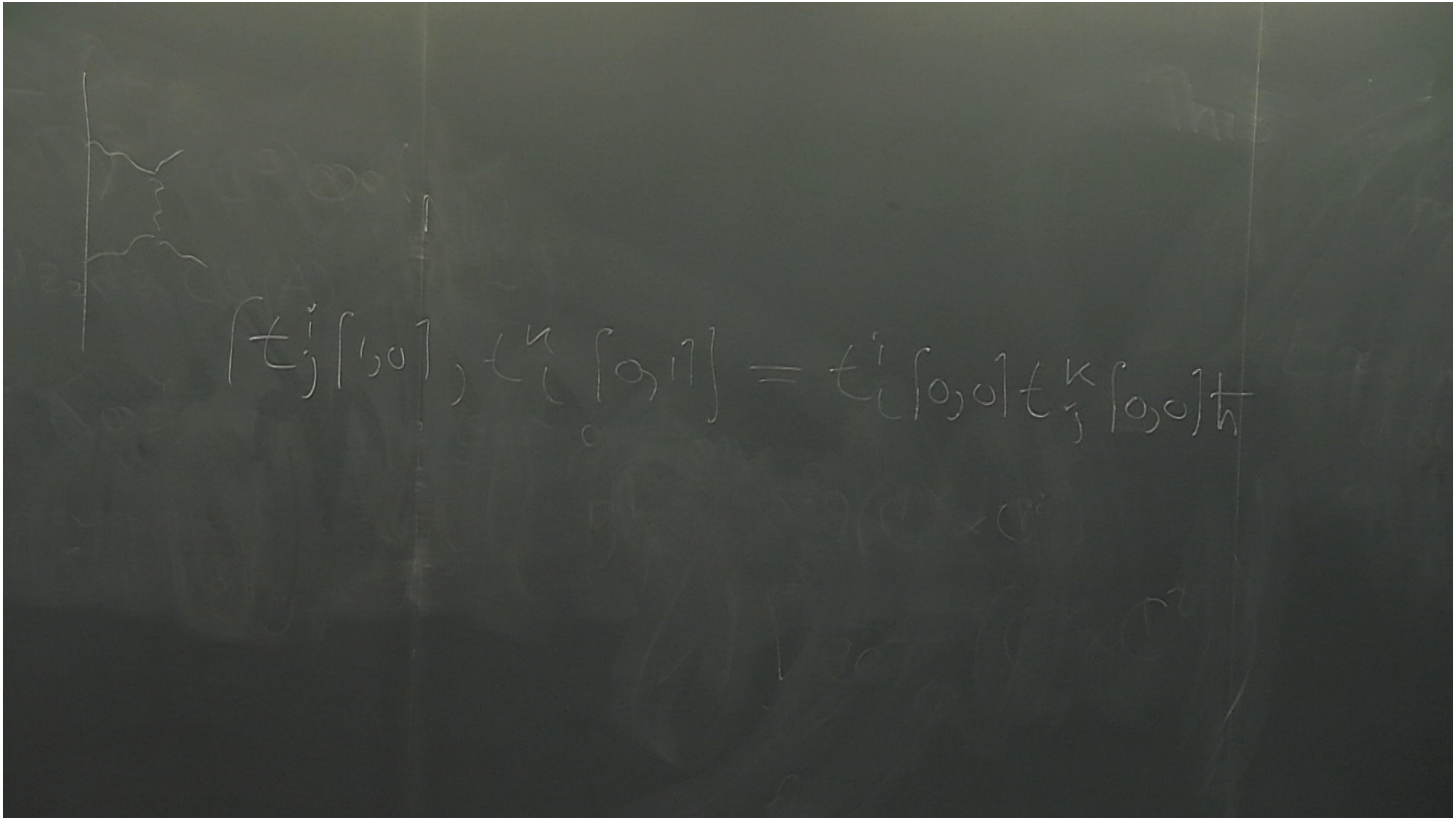
$U_\hbar(gl_N \otimes \mathrm{Diff}\,\mathbb{C})$ is End of fibre

$\mathbb{CP}^2 \longrightarrow \mathbb{R}$

Equivalently,
Koszul dual of local
operators
(which is an E_1 alg.)


$\mathbb{CP}^2 \longrightarrow \mathbb{R}$
Equivalently,
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operators
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More physical
Imagine coupling
some QFT system
at $z_1 = 0$
QFT system has operators
$\mathcal{B}$.
(coupling
$\int \partial_{z_1}^{k} \partial_{z_2}^{\ell} A^a \, t_a[k,\ell]$
$t_a[k,\ell] \in \mathcal{B}$


To be anomaly free,
$t_a(k,\ell)$ must satisfy some
commutation relation
This defines the algebra
$R \times \mathbb{C}^2$
$R \times \mathbb{C}^\times \times \mathbb{C}$?
$R \times \mathbb{C}^\times \times \mathbb{C}^\times$?


$[t_j^i[1,0], t_l^k[0,1]] = t_l^i[0,0]\, t_j^k[0,0]\, h$

Relation To COHAs

X is a toric CY3

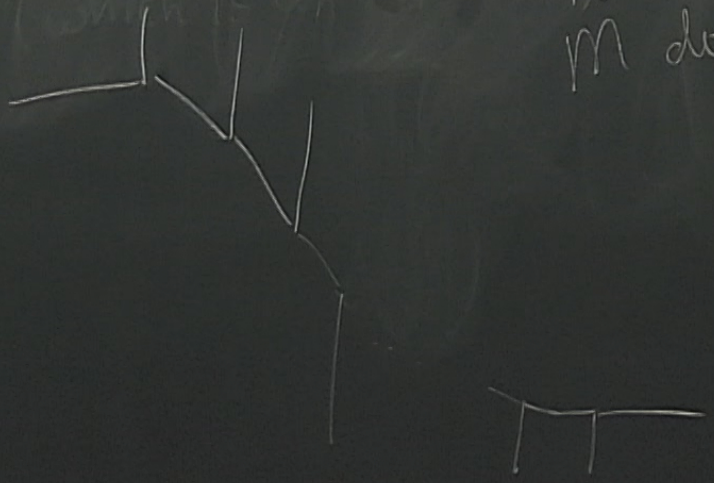
M-theory on X, w. equivalent parameters $\vec{\xi}$ we get a 5d theory: $T[X_{\vec{\xi}}]$

You can show

This 5d theory is the one we discussed

eg. params $\longleftrightarrow$ (Non-commutativity, coupling constant)

$\mathbb{X}$ toric diagram



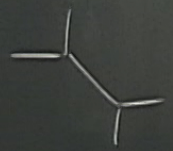
N up
M down
$\longrightarrow$ 5d for $g^1_{N/M}$

$m = 0$

$\tilde{\mathbb{C}}/\mathbb{Z}_N \times \mathbb{C}$

$N = m = 1$

$\Rightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$



commutativity,
oling constant)

$\longrightarrow$ 5d for $g_{N/m}$

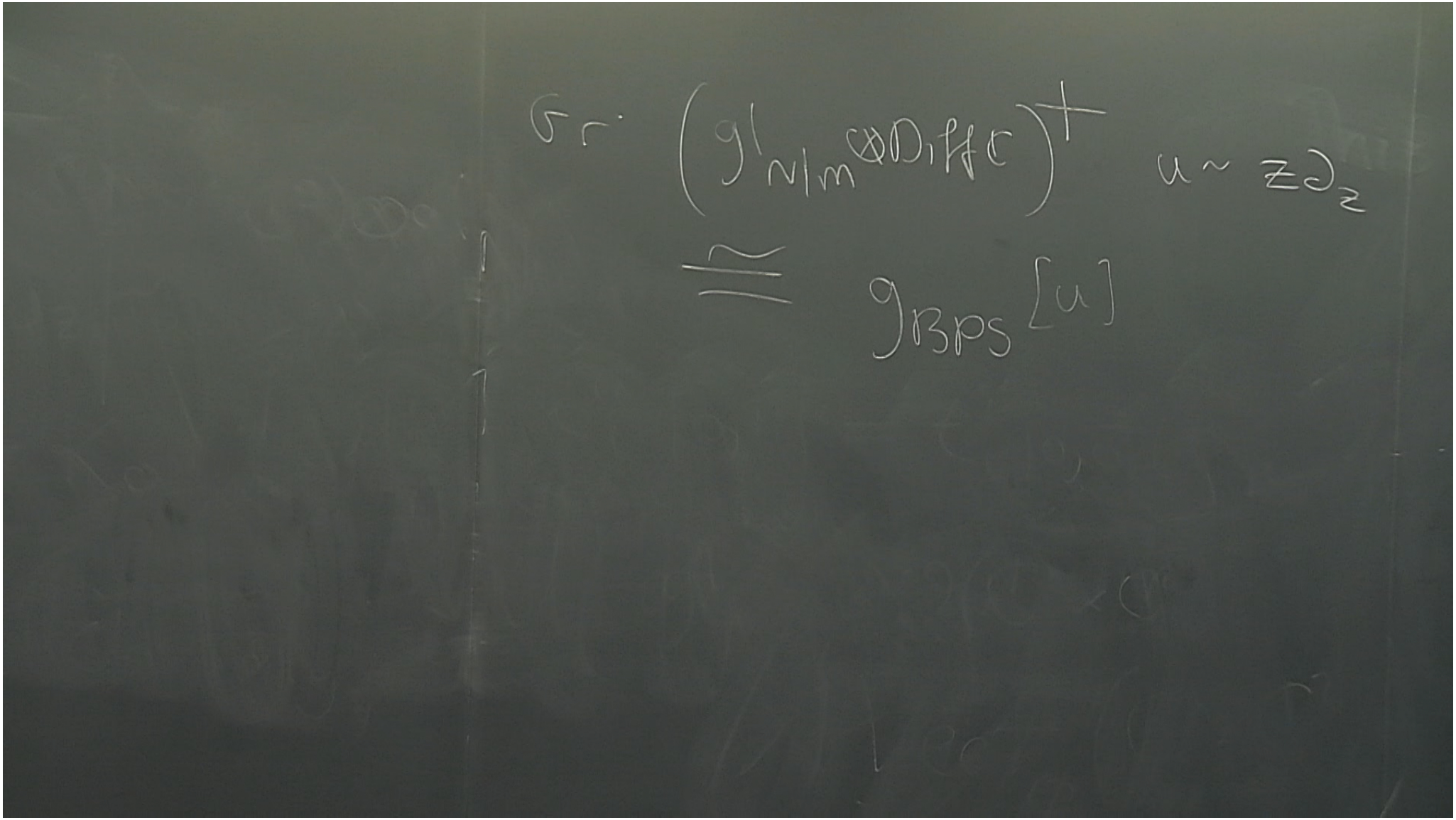
$X_{N,m} \rightsquigarrow gl_{N/m} \otimes \mathrm{Diff}\mathbb{C}$

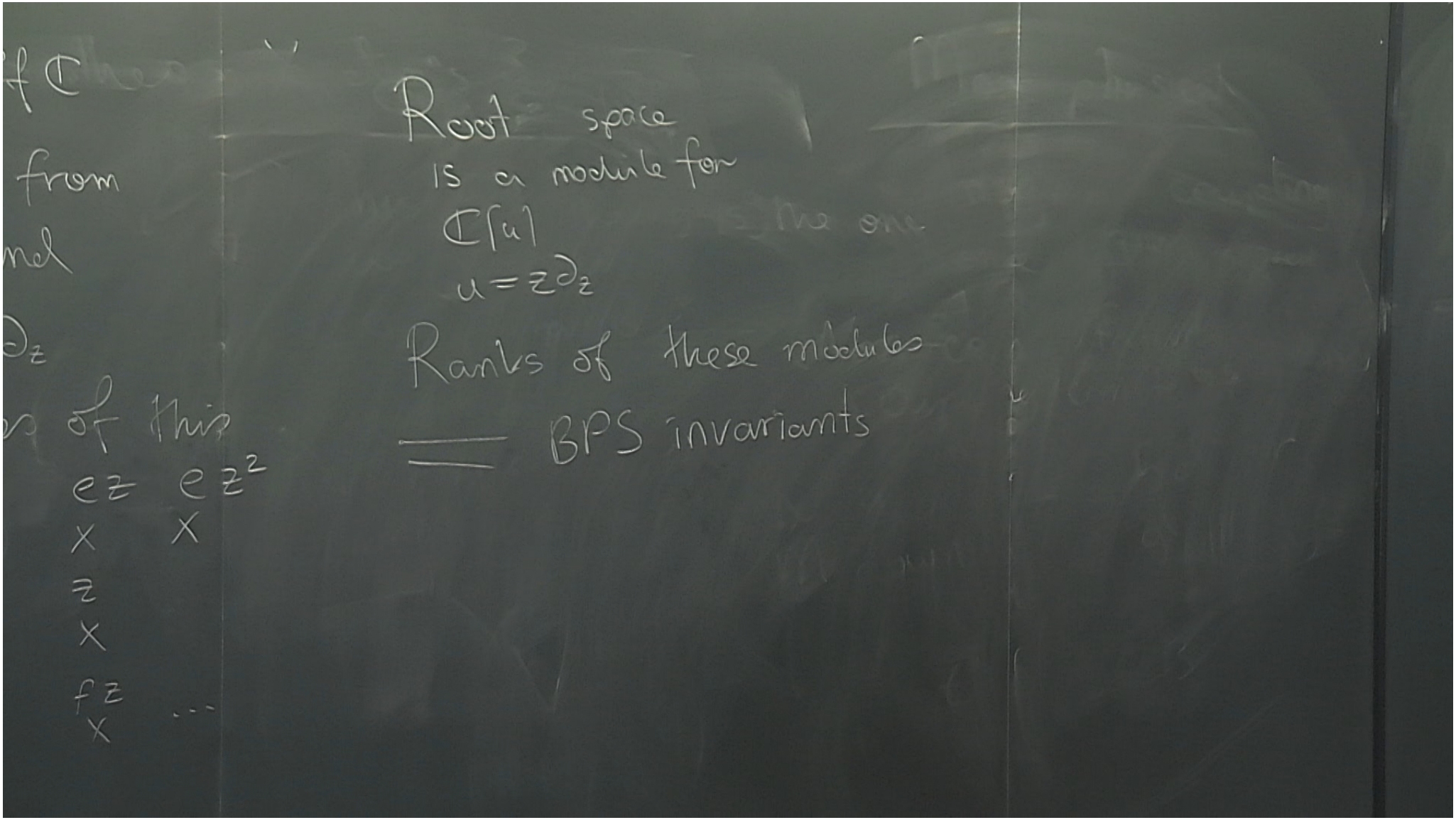
Notice $H^2(X_{N,m}) =$ coweight lattice of $gl_{N/m}$

$H^0(X_{N,m}) \Rightarrow$ an extra cocharacter of $gl_{N/m}$; namely $\mathrm{Id}\mathbb{Z}\partial_z \in gl_{N/m} \otimes \mathrm{Diff}\mathbb{C}$

If we choose a real linear comb. of these cocharacters, then consider -ve part.

Conj This is $g_{RSPS}(a)$


Gr. $(g'_{N|m} \otimes Diff_C)^+$ $u \sim z\partial_z$
$\cong$ $g_{BPS}[u]$

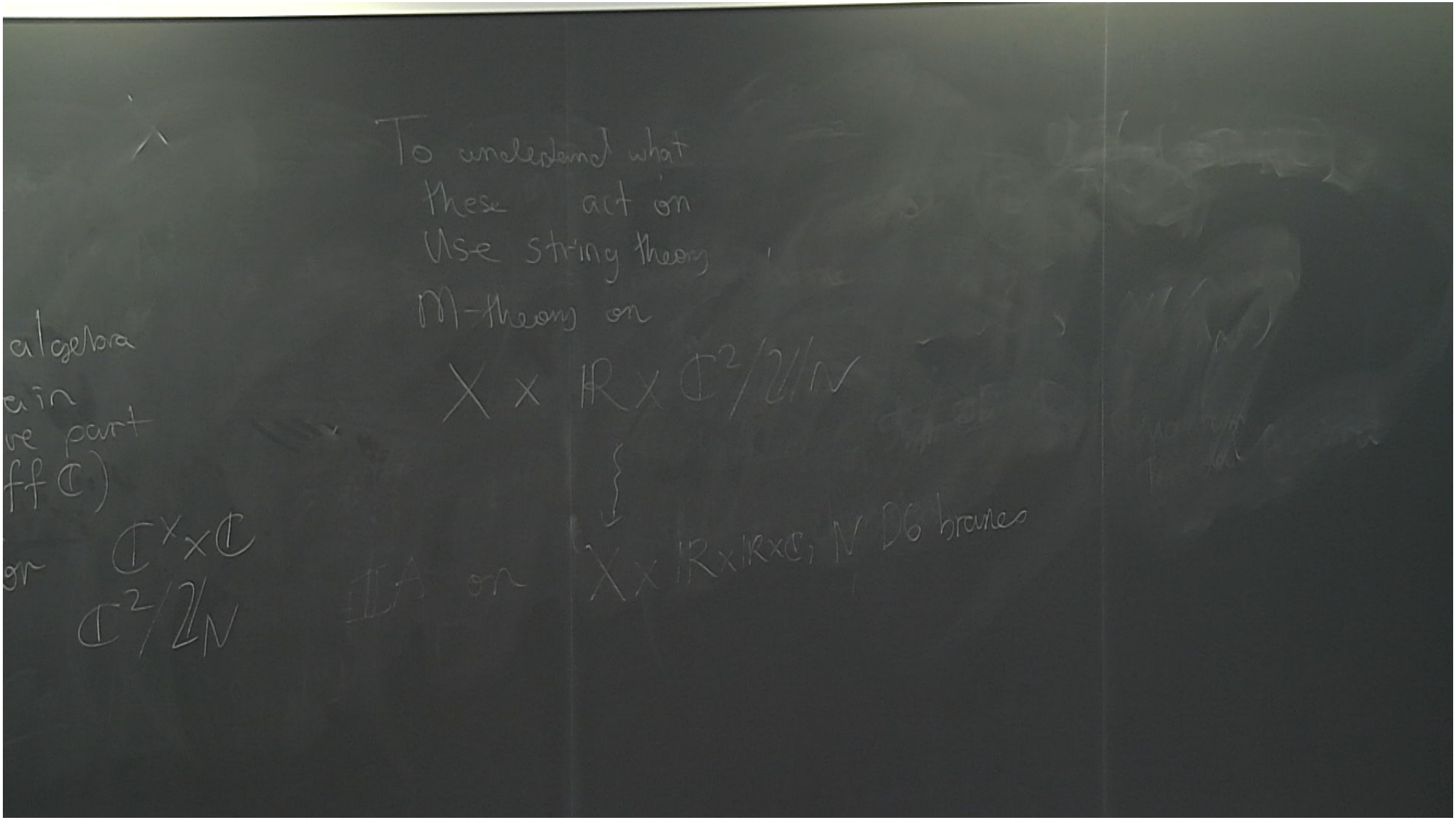

Root space
is a module for
$\mathbb{C}[u]$
$u = z\partial_z$
Ranks of these modules
= BPS invariants

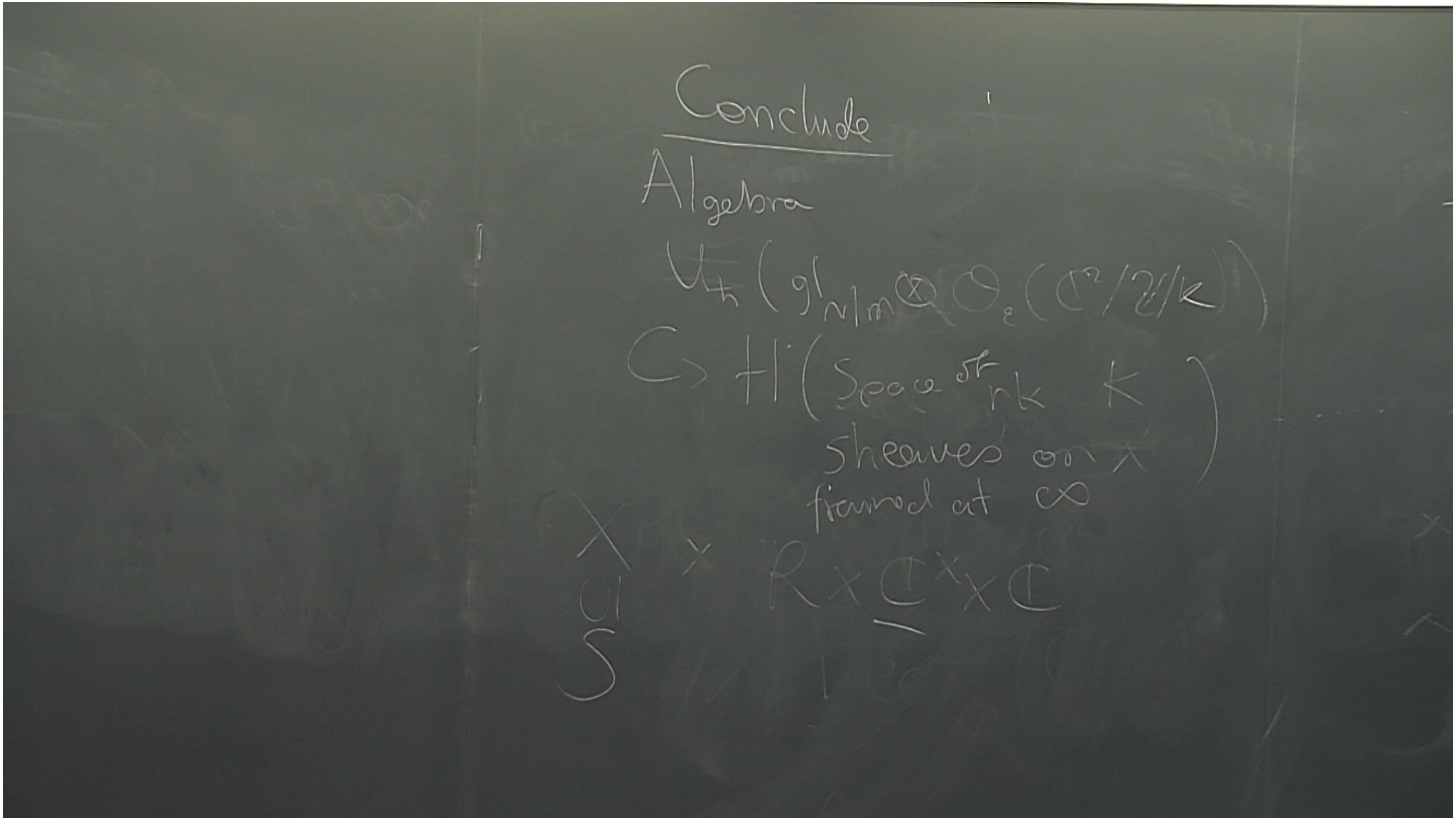
Doubles

5d theory on
$\mathbb{R} \times \mathbb{C}^2$

$\rightsquigarrow$ a particular algebra
seems to contain
COHA as +ve part
$U_+(gl_{N/m} \otimes \mathrm{Diff}\,\mathbb{C})$

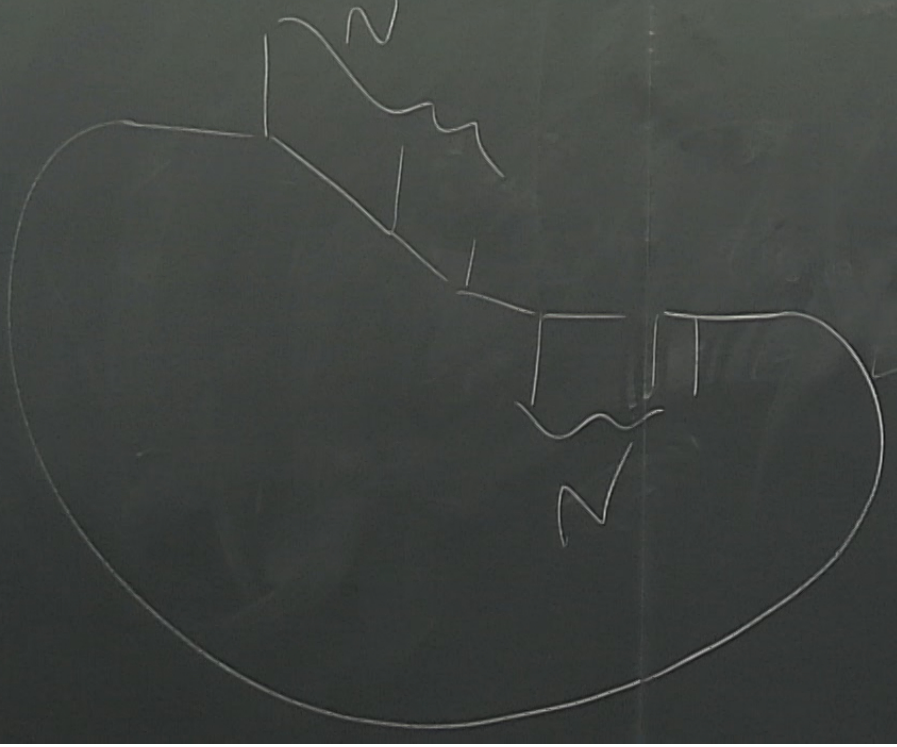
Same +ve part for $\mathbb{C}^\times \times \mathbb{C}$
$\mathbb{C}^2/\mathbb{Z}_N$


To understand what
these act on
Use string theory
M-theory on
$X \times \mathbb{R} \times \mathbb{C}^2/\mathbb{Z}_N$
algebra
ein
ve part
ff $\mathbb{C}$)
or $\mathbb{C}^\times \times \mathbb{C}$
$\mathbb{C}^2/\mathbb{Z}_N$
IIA on
$X \times \mathbb{R} \times \mathbb{R} \times \mathbb{C}$; N D6 branes


Conclude
Algebra
$U_h(gl_{N|m} \otimes \mathcal{O}_\epsilon(\mathbb{C}^2/\mathbb{Z}/k))$
$\hookrightarrow H^i$(Space of rk k sheaves on X framed at ∞)
X
$\downarrow$
S
$\times R \times \mathbb{C}^\times \times \mathbb{C}$

6d Theory

Consider the quasi-toric CY3 which is

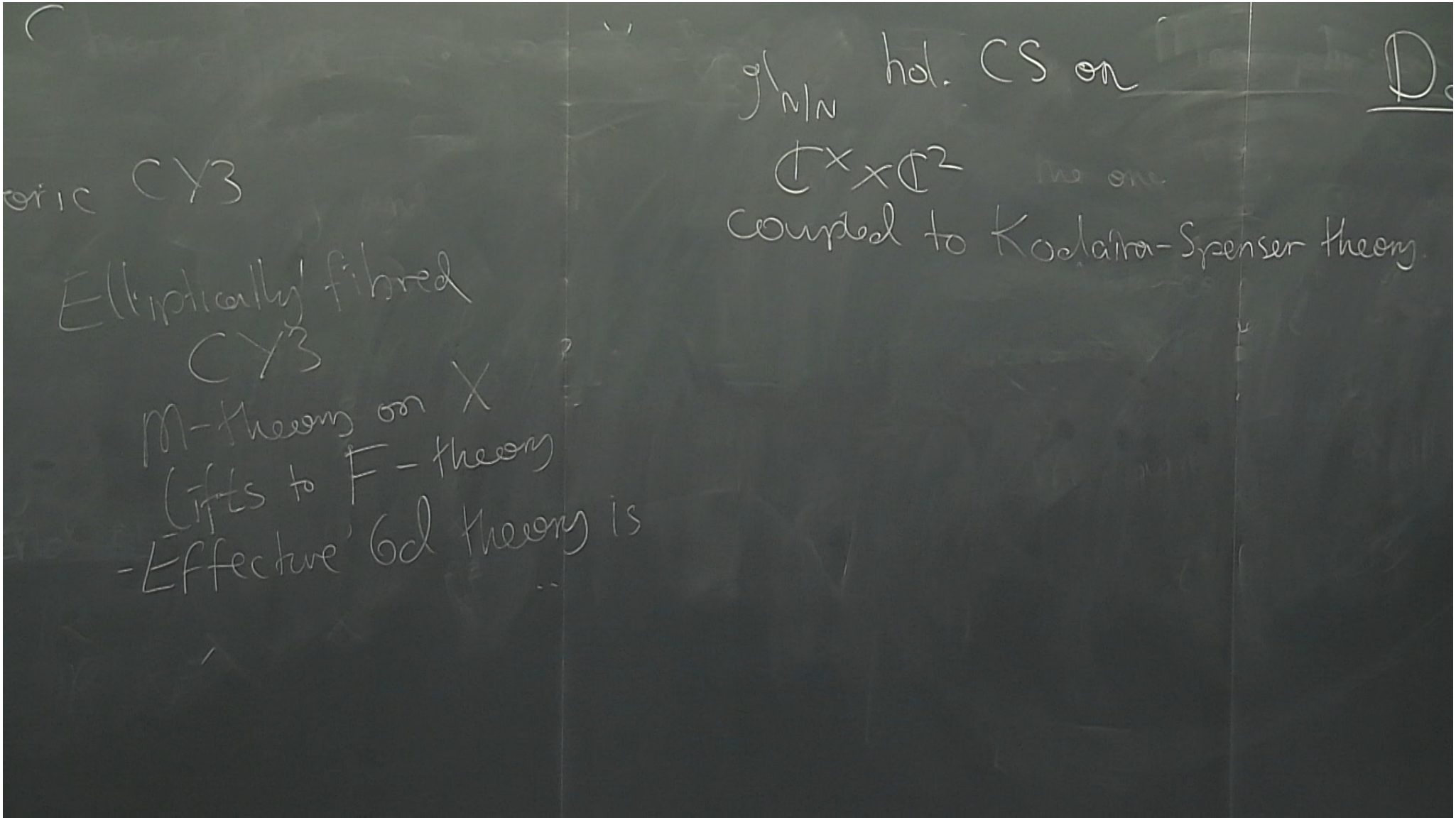


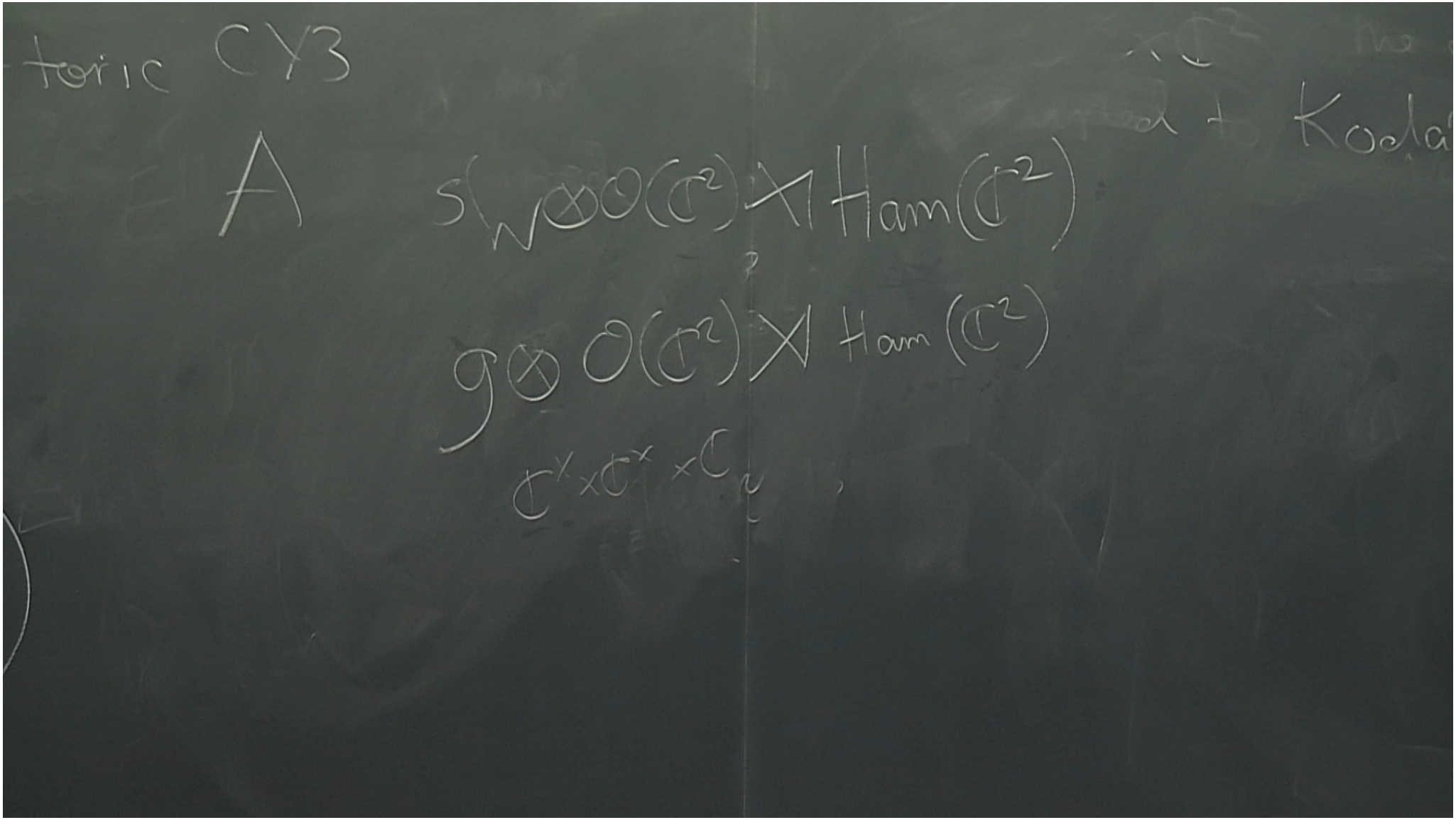
Elliptically fibred CY3

M-theory on X

Gets to F-theory

-Effective 6d theory is


$gl_{N|N}$ hol. CS on
$\mathbb{C}^\times \times \mathbb{C}^2$
coupled to Kodaira-Spencer theory
Elliptically fibred
CY3
M-theory on X
(IIA) to F-theory
-Effective 6d theory is


toric CY3
A
$SU(N) \otimes O(\mathbb{C}^2) \times Ham(\mathbb{C}^2)$
$SO(N) \otimes O(\mathbb{C}^2) \times Ham(\mathbb{C}^2)$
$\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}_R$