Title: Yangians from cohomological Hall algebras

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Collection: Cohomological Hall Algebras in Mathematics and Physics

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Abstract: I will explain various features of the preprojective CoHA, a kind of universal algebra of correspondences generalising the algebras of endomorphisms of cohomology of quiver varieties considered by Nakajima. In particular I will focus on features of this algebra that become visible after viewing it as a dimensional reduction of the Kontsevich-Soibelman) critical CoHA associated to a related 3-dimensional Calabi-Yau category. Many nice features emerge from this view, e.g. an embedding in a related shuffle algebra, a formula for the graded dimension of the algebra, a flat non cocommutative deformation, a cocommutative coproduct, a geometric doubling procedure, the PBW theorem, an isomorphism with the Yangian considered by Maulik and Okounkov... I will focus on the perverse filtration, which is the key feature of my joint work with Sven Meinhardt, and gives rise to most of the above

CoHAs and Yangians (joint w Meinhardt)

1) Perverse filtrations

Q a quiver $s,t: Q_1 \rightrightarrows Q_0$

arrows — vertices

$\gamma \in \mathbb{N}^{Q_0}$ - a dimension vector

$$\mathbb{A}_{Q,\gamma} = \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\gamma(s(a))}, \mathbb{C}^{\gamma(t(a))})$$

$$G_\gamma = \prod_{i \in Q_0} Gl(\mathbb{C}^{\gamma(i)})$$

$$\begin{array}{ccc} \mathrm{Rep}_\gamma Q & \xrightarrow{JH} & X_\gamma(Q) \\ \| & & \| \\ \mathbb{A}_{Q,\gamma}/G_\gamma & \xrightarrow{\text{affinization}} & \mathrm{Spec}(\Gamma(\mathbb{A}_{Q,\gamma})^{G_\gamma}) \end{array}$$

coarse moduli space of γ-dim'l reps

pts in $X_\gamma(Q) \xleftrightarrow{1:1}$ semisimple γ-dim'l reps

JH Jordan-Holder map

$[M] \longrightarrow$ [semisimplification of M]

*)

Q_0

vertices

$\left(\mathbb{C}^{\gamma(t(a))}\right)$

$$\begin{array}{ccc} \mathrm{Rep}_\gamma Q & \xrightarrow{\ JH\ } & X_\gamma(Q) \\ \| & & \| \\ \mathbb{A}_{Q,\gamma}/G_\gamma & \xrightarrow{\text{affinization}} & \mathrm{Spec}\left(\Gamma(\mathbb{A}_{Q,\gamma})^{G_\gamma}\right) \end{array}$$

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$[M] \longrightarrow [\text{semisimplification of } M]$

Calculating $H(JH_* \underline{\mathbb{Q}})$

Fix $N \gg 0$. Let V_N be G_γ-rep'n, $U_N \subset V_N$ open subvariety s.t. $G_\gamma \curvearrowright U_N$ is free, $\mathrm{codim}_{V_N}(V_N \setminus U_N) > N$.

$$
\begin{tikzcd}
(\mathbb{A}_{Q\gamma} \times U_N)/G_\gamma \arrow[r, bend left, "\iota \cdot \sim \text{ in low degree}"] \arrow[dr, bend right] & (\mathbb{A}_{Q\gamma} \times V_N)/G_\gamma \arrow[r, "q", "\sim"'] & \mathrm{Rep}_\gamma Q \arrow[dd, "JH"] \\
 & \mathrm{Rep}_\gamma^{N\text{-sfr}} Q \arrow[u, "\iota'"] \arrow[dr, "GIT\ (\text{proper map})"'] & \\
 & & X_\gamma(Q)
\end{tikzcd}
$$

Smooth

$\tau_{\leq c}\, JH_*(\underline{\mathbb{Q}}_{\mathrm{Rep}_\gamma Q} \to q_* \underline{\mathbb{Q}})$ is an isomorphism

plus same is true for ι'

Let $\mathrm{Rep}_\gamma^{N\text{-sfr}} Q$ be moduli space of pairs $\{(\rho \text{ a } \gamma\text{-dim'l } Q \text{ rep}, f : \mathbb{C}^N \dashrightarrow \rho) \mid \mathrm{Image}(f)$ generates ρ as a $\mathbb{C}Q$-module$\}/N$

li Space

Q rep

$) \mid \mathrm{Image}(f)$

$\mathbb{C}Q\text{-module}\}/N$

$$H^i(JH_*\underline{\mathbb{Q}}) \cong H^i(q_*\underline{\mathbb{Q}})$$

JH is "approximated by proper maps"

$\Rightarrow JH_*, JH_!$ commute with Φ_{trW}

$\Rightarrow H(\mathrm{Rep}_\gamma Q, \mathbb{Q})$ carries perverse filtration

$\Rightarrow H(\mathrm{Rep}_\gamma Q, \Phi_{trW})$ " " "

② KS CoHA

Assume $\tilde{Q}$ is a symmetric quiver

$$\mathcal{H}_{\tilde{Q},W} = \bigoplus_{\gamma\in\mathbb{N}^{Q_0}} H(\mathrm{Rep}_\gamma \tilde{Q}, \phi_{\mathrm{tr}W} Q)$$

$\gamma' + \gamma'' = \gamma$

Stack of SESs

$$\begin{tikzcd} & A = \mathrm{Rep}_{\gamma',\gamma''} Q \arrow[dl, "p_{02}"'] \arrow[dr, "p_1"] & \\ B = \mathrm{Rep}_{\gamma'} Q \times \mathrm{Rep}_{\gamma''} Q & & C = \mathrm{Rep}_\gamma Q \end{tikzcd}$$

I.

$\gamma' + \gamma'' = \gamma$

Stack of SESs

$$\begin{tikzcd}
A = \mathrm{Rep}_{\gamma',\gamma''}\tilde{Q} \arrow[d, "p_{02}"'] \arrow[dr, "p_1"] & \\
B = \mathrm{Rep}_{\gamma'}\tilde{Q} \times \mathrm{Rep}_{\gamma''}\tilde{Q} & \mathrm{Rep}_{\gamma}\tilde{Q}
\end{tikzcd}$$

T, T' maps to apt

$$T_* \varphi_{\mathrm{tw}}\left(\underline{\mathbb{Q}}_B \longrightarrow p_{02,*}\underline{\mathbb{Q}}_A\right)$$

$$H(\mathrm{Rep}_{\gamma'}\tilde{Q} \times \mathrm{Rep}_{\gamma''}\tilde{Q}, \varphi) \longrightarrow H(A, \varphi_{\mathrm{tw}})$$

$$\text{IIS TS}$$

$$\mathcal{H}_{\tilde{Q},W,\gamma'} \otimes \mathcal{H}_{\tilde{Q},W,\gamma''}$$

$$\underline{\mathbb{Q}}_C \longrightarrow p_{1,*}\underline{\mathbb{Q}}_A$$

$$\Big\downarrow D$$

$$T_* \varphi_{\mathrm{tw}}\left(p_{1,*}\underline{\mathbb{Q}}_A \longrightarrow \underline{\mathbb{Q}}_C[\cdot]\right) \cdot H(A, \varphi_{\mathrm{tw}}) \to \mathcal{H}_{\tilde{Q},W,\gamma}$$

a Symmetric quiver

$$H(\mathrm{Rep}_\gamma \tilde{Q}, \phi_{\mathrm{Tr}W}\mathbb{Q})$$

$$\gamma' + \gamma'' = \gamma$$

Stack of SESs

$$\begin{tikzcd}
A = \mathrm{Rep}_{\gamma',\gamma''}\tilde{Q} \arrow[r, "p_1"] \arrow[d, "p_{0,2}"'] & \mathrm{Rep}_\gamma \tilde{Q} \arrow[dd, "JH"] \\
B = \mathrm{Rep}_{\gamma'}\tilde{Q} \times \mathrm{Rep}_{\gamma''}\tilde{Q} \arrow[d, "JH \times JH"'] & \\
X_{\gamma'}(\tilde{Q}) \times X_{\gamma''}(\tilde{Q}) \arrow[r, "\oplus"] & X_\gamma(\tilde{Q})
\end{tikzcd}$$

τ, τ' maps to a pt

$$\tau_* \phi_{\mathrm{Tr}W}(\mathbb{Q}_B \longrightarrow p_{0,2,*}\mathbb{Q}_A)$$

$$H(\mathrm{Rep}_{\gamma'}\tilde{Q} \times \mathrm{Rep}_{\gamma''}\tilde{Q}, \phi) \text{ —}$$

$$\cong \text{TS}$$

$$\mathcal{H}_{\tilde{Q},W,\gamma'} \otimes \mathcal{H}_{\tilde{Q},W,\gamma''}$$

$$\mathbb{Q}_C \longrightarrow p_{1,*}\mathbb{Q}_A$$

$$\downarrow D$$

$$\tau'_* \phi_{\mathrm{Tr}W}(p_{1*}\mathbb{Q}_A \longrightarrow \mathbb{Q}_C[\cdot])$$

$H(A,\varphi_{tw})$

Observation

maps defining multiplication define morphisms of complexes (of sheaves) or MMHMs

$X_\gamma(\tilde{Q}) \in \mathcal{C}$

$D^+(MMHM(X(\tilde{Q})))$ carries a symm monoidal str.

$$\mathcal{F} \otimes \mathcal{G} = \oplus_+(\mathcal{F} \boxtimes \mathcal{G})$$

$): H(A,\varphi_{tw}) \to H_{\tilde{Q},w,\gamma}$ and $JH_+\varphi_{tw}$ is a monoid in $\mathcal{C}$

$\therefore$ product preserves perv filtration

Thm (-M)

- $\mathrm{Gr}^P \mathcal{H}_{\tilde{Q},W}$ is a free supercommutative $\mathbb{Z}^{Q_0} \times \underset{\text{Cohom}}{\mathbb{Z}}$-graded algebra
- Perv filt'n starts in degree 1

Set $\mathfrak{g}_{\tilde{Q},W} = P_{\leq 1} \mathcal{H}_{\tilde{Q},W}$

The map $\mathrm{Sym}\left(\mathfrak{g}_{\tilde{Q},W} \otimes \mathbb{Q}[\underset{c_1(\mathrm{Taut})}{u}]\right) \xrightarrow{\text{mult}} \mathcal{H}_{\tilde{Q},W}$

is a (PBW) isomorphism

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filt'n starts in degree 1

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$u = c_1(T_{\text{taut}})$

a (PBW) isomorphism

• $\mathfrak{g}_{\tilde{Q},W}$ closed under [,]

"BPS Lie algebra"

filtration

Thm (-M)

Cohom

$\mathbb{Z}^{Q_0} \times \mathbb{Z}$-graded

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Set $\mathfrak{g}_{\tilde{Q},W} = P_{\leq 1} \mathcal{H}_{\tilde{Q},W} \overset{\chi}{\rightsquigarrow}$ DT invariants

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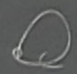
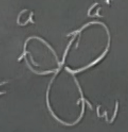
"BPS Lie algebra"

3 3d Propaganda

Set $\tilde{Q} = \bar{\bar{Q}}$ - tripled quiver assoc to Q

$\quad\; = \bar{Q} \amalg \{\omega_i \mid i \in Q_0\}$

$$W = \sum_{i \in Q_0} \omega_i \sum_{a \in Q_1} [a, a^*]$$

Ex $Q =$  $\longrightarrow$ $\tilde{Q} =$ 

$W = \omega[a, a^*]$

Fact

$$\mathrm{Jac}(\tilde{Q}, W) \cong \Pi_Q[z]; \quad z = \sum_i \omega_i$$

$$\mathbb{C}^* \curvearrowright \quad \begin{array}{ccc} a, & a^*, & \omega \\ 1 & 1 & -2 \end{array}$$

Dim red'n

Schiffmann + Vasserot

$$\mathcal{H}^{(\mathbb{C}^*)}_{\tilde{Q}, W} \cong \mathcal{H}^{(\mathbb{C}^*)}_{\Pi_Q}$$

$(\tilde{Q}, W) \cong \Pi_Q[z]$; $z = \sum_i \omega_i$

$a, \; a^*, \; \omega$

$1 \quad 1 \quad -2$

Schiffmann + Vasserot

$(\mathbb{C}^*)$

$,W \cong \mathcal{H}^{(\mathbb{C}^*)}_{\Pi_Q}$

RHS more natural

LHS is where we can prove things

1) $\mathcal{H}^{(\mathbb{C}^*)}_{\Pi_Q}$ pure of Tate type

2) $\mathcal{H}^{\mathbb{C}^*}_{\Pi_Q}$ flat deformation of $\mathcal{H}_{\Pi_Q}$

3) $\mathcal{H}_{\Pi_Q} \overset{\text{graded v.s.}}{\cong} \mathrm{Sym}\left(\bigoplus_{0 \neq \gamma \in \mathbb{N}^{Q_0}} V_\gamma \otimes \mathbb{Q}[u]\right)$

$\chi(V_\gamma) = a_{Q,\gamma}(q^{-1})$

4) $\mathcal{H}^{\mathbb{C}^*}_{\Pi_Q} \longrightarrow Sh^{\mathbb{C}^*}_{\overline{Q}}$ is an injection

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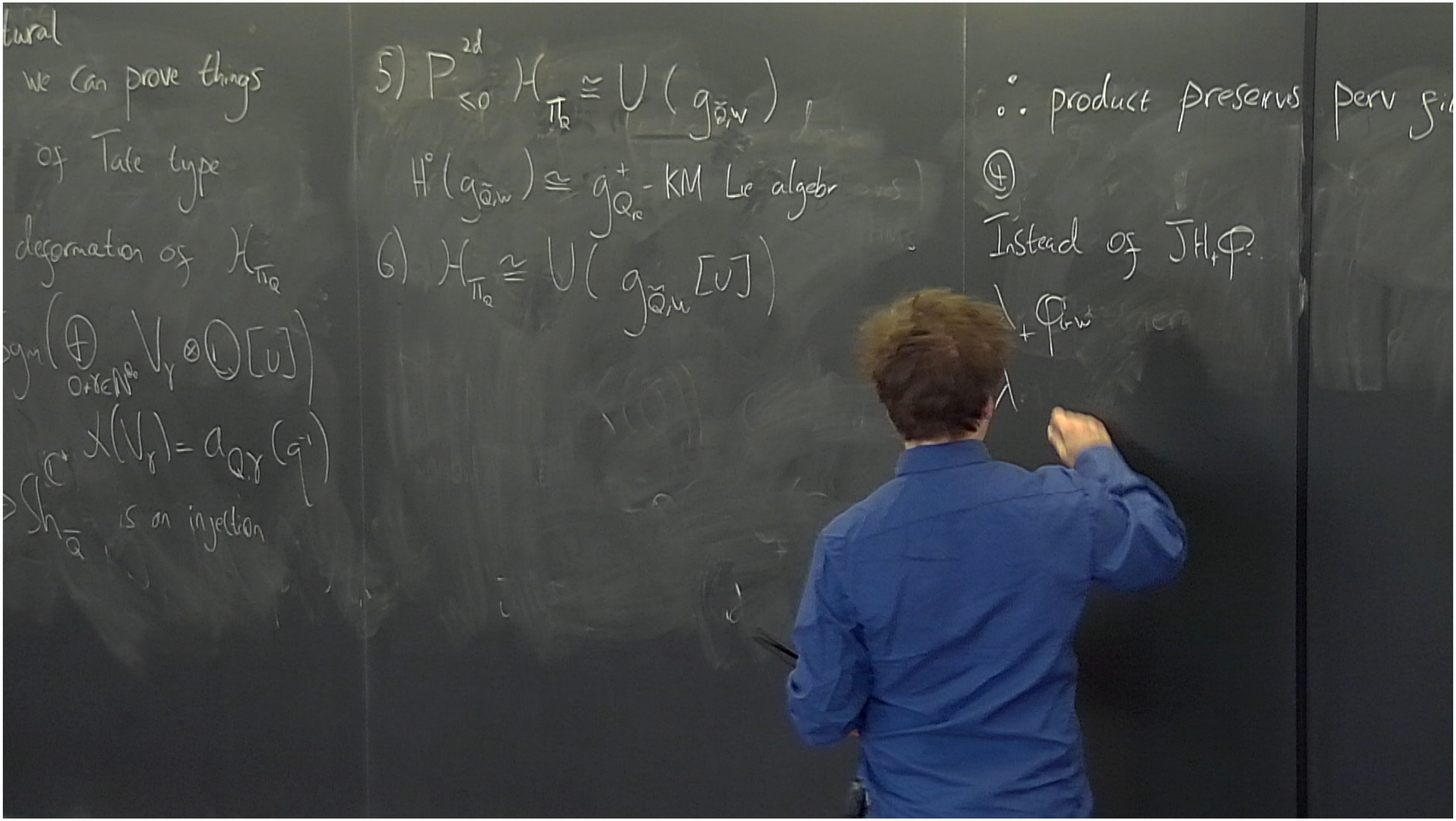
4) $\mathcal{H}_{\Pi_Q}^{\mathbb{C}^*} \longrightarrow Sh_{\tilde{Q}}^{\mathbb{C}^*}$ is an injection

5) $P^{2d}_{\leq 0} \mathcal{H}_{\Pi_Q} \cong U(\mathfrak{g}_{\tilde{Q},W})$

$H^0(\mathfrak{g}_{\tilde{Q},W}) \cong \mathfrak{g}_Q^+$

6) $\mathcal{H}_{\Pi_Q} \cong U(\mathfrak{g}_{\tilde{Q},W}[u])$

$\therefore$ product pr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$H^0(\mathfrak{g}_{\tilde{Q},w}) \cong \mathfrak{g}^+_{Q_r}$ - KM Lie algebr
6) $\mathcal{H}_{\Pi_Q} \cong U(\mathfrak{g}_{\tilde{Q},w}[u])$
∴ product preserves perv fi
Instead of $JH_+\varphi$?

s perv filtration

$\mathbb{A}^1$

es

• Let U be a ball in $\mathbb{A}^1$

$\lambda^{-1}(S_{\mathrm{sym}} U)$

$$H(\mathrm{Rep}_\gamma \mathbb{C}Q, \varphi_{\mathrm{Tr}W}) \simeq H(\mathrm{Rep}_\gamma^{\boxed{U}} \mathbb{C}\tilde{Q}, \varphi_{\mathrm{Tr}W})$$

($\mathbb{C}^*$-invariance of $\mathrm{tr}\, W$)

• Same is true for $U \to V$ disjoint from an

U

$\mathbb{A}^1$ invariance of $\mathrm{tr} W$

$\mathcal{H}_{\tilde{Q},W} \xrightarrow{\sim} H($

$\lambda^{-1}(S_{sym} V)$

$\boxed{V}$

$_{\gamma}(\mathbb{C}\tilde{Q}, \varphi_{trw})$

V disjoint from

and

$$\mathcal{H}_{\tilde{Q},W} \xrightarrow{\sim} H(\mathrm{Rep}^{V \oplus V}\mathbb{C}\tilde{Q}, \mathrm{tr}W)$$

$$\cong \mathrm{TS}$$

$$H(\mathrm{Rep}^{V}\mathbb{C}\tilde{Q}, \mathrm{tr}W) \otimes H(\mathrm{Rep}^{V}\mathbb{C}\tilde{Q}, \varphi)$$

$$\downarrow \cong$$

$$\mathcal{H}_{\tilde{Q},W} \otimes \mathcal{H}_{\tilde{Q},W}$$

cocommutative coproduct

$\longrightarrow$ connected bialgebra $\rightsquigarrow U(V.)$

$\therefore\ U(\mathfrak{g}_{\tilde{Q},W}[u])$ $\square$