Title: Short star-products for filtered quantizations

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Collection: Cohomological Hall Algebras in Mathematics and Physics

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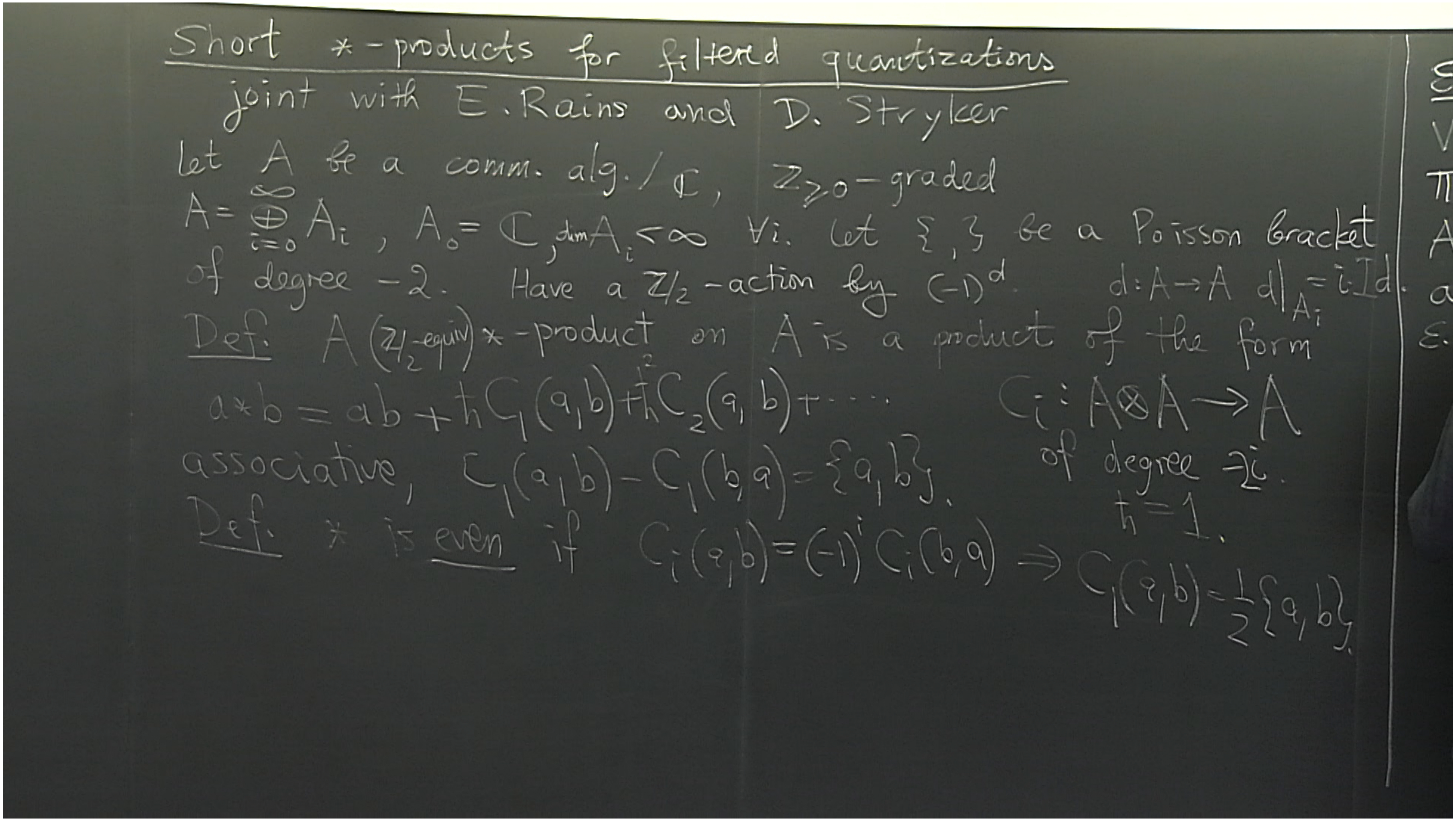
Abstract: Let A be a filtered Poisson algebra with Poisson bracket ${ , }$ of degree -2. A star product on A is an associative product $*: A\otimes A\to A$ given by $a*b=ab+\sum_{i\ge 1}C_i(a,b)$, where C_i has degree $-2i$ and $C_1(a,b)-C_1(b,a)={a, b}$. We call the product * "even", if $C_i(a,b)=(-1)^iC_i(b,a)$ for all i, and call it "short", if $C_i(a,b)=0$ whenever $i> min(deg(a),deg(b))$.

Motivated by three-dimensional $N=4$ superconformal field theory, In 2016 Beem, Peelaers and Rastelli considered short even star-products for homogeneous symplectic singularities (more precisely, hyperK\"ahler cones) and conjectured that that they exist and depend on finitely many parameters. We prove the dependence on finitely many parameters in general and existence for a large class of examples, using the connection of this problem with zeroth Hochschild homology of quantizations suggested by Kontsevich.

Beem, Peelaers and Rastelli also computed the first few terms of short quantizations for Kleinian singularities of type A, which were later computed to all orders by Dedushenko, Pufu and Yacoby. We will discuss some generalizations of these results.

This is joint work with Eric Rains and Douglas Stryker.



Short $*$-products for filtered quantizations

joint with E. Rains and D. Stryker

Let A be a comm. alg. / $\mathbb{C}$, $\mathbb{Z}_{\geq 0}$-graded $A = \bigoplus_{i=0}^{\infty} A_i$, $A_0 = \mathbb{C}$, $\dim A_i < \infty$ $\forall i$. Let $\{ , \}$ be a Poisson bracket of degree -2. Have a $\mathbb{Z}/2$-action by $(-1)^d$. $d: A \to A$ $d|_{A_i} = i \cdot \mathrm{Id}$.

<u>Def.</u> A ($\mathbb{Z}/2$-equiv) $*$-product on A is a product of the form

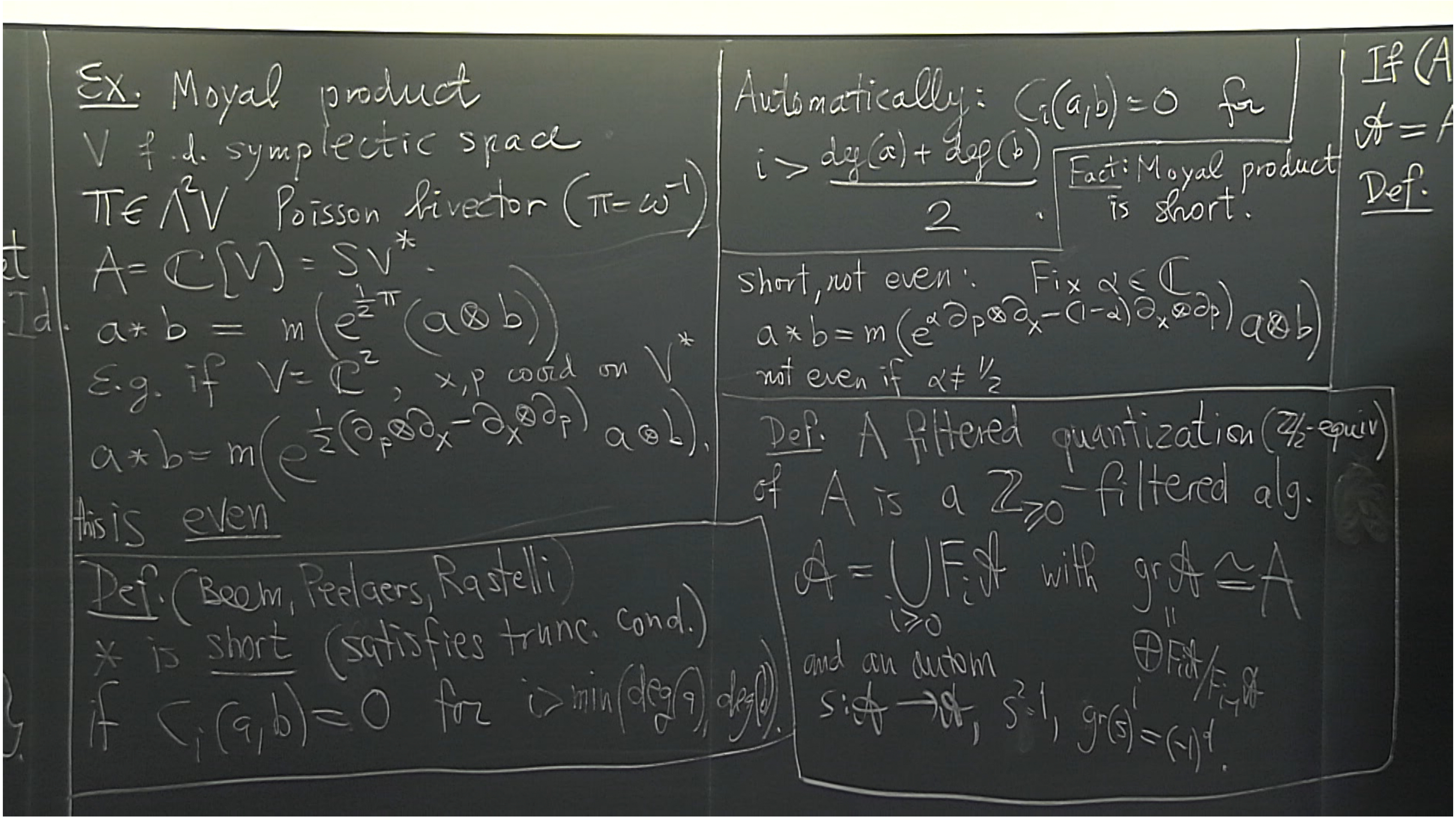
$$a * b = ab + \hbar C_1(a,b) + \hbar^2 C_2(a,b) + \cdots \qquad C_i : A \otimes A \to A$$

of degree $-2i$.

associative, $C_1(a,b) - C_1(b,a) = \{a,b\}$.

$\hbar = 1$.

<u>Def.</u> $*$ is <u>even</u> if $C_i(a,b) = (-1)^i C_i(b,a) \Rightarrow C_1(a,b) = \frac{1}{2}\{a,b\}$.



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Id.

y.

Ex. Moyal product

V f.d. symplectic space

$\pi \in \Lambda^2 V$ Poisson bivector $(\pi = \omega^{-1})$

$A = \mathbb{C}[V] = SV^*$.

$$a * b = m\left(e^{\frac{1}{2}\pi}(a \otimes b)\right)$$

E.g. if $V = \mathbb{C}^2$, x, p coord on V^*

$$a * b = m\left(e^{\frac{1}{2}(\partial_p \otimes \partial_x - \partial_x \otimes \partial_p)} a \otimes b\right).$$

this is even

Def. (Beem, Peelaers, Rastelli)

$*$ is short (satisfies trunc. cond.)

if $C_i(a,b) = 0$ for $i > \min(\deg(a), \deg(b))$.

Automatically: $C_i(a,b) = 0$ for $i > \dfrac{\deg(a) + \deg(b)}{2}$.

Fact: Moyal product is short.

short, not even: Fix $\alpha \in \mathbb{C}$

$$a * b = m\left(e^{\alpha \partial_p \otimes \partial_x - (1-\alpha)\partial_x \otimes \partial_p} a \otimes b\right)$$

not even if $\alpha \neq 1/2$

Def. A filtered quantization ($\mathbb{Z}/2$-equiv) of A is a $\mathbb{Z}_{\geq 0}$-filtered alg.

$$\mathcal{A} = \bigcup_{i \geq 0} F_i \mathcal{A} \text{ with } \underset{\underset{\bigoplus_i F_i\mathcal{A}/F_{i-1}\mathcal{A}}{\parallel}}{\mathrm{gr}\,\mathcal{A}} \cong A$$

and an autom $s: \mathcal{A} \to \mathcal{A}$, $s^2 = 1$, $\mathrm{gr}(s) = (-1)^d$.

If (A

$\mathcal{A} = A$

Def.

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…deg(b)

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…tom

$\mathcal{A}$, $s^2=1$, $\mathrm{gr}(s)=(-1)^d$.

If $(A,*)$ is a $*$-prod then $\mathcal{A}=A$, mult $=*$, $s=(-1)^d$.

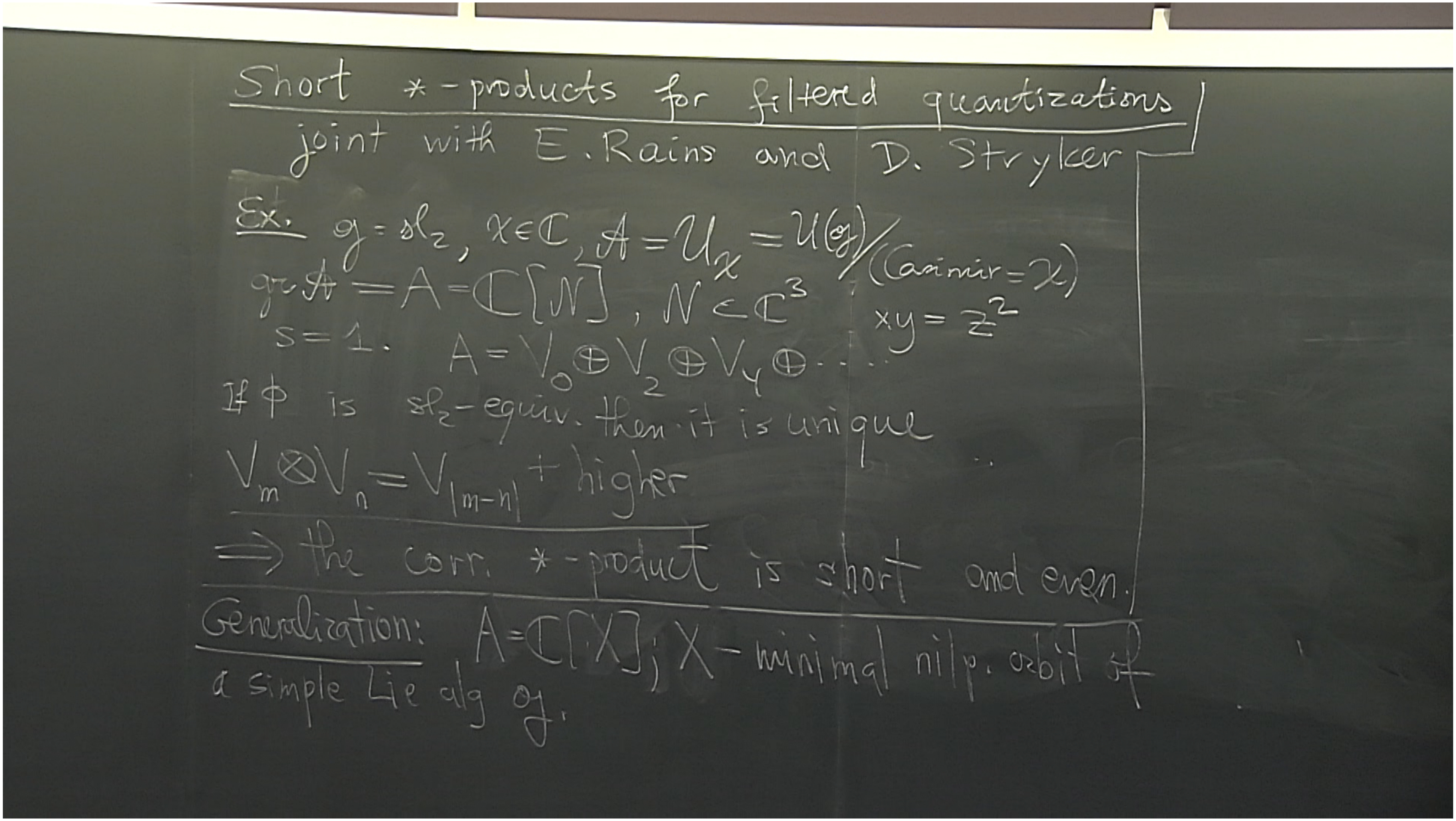
Def. A quantiz map is a linear map $\phi: A\to\mathcal{A}$, $\mathrm{gr}\,\phi=\mathrm{Id}_A$, $\phi\circ(-1)^d=s\circ\phi$.

Then get $*$ by $a*b=\phi^{-1}(\phi(a)\phi(b))$

$(\phi(a)=\hat{a})$

Lemma. This is a bijection.

Lemma: $*$ even $\Leftrightarrow$ $\mathcal{A}$ is equipped with an antiautom $\sigma:\mathcal{A}\to\mathcal{A}$ s.t. $\phi\circ(\sqrt{-1})^d=\sigma\circ\phi$, $\sigma^2=s$



Short $*$-products for filtered quantizations

joint with E. Rains and D. Stryker

Ex. $\mathfrak{g}=\mathfrak{sl}_2$, $\chi\in\mathbb{C}$, $\mathcal{A}=\mathcal{U}_\chi=\mathcal{U}(\mathfrak{g})/(\text{Casimir}=\chi)$

$\mathrm{gr}\,\mathcal{A}=A=\mathbb{C}[\mathcal{N}]$, $\mathcal{N}\subset\mathbb{C}^3$ $\quad xy=z^2$

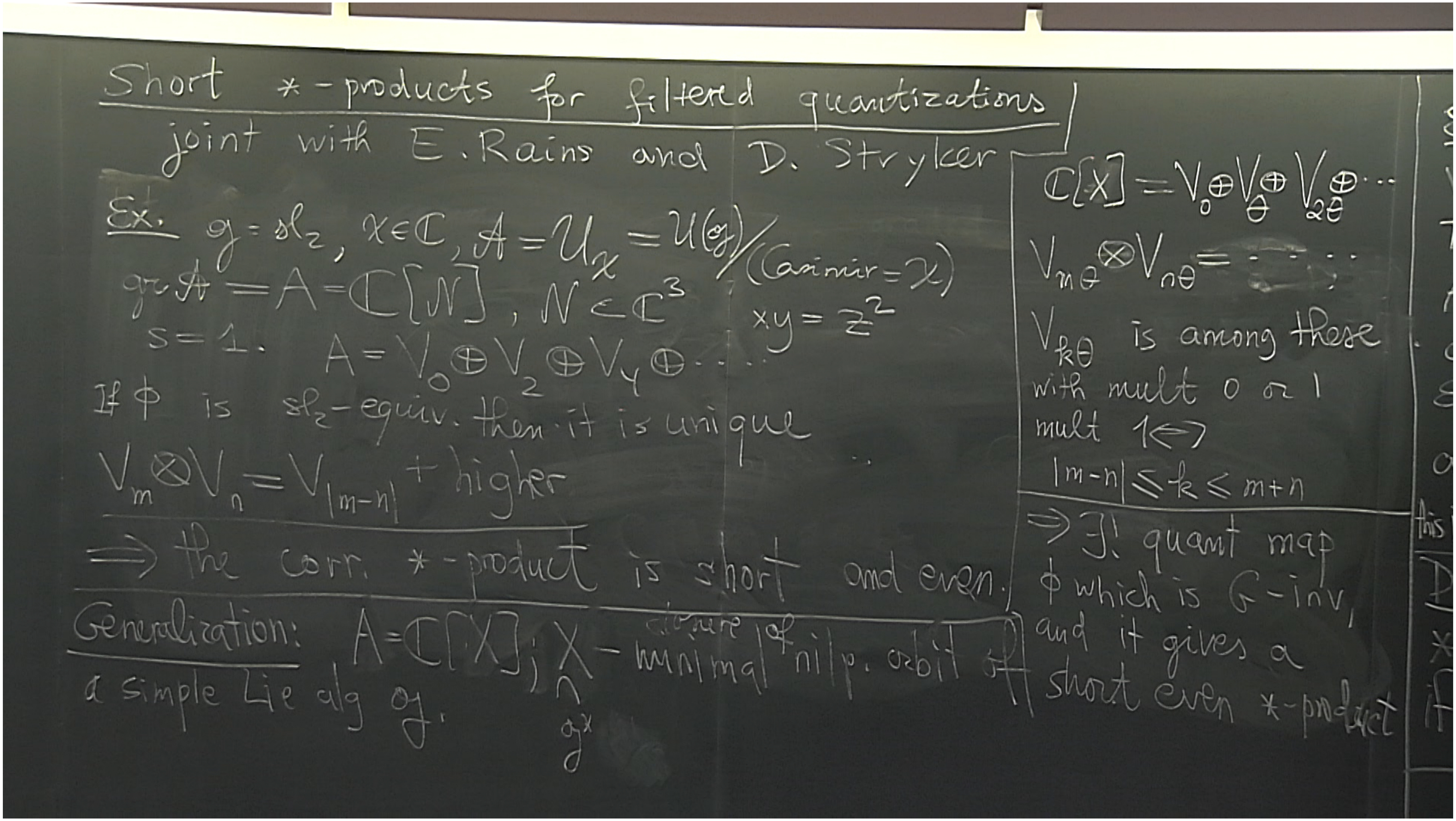
$s=1$. $\quad A=V_0\oplus V_2\oplus V_4\oplus\cdots$

If ϕ is $\mathfrak{sl}_2$-equiv. then it is unique

$V_m\otimes V_n=V_{|m-n|}$ + higher

$\Rightarrow$ the corr. $*$-product is short and even.

Generalization: $A=\mathbb{C}[X]$; X - minimal nilp. orbit of a simple Lie alg $\mathfrak{g}$.



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Generalization: $A=\mathbb{C}[X]$; $\begin{matrix}X\\ \cap\\ \mathfrak{g}^*\end{matrix}$ - closure of minimal nilp. orbit of a simple Lie alg $\mathfrak{g}$.

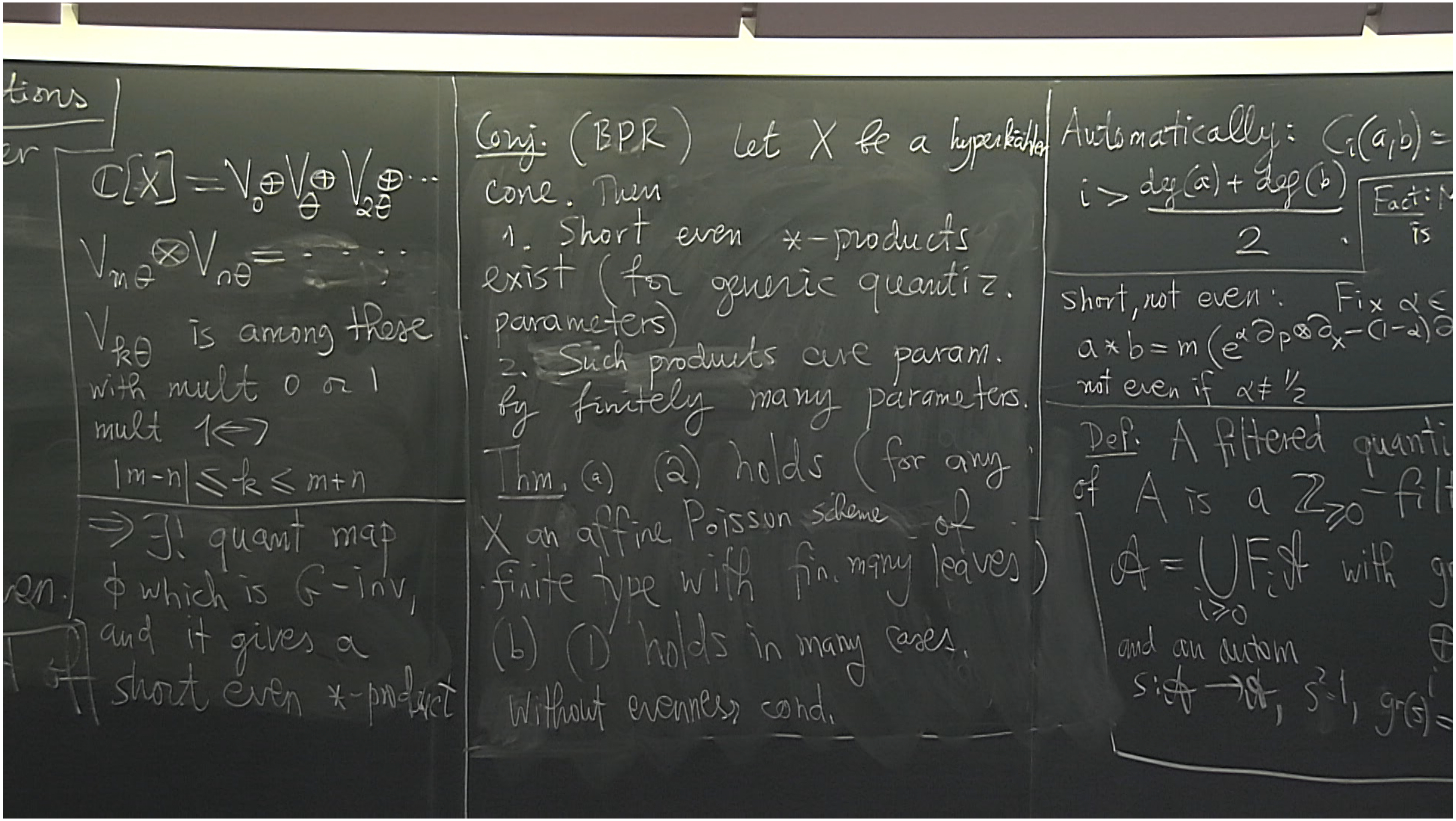
$\mathbb{C}[X]=V_0\oplus V_\theta\oplus V_{2\theta}\oplus\cdots$

$V_{m\theta}\otimes V_{n\theta}=\cdots$

$V_{k\theta}$ is among these with mult 0 or 1

mult 1 $\Leftrightarrow$ $|m-n|\leq k\leq m+n$

$\Rightarrow\exists!$ quant map ϕ which is G-inv, and it gives a short even $*$-product



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Conj. (BPR) Let X be a hyperkähler cone. Then

1. Short even $*$-products exist (for generic quantiz. parameters)
2. Such products are param. by finitely many parameters.

Thm. (a) (2) holds (for any X an affine Poisson scheme of finite type with fin. many leaves)

(b) (1) holds in many cases.

Without evenness cond.

Automatically: $C_i(a,b) =$

$i > \dfrac{\deg(a) + \deg(b)}{2}$.

Fact:

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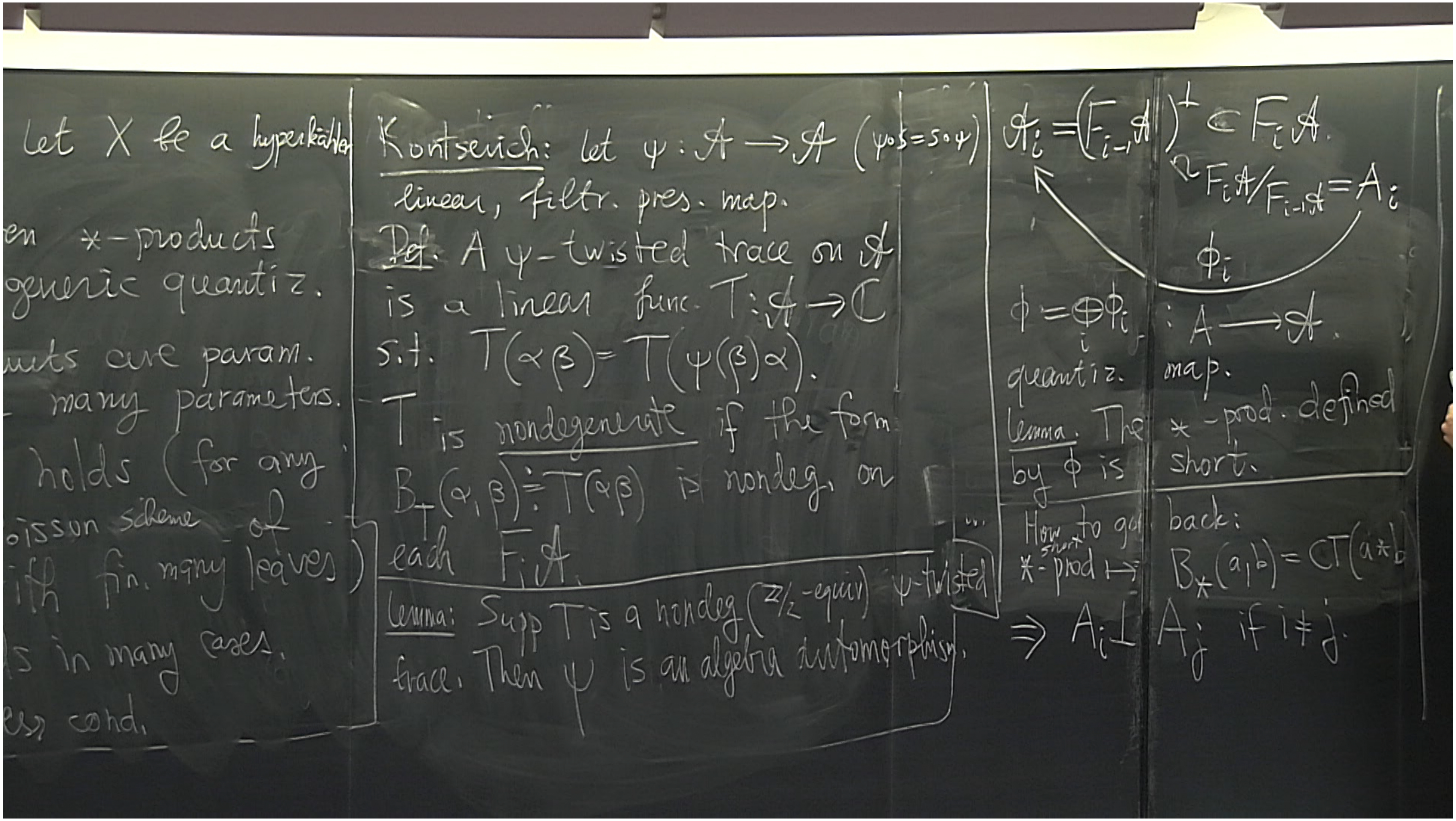
Def. A ψ-twisted trace on $\mathcal{A}$ is a linear func. $T : \mathcal{A} \to \mathbb{C}$ s.t. $T(\alpha\beta) = T(\psi(\beta)\alpha)$.

T is nondegenerate if the form $B_T(\alpha, \beta) \doteq T(\alpha\beta)$ is nondeg. on each $F_i\mathcal{A}$.

Lemma: Supp T is a nondeg ($\mathbb{Z}/2$-equiv) ψ-twisted trace. Then ψ is an algebra automorphism.



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$\mathcal{A}_i = (F_{i-1}\mathcal{A})^\perp \subset F_i\mathcal{A}$.

$\cong F_i\mathcal{A}/F_{i-1}\mathcal{A} = A_i$

$$\begin{tikzcd} \mathcal{A}_i & A_i \arrow[l, bend left, "\phi_i"] \end{tikzcd}$$

$\phi = \bigoplus_i \phi_i : A \longrightarrow \mathcal{A}$.

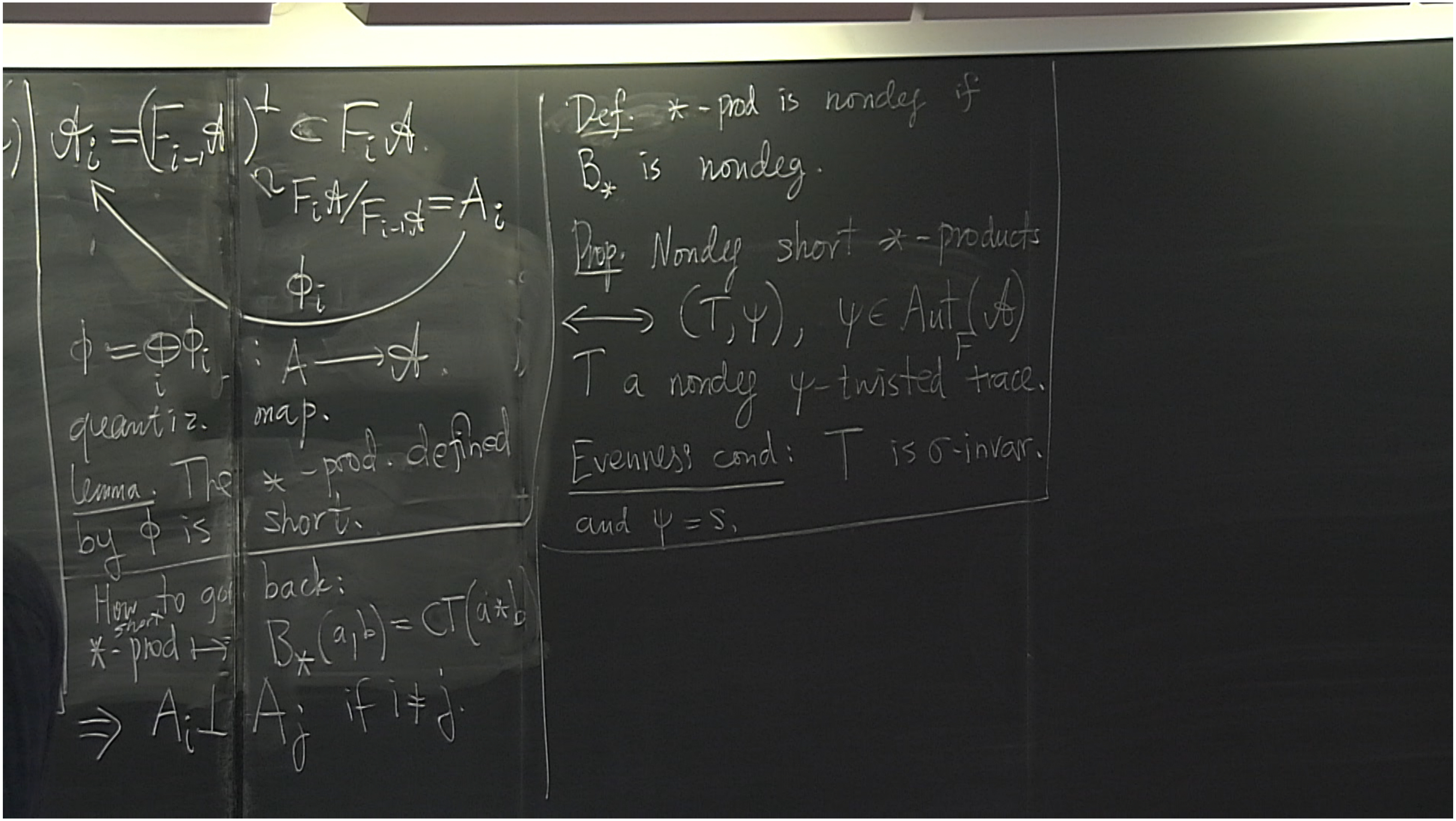
quantiz. map.

Lemma. The $*$-prod. defined by ϕ is short.

How to go back:

short $*$-prod $\mapsto$ $B_*(a,b) = CT(a*b)$

$\Rightarrow A_i \perp A_j$ if $i \neq j$.



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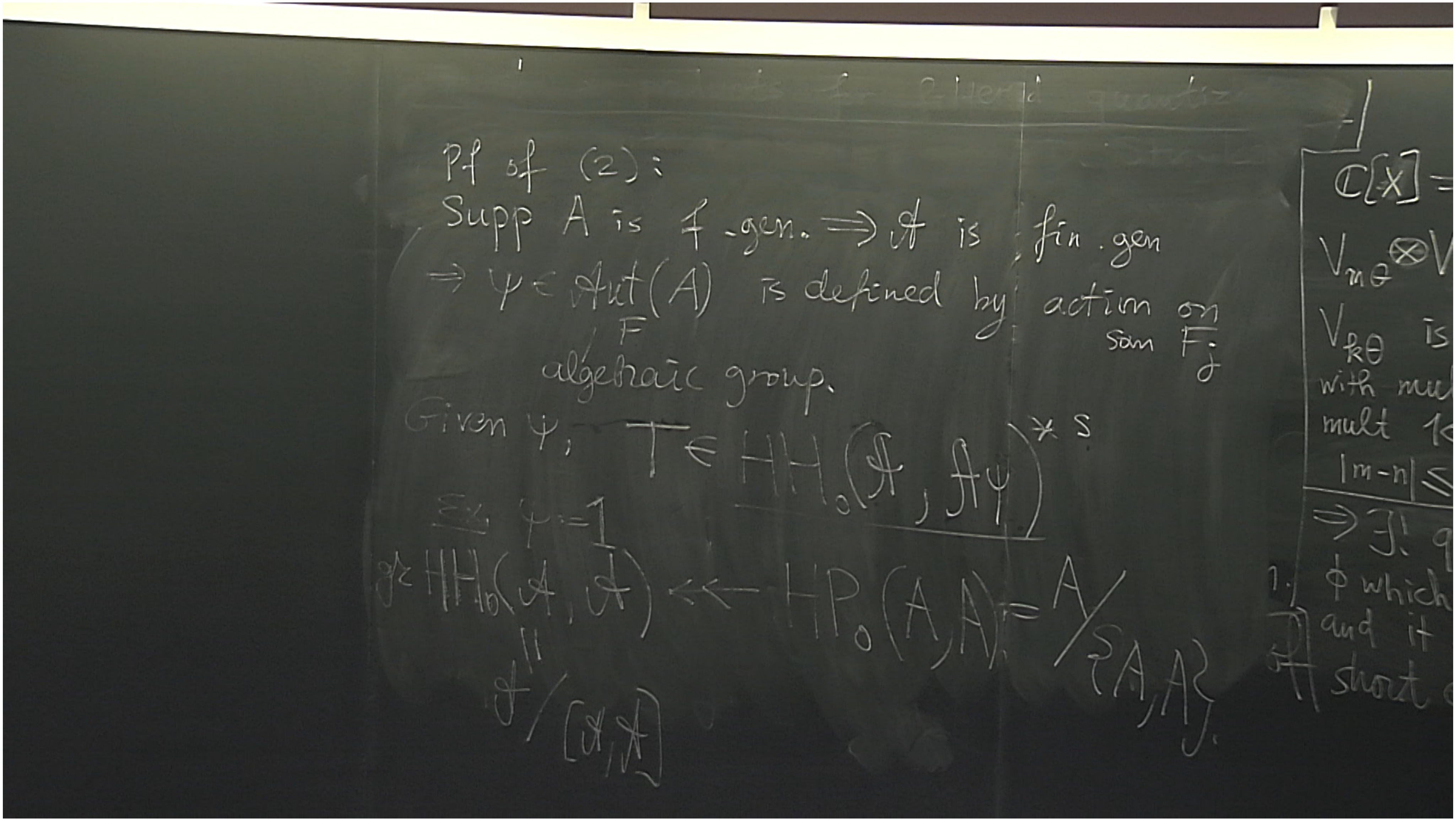
Def. $*$-prod is nondeg if B_* is nondeg.

Prop. Nondeg short $*$-products $\longleftrightarrow (T, \psi)$, $\psi \in \mathrm{Aut}_F(\mathcal{A})$

T a nondeg ψ-twisted trace.

Evenness cond: T is σ-invar. and $\psi = s$.



Pf of (2):

Supp A is f-gen. $\Longrightarrow \mathcal{A}$ is fin. gen

$\Rightarrow \psi \in \mathrm{Aut}_F(\mathcal{A})$ is defined by action on som F_j

algebraic group.

Given ψ, $T \in \underline{HH_0(\mathcal{A}, \mathcal{A}_\psi)^*}$ s

$\underline{\text{Ex.}}$ $\psi = 1$

$$\begin{tikzcd} gr\, HH_0(\mathcal{A}, \mathcal{A}) \arrow[d, equal] & HP_0(A, A) \arrow[l, two heads] = A/\{A, A\}. \\ \mathcal{A}/[\mathcal{A}, \mathcal{A}] & \end{tikzcd}$$

$\mathbb{C}[X] =$

$V_{m\theta} \otimes V$

$V_{k\theta}$ is

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mult 1

$|m-n| \leq$

$\Rightarrow \exists!$ q

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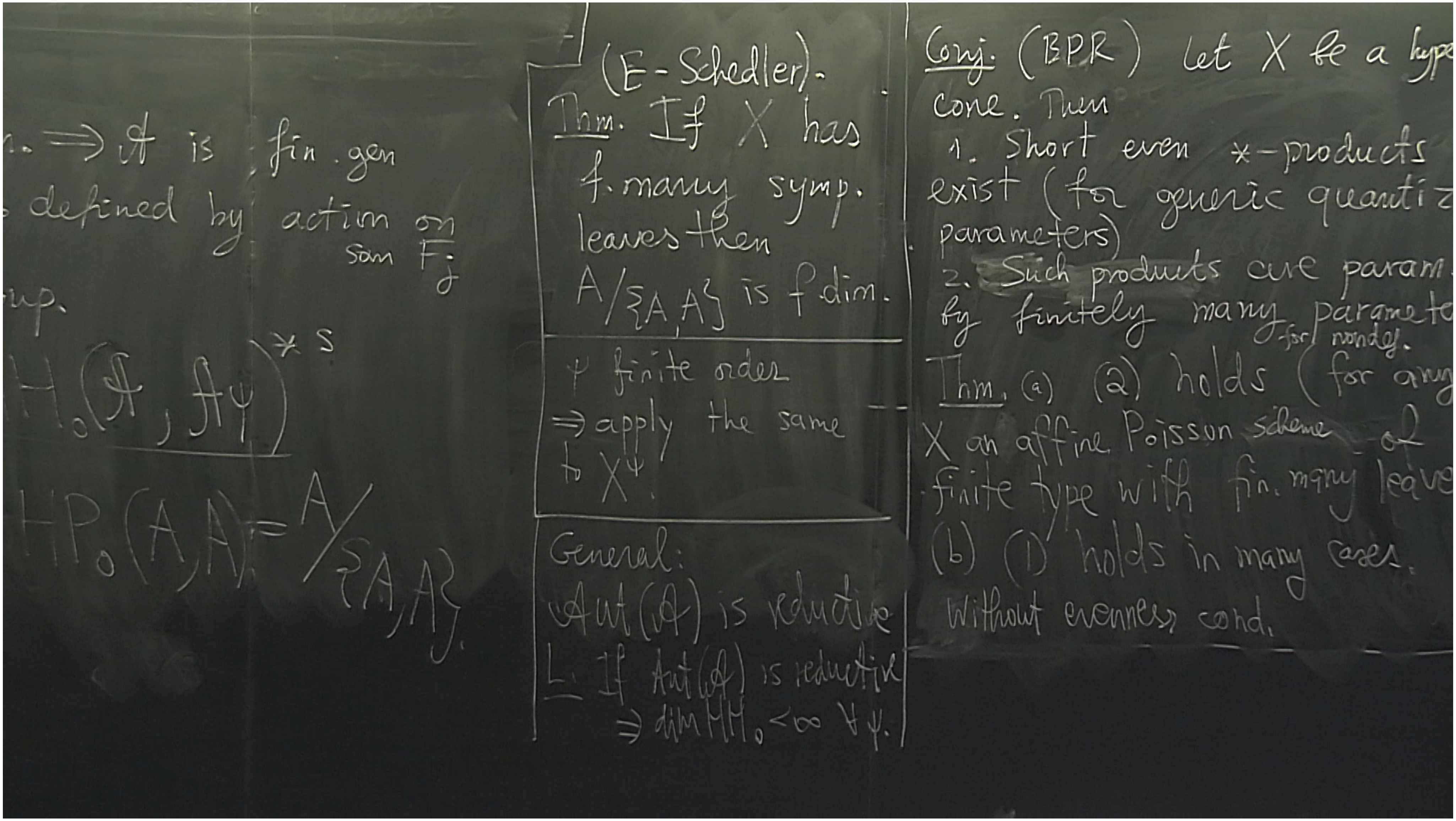

n. ⇒ $\mathcal{A}$ is fin. gen

defined by action on som F_j

up.

$$H_0(\mathcal{A}, \mathcal{A}\psi)^{*S}$$

$$HP_0(A,A) = A/\{A,A\}.$$

(E-Schedler).

Thm. If X has f. many symp. leaves then $A/\{A,A\}$ is f.dim.

ψ finite order ⇒ apply the same to X^ψ.

General: $\mathrm{Aut}(\mathcal{A})$ is reductive

L. If $\mathrm{Aut}(\mathcal{A})$ is reductive ⇒ $\dim HH_0 < \infty$ $\forall \psi$.

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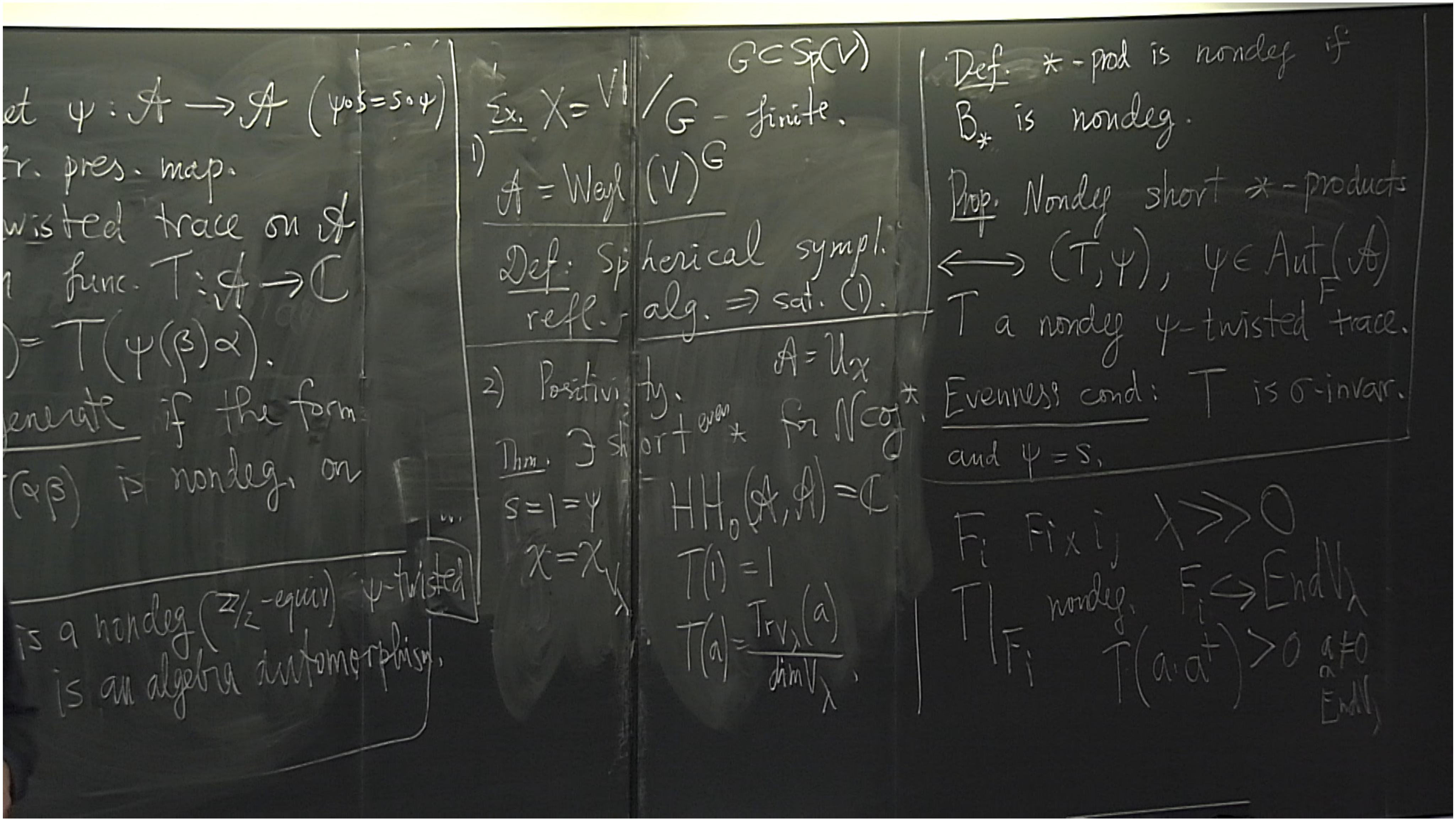

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$) = T(\psi(\beta)\alpha)$.

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$(\alpha\beta)$ is nondeg. on

...

is a nondeg ($\mathbb{Z}/2$-equiv) ψ-twisted

is an algebra automorphism.

Ex. $X = V / G$ - finite. $G \subset Sp(V)$

1) $\mathcal{A} = \text{Weyl}(V)^G$

Def: Spherical sympl. refl. alg. $\Rightarrow$ sat. (1).

2) Positivity. $\mathcal{A} = \mathcal{U}_\chi$

Thm. $\exists$ short $^{\text{even}}$ $*$ for $N \subset \mathfrak{g}^*$.

$s = 1 = \psi$ $\quad HH_0(\mathcal{A}, \mathcal{A}) = \mathbb{C}$

$\chi = \chi_{V_\lambda}$ $\quad T(1) = 1$

$T(a) = \frac{\text{Tr}_{V_\lambda}(a)}{\dim V_\lambda}$.

Def. $*$-prod is nondeg if B_* is nondeg.

Prop. Nondeg short $*$-products $\longleftrightarrow (T, \psi)$, $\psi \in \text{Aut}_F(\mathcal{A})$

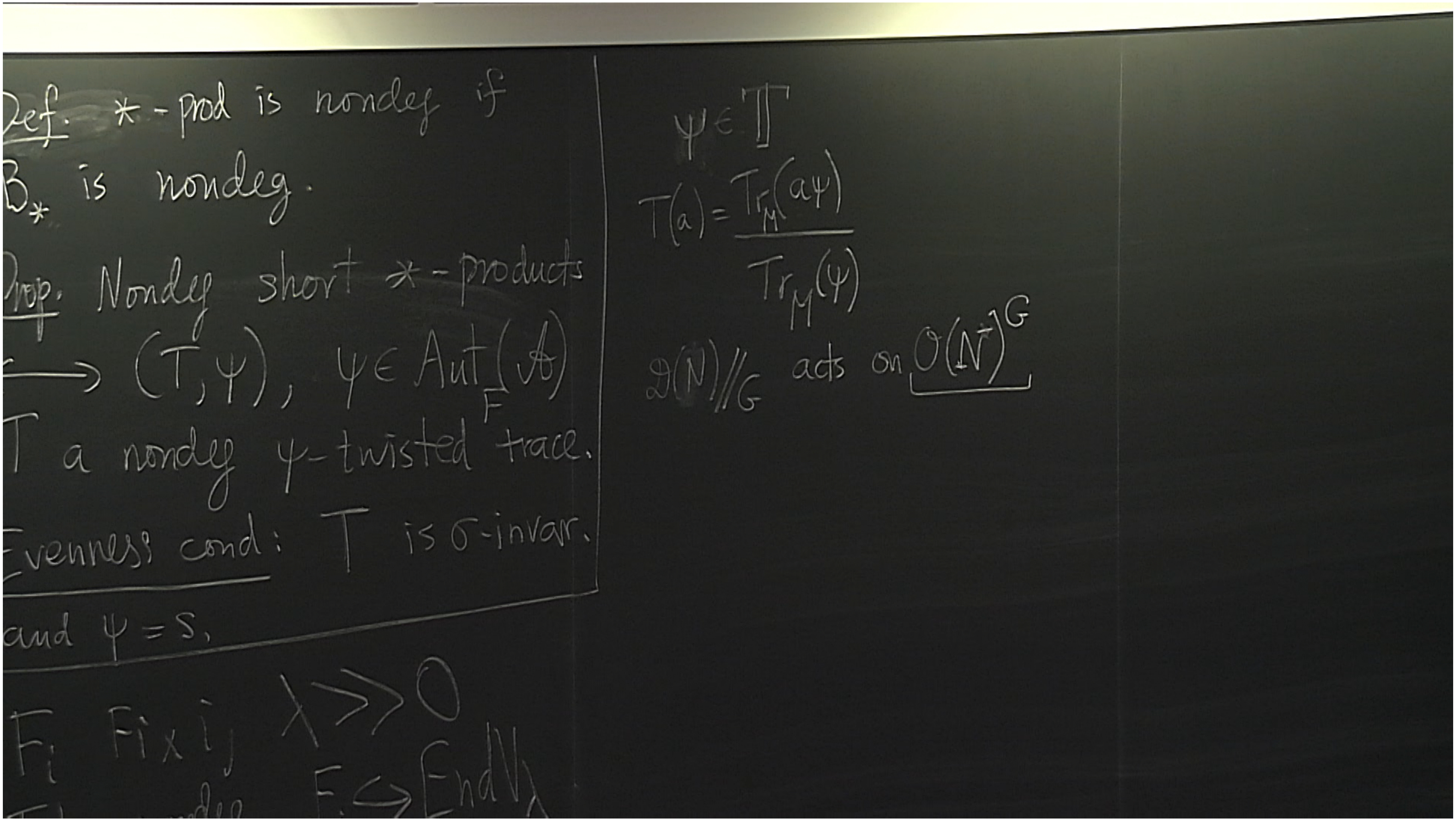
T a nondeg ψ-twisted trace.

Evenness cond: T is σ-invar. and $\psi = s$.

F_i $\quad F_i \times i$, $\quad \lambda >> 0$

$T|_{F_i}$ nondeg. $\quad F_i \hookrightarrow \text{End} V_\lambda$

$T(a \cdot a^+) > 0$ $\quad a \neq 0 \in \text{End} V_\lambda$



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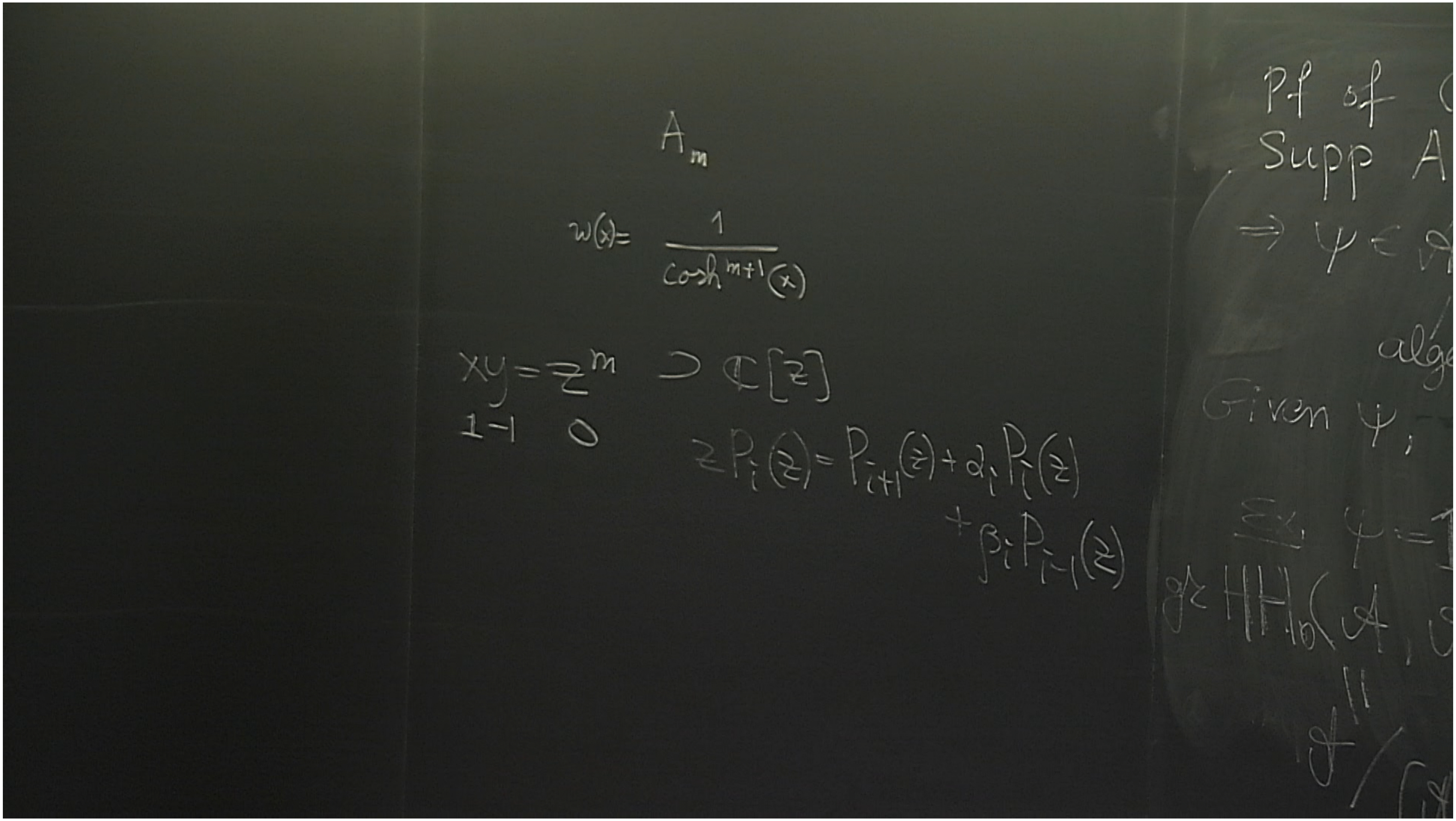
$F_i \hookrightarrow \mathrm{End}\, V_\lambda$

$\psi \in \mathbb{T}$

$$T(a) = \frac{\mathrm{Tr}_M(a\psi)}{\mathrm{Tr}_M(\psi)}$$

$\mathcal{D}(N)/\!/G$ acts on $\underbrace{\mathcal{O}(N)^G}$



A_m

$$w(x) = \frac{1}{\cosh^{m+1}(x)}$$

$xy = z^m \supset \mathbb{C}[z]$

$1 \; {-1} \quad 0$

$$z P_i(z) = P_{i+1}(z) + \alpha_i P_i(z) + \beta_i P_{i-1}(z)$$

Pf of (

Supp A

$\Rightarrow \psi \in \mathcal{A}$

alge

Given ψ ;

Ex. $\psi =$

$\mathrm{gr}\, HH_0(\mathcal{A},$

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$g\,/\,($