Title: An introduction to Cohomological Hall algebras and their representations

Speakers: Yan Soibelman

Collection: Cohomological Hall Algebras in Mathematics and Physics

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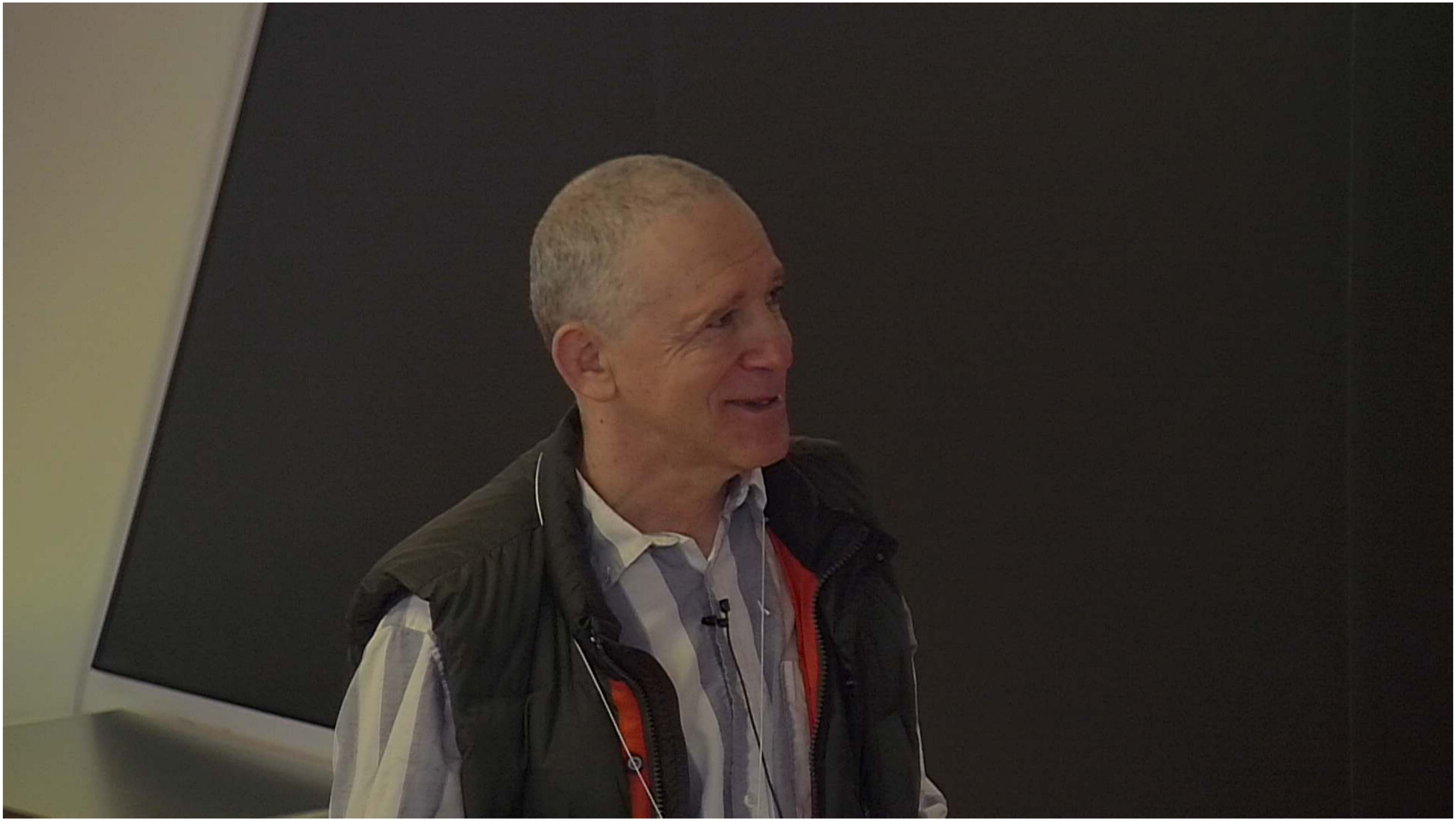
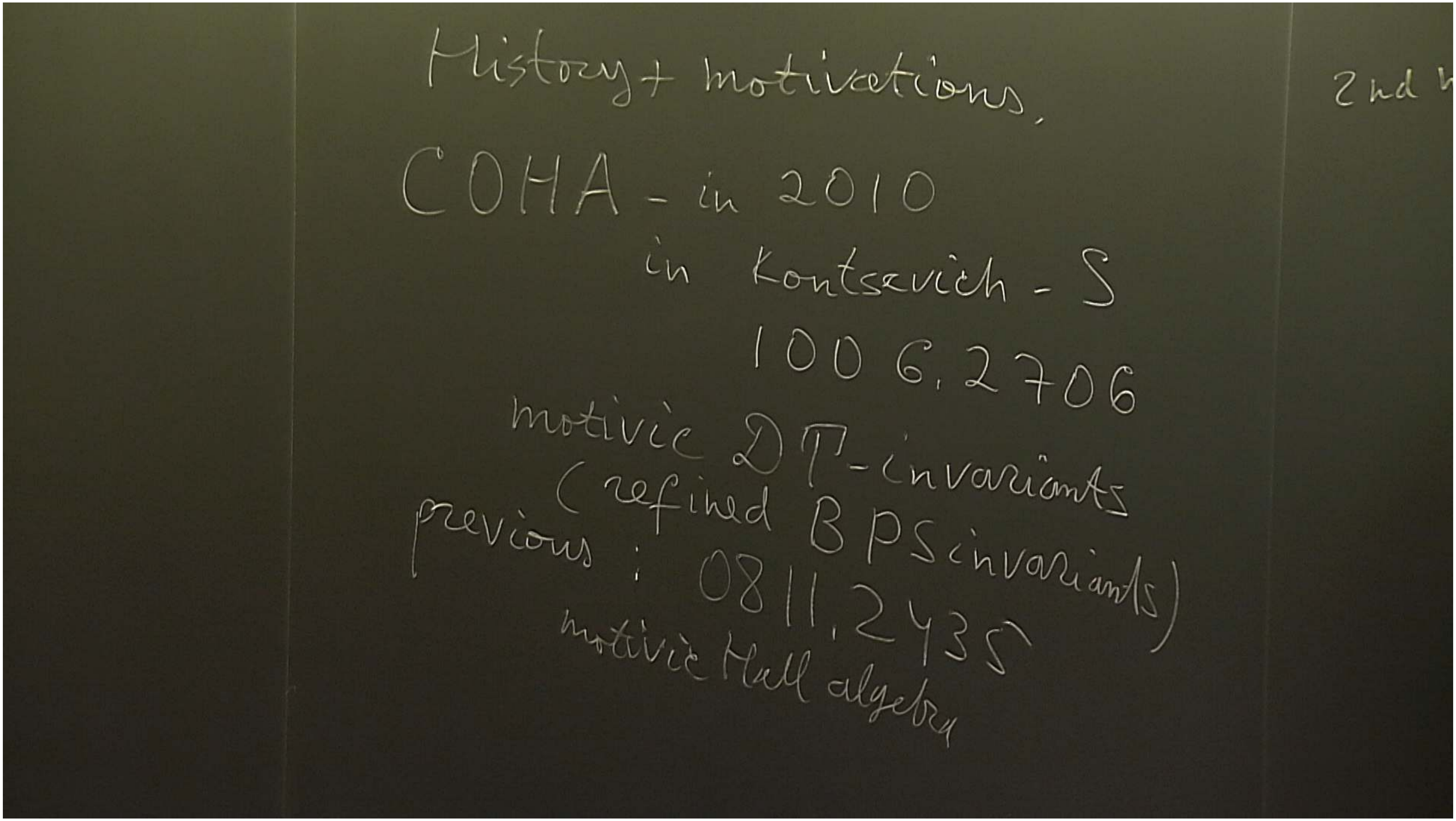
Abstract: I am going to review for non-experts the notion of Cohomological Hall algebra (COHA) introduced in my joint paper with Maxim Kontsevich in 2010 (see arXiv:1006.2706).Restricting considerations to the case of COHA of quivers with potential I will recallsome structural results, like e.g. dimensional reduction of COHA from 3d to 2d.

Then I plan to discuss a class of representations of COHA in the cohomology of the modulispaces of framed stable objects of a 3d Calabi-Yau category endowed with a stability structure, following my paper arXiv:1404.1606.

Finally, if time permits, I will discuss some examples of representations of COHA and its double, including my recent joint work with Miroslav Rapcak, Yaping Yang and Gufang Zhao on the relation of COHA with affine Yangians and moduli spaces of Nekrasov spiked instantons (see arXiv:1810.10402). As will be explained in other talks at this conference this relation gives rise to a class of representations of the ``vertex algebra at the corner" (see Gaiotto and Rapcak, arXiv:1703.00982). Other classes of representations of the VOA at the corner are conjecturally related to the action of the double of spherical COHA on the cohomology of Hilbert schemes of non-reduced divisors in toric Calabi-Yau 3-folds.





History + motivations.

COHA - in 2010

in Kontsevich - S

1006.2706

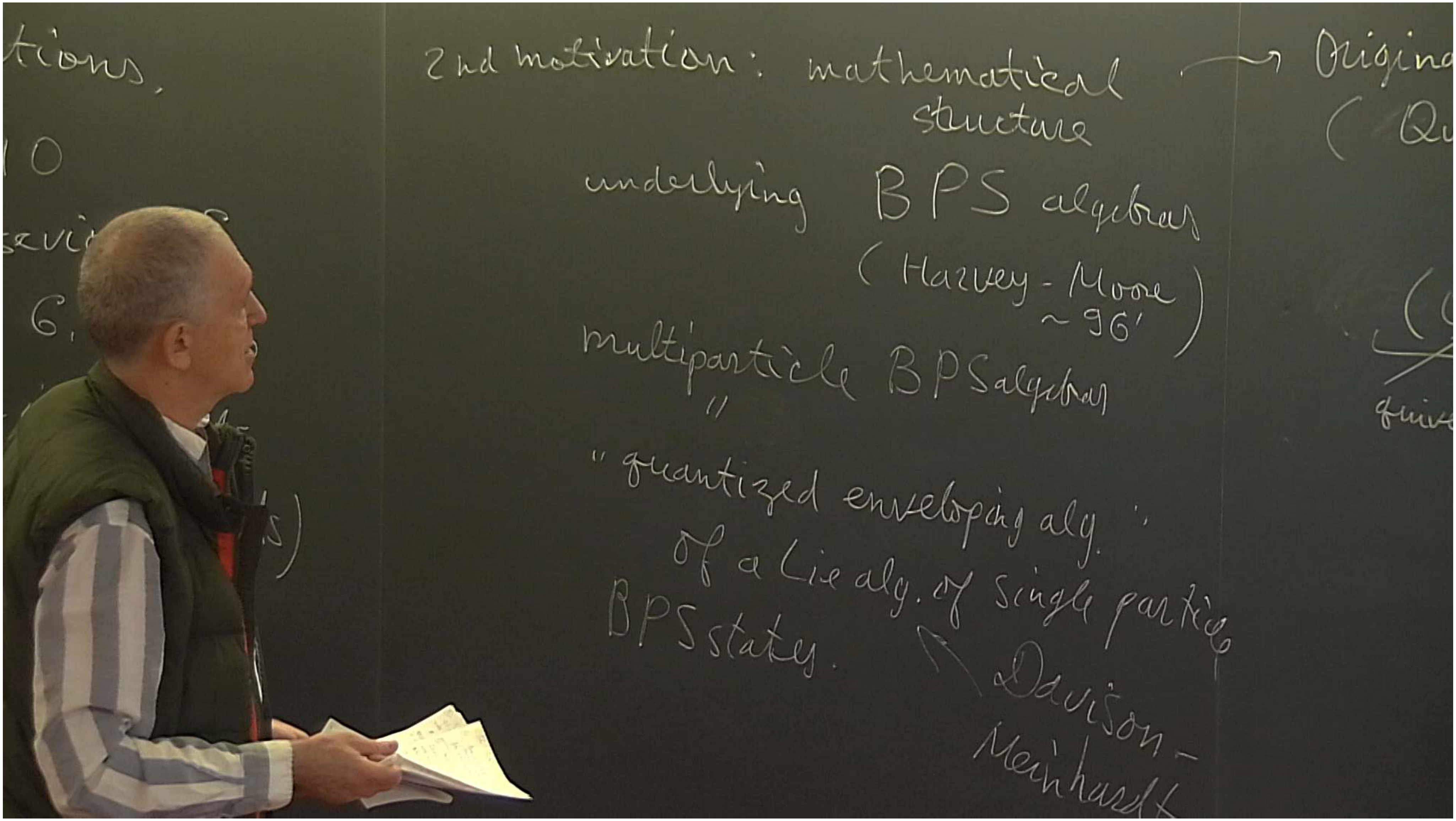
motivic DT-invariants

(refined BPS invariants)

previous: 0811.2435

motivic Hall algebra

2nd



tions,

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6,

2nd motivation: mathematical structure

underlying BPS algebras

(Harvey-Moore ~96')

multiparticle BPS algebras

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"quantized enveloping alg."

of a Lie alg. of single particle BPS states.

Davison-Meinhardt

→ Origina

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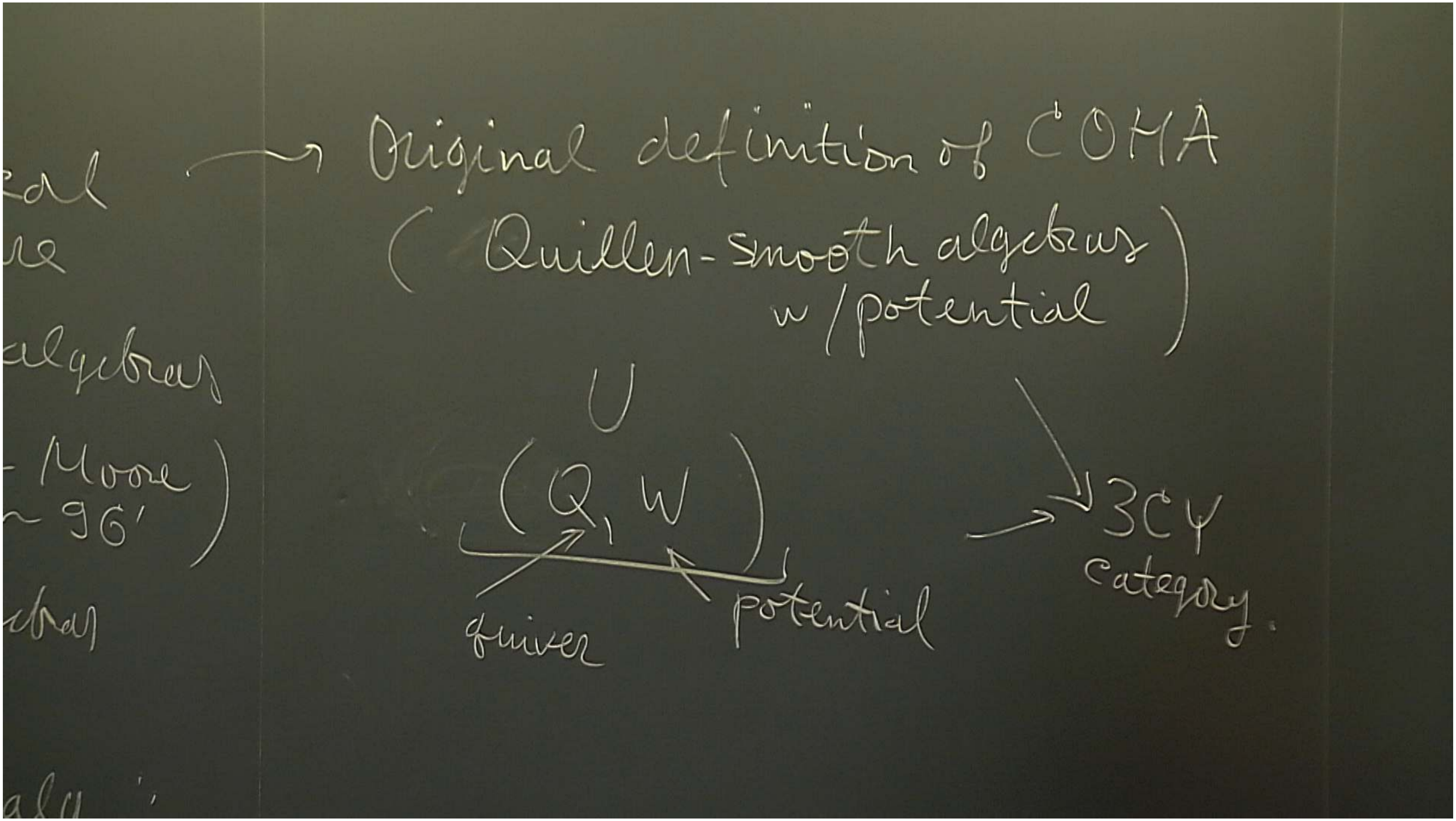


cal

re

algebras

- Moore
~ 96′)

ebras

alg

→ Original definition of COHA

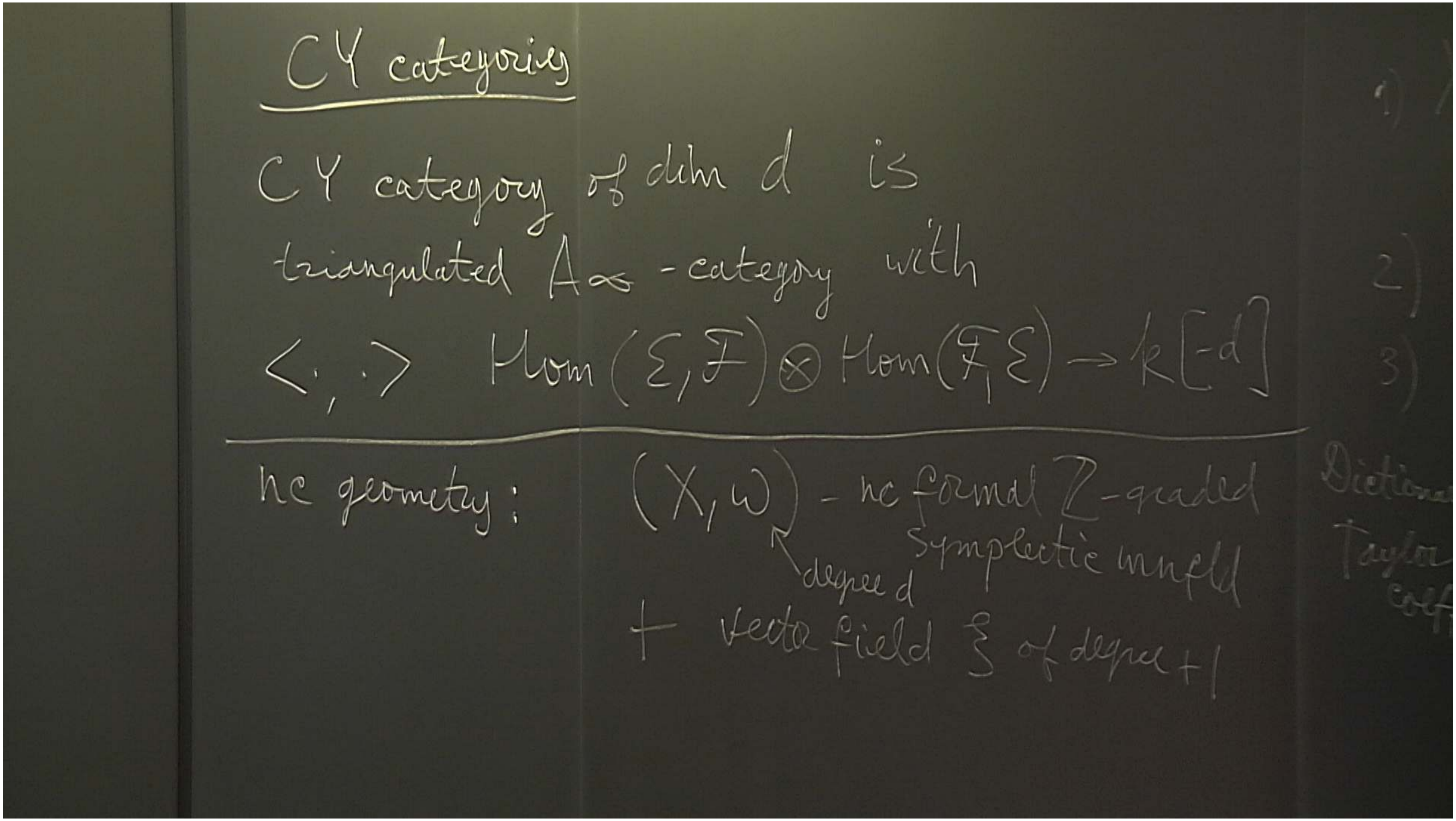
(Quillen-smooth algebras w/potential)

$\cup$

$\underbrace{(Q, W)}$ → 3CY category.

quiver

potential



CY categories

CY category of dim d is triangulated A_∞-category with

$$\langle\cdot,\cdot\rangle \quad \mathrm{Hom}(\mathcal{E},\mathcal{F})\otimes\mathrm{Hom}(\mathcal{F},\mathcal{E})\to k[-d]$$

nc geometry: (X,ω) - nc formal $\mathbb{Z}$-graded symplectic mnfld

degree d

+ vector field ξ of degree +1

1)

2)

3)

Diction

Taylo

coef

n

$\mathcal{F}_1 \xi) \to k[-d]$

---

rmal $\mathbb{Z}$-graded
mplectic mnfld
ξ of degree +1

1) X is a formal completion of $Z = \mathrm{Zeros}(\xi)$

2) $\mathrm{Lie}_\xi(\omega) = 0$

3) $[\xi, \xi] = 0$

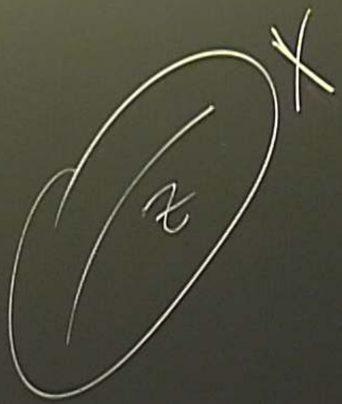
Dictionary: $Z \Longleftrightarrow \mathrm{Ob}(\mathcal{C})$

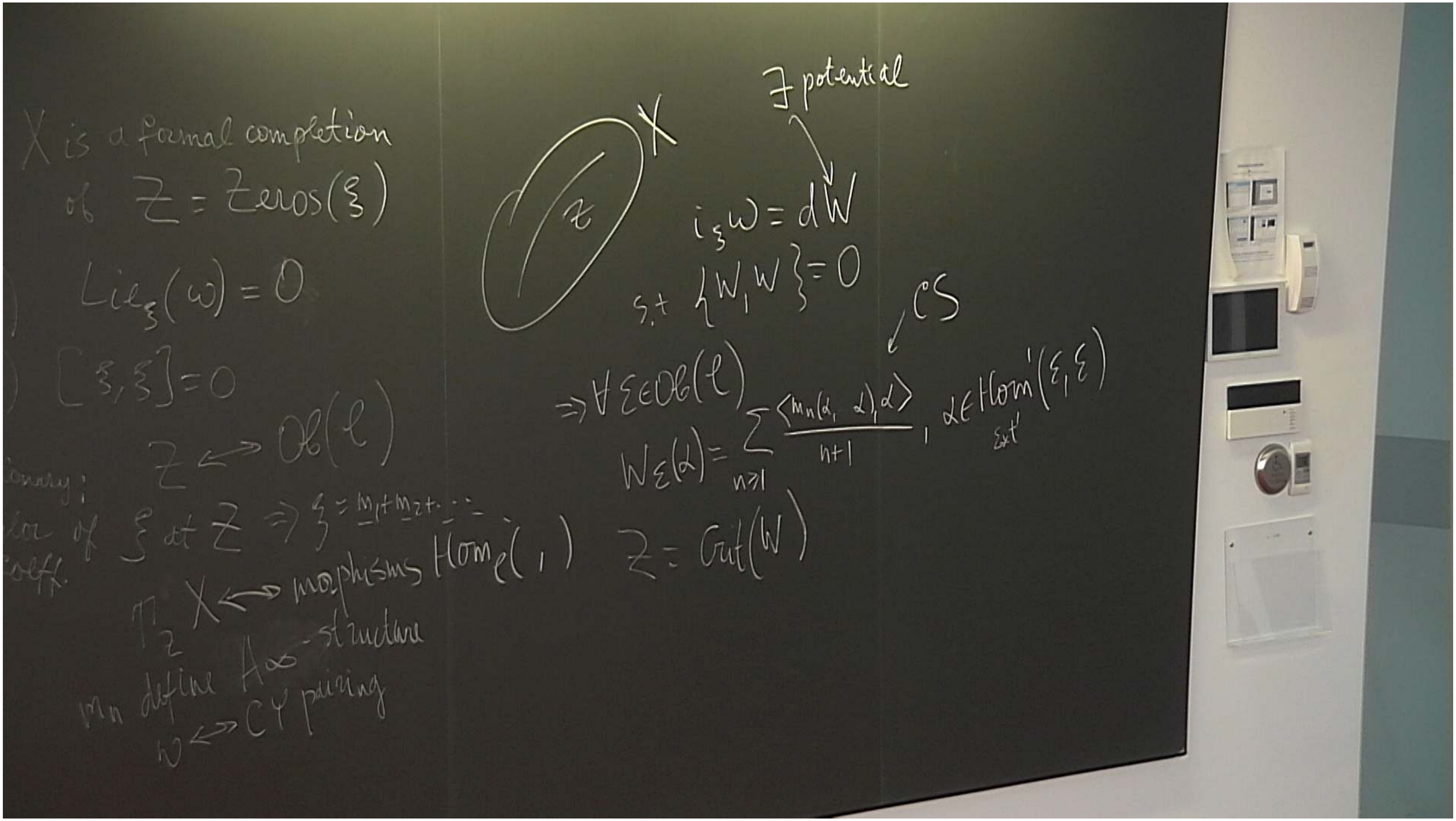
Taylor coeff. of ξ at Z $\Rightarrow$ $\xi = \underline{m}_1 + \underline{m}_2 + \cdots$

$\pi_2 X \Longleftrightarrow$ morphisms $\mathrm{Hom}_{\mathcal{C}}(,)$

m_n define A_∞-structure

$\omega \Longleftrightarrow$ CY pairing





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coeff.

$T_Z X \longleftrightarrow$ morphisms $\mathrm{Hom}_{\mathcal{C}}(\ ,\ )$

m_n define A_∞-structure

$\omega \longleftrightarrow$ CY pairing

$\exists$ potential

$i_{\xi}\omega = dW$

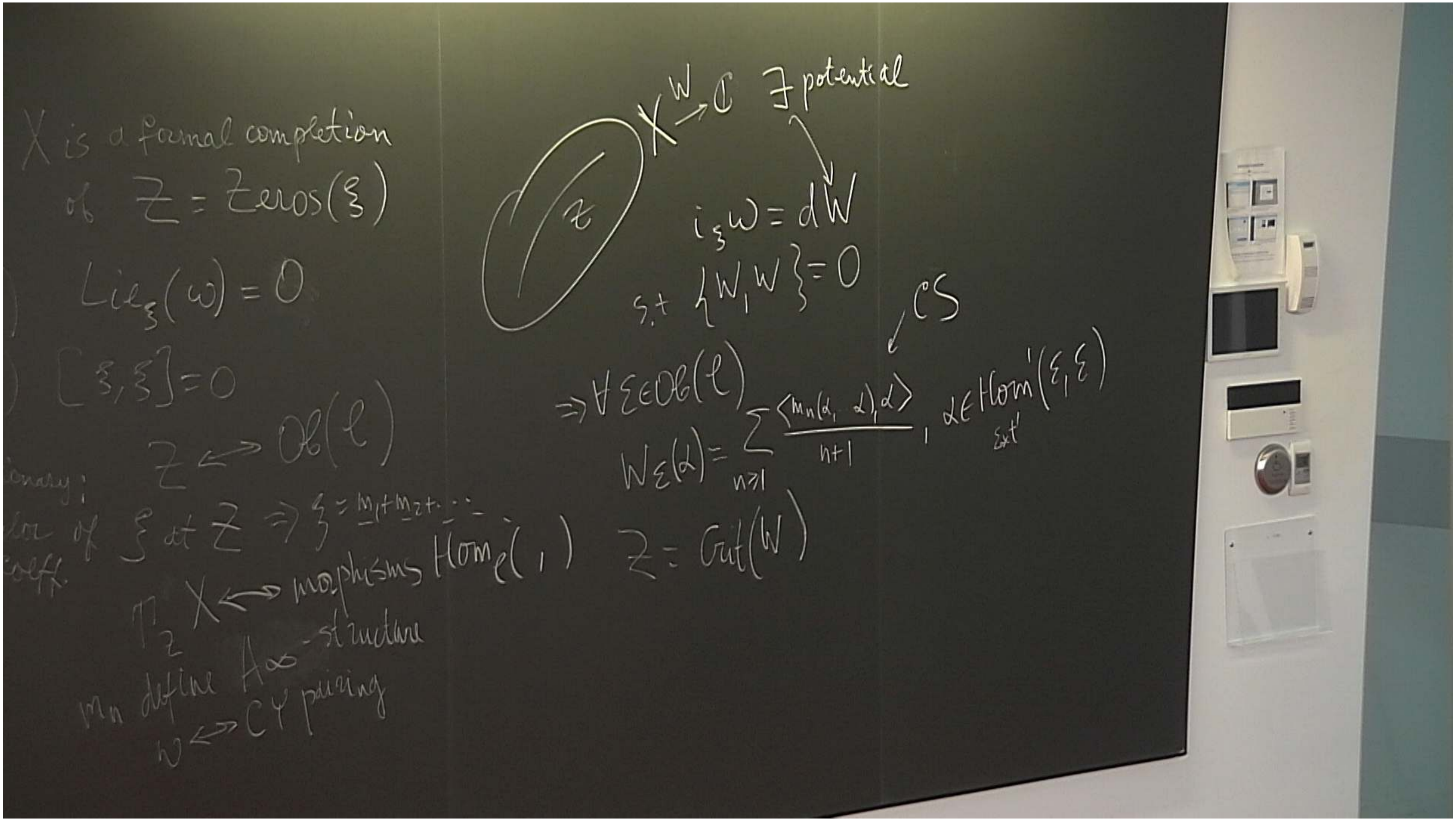
s.t. $\{W, W\} = 0$

CS

$\Rightarrow \forall \mathcal{E} \in \mathrm{Ob}(\mathcal{C})$

$$W_{\mathcal{E}}(\alpha) = \sum_{n \geq 1} \frac{\langle m_n(\alpha, \ \alpha), \alpha \rangle}{n+1}, \quad \alpha \in \underset{\mathrm{Ext}^1}{\mathrm{Hom}^1}(\mathcal{E}, \mathcal{E})$$

$Z = \mathrm{Crit}(W)$



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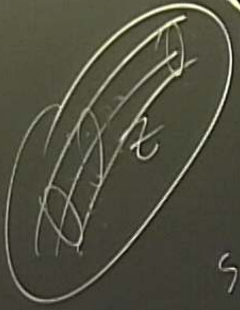

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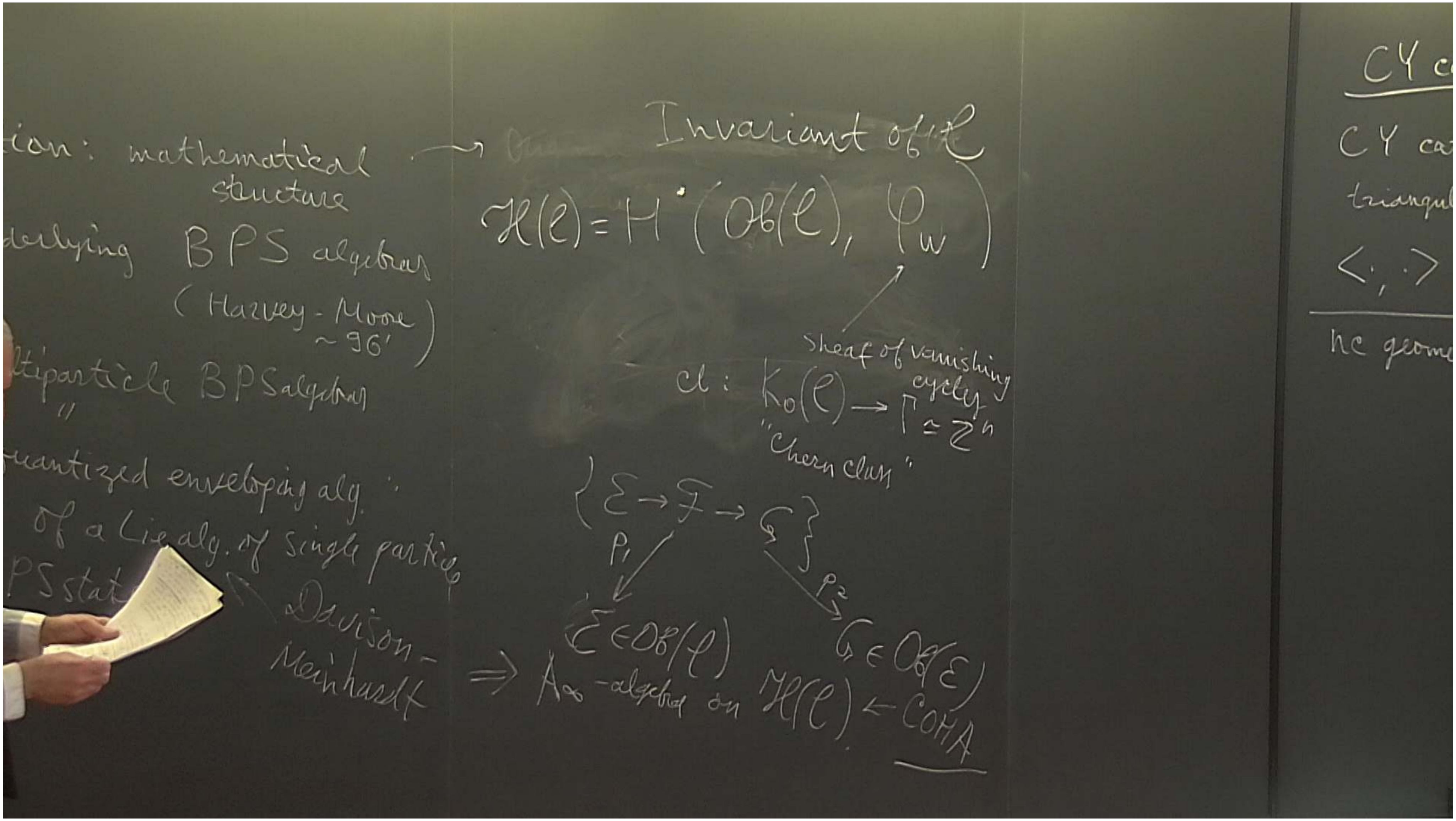

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PS stat

Davison-Meinhardt ⇒ A_∞-algebra on $\mathcal{H}(\mathcal{C}) \leftarrow$ COHA

Invariant of $\mathcal{C}$

$$\mathcal{H}(\mathcal{C}) = H^\bullet(Ob(\mathcal{C}), \varphi_W)$$

Sheaf of vanishing cycles

$$cl : K_0(\mathcal{C}) \to \Gamma \simeq \mathbb{Z}^n$$

"Chern class"

$$\begin{tikzcd} & \{\mathcal{E} \to \mathcal{F} \to \mathcal{G}\} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{E} \in Ob(\mathcal{C}) & & \mathcal{G} \in Ob(\mathcal{C}) \end{tikzcd}$$

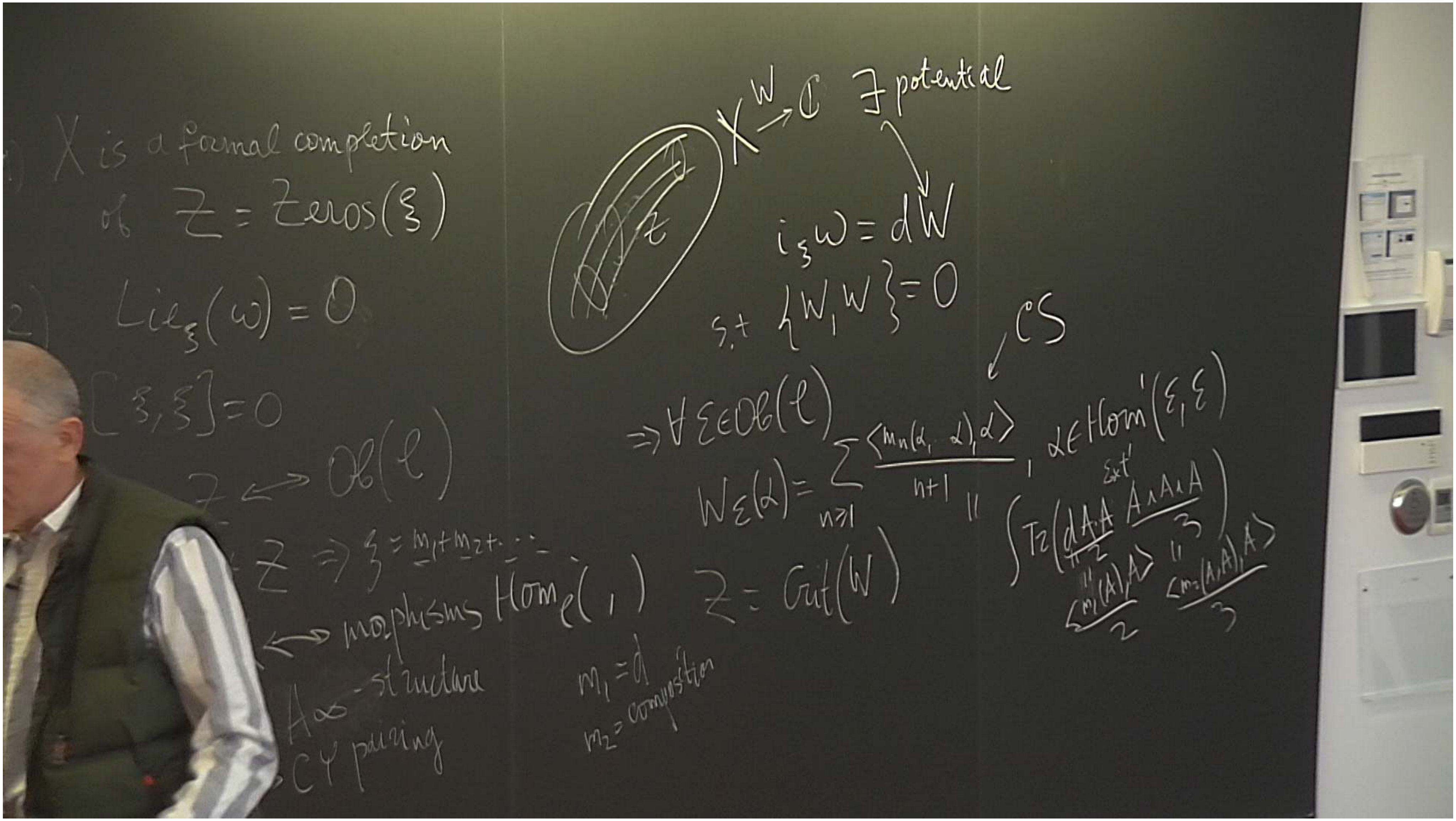
CY c

CY ca

triangul

$\langle \cdot , \cdot \rangle$

he geome



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$\longleftrightarrow$ morphisms $\text{Hom}_{\mathcal{C}}(\ , \ )$

A_∞-structure

CY pairing

$m_1 = d$

$m_2 =$ composition

$X \xrightarrow{W} \mathbb{C}$ $\exists$ potential

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Ext^1

$$\parallel$$

$$\int \text{Tr}\left(\frac{dA \wedge A}{2} + \frac{A \wedge A \wedge A}{3} \right)$$

$$\frac{\langle m_1(A), A \rangle}{2} \qquad \frac{\langle m_2(A, A), A \rangle}{3}$$

$Z = \text{Crit}(W)$

In reality:

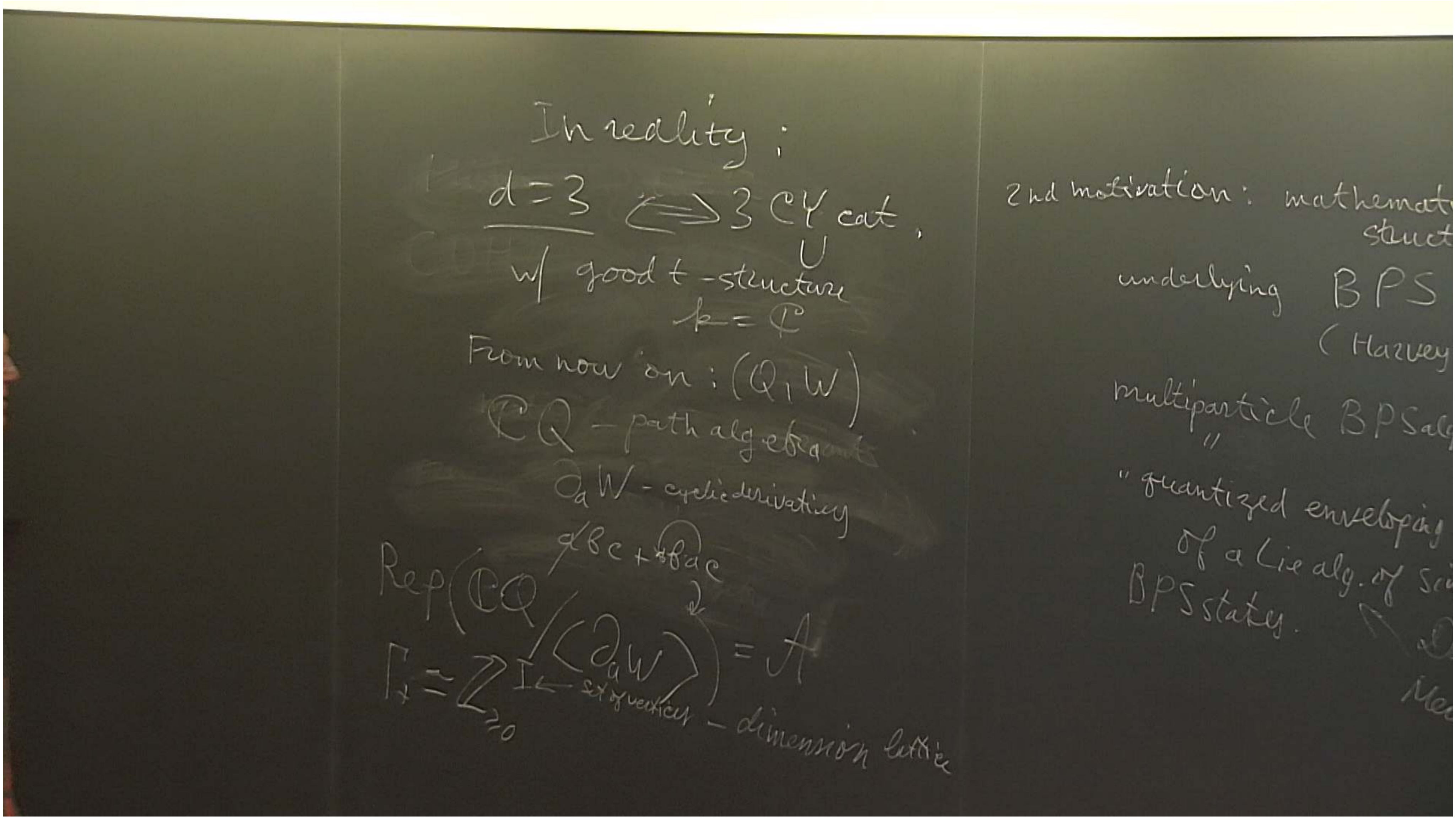
$\underline{d=3} \Longleftrightarrow 3\ \mathrm{CY}$ cat,

$\cup$

w/ good t-structure

2nd

From now on: (Q, W)



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w/ good t-structure

$k=\mathbb{C}$

From now on: (Q, W)

$\mathbb{C}Q$ – path algebra

$\partial_a W$ – cyclic derivatives

$\alpha\beta c+\beta\alpha c$

$$\mathrm{Rep}\left(\mathbb{C}Q/\left(\langle\partial_a W\rangle\right)\right)=\mathcal{A}$$

$\Gamma_+ \cong \mathbb{Z}^{I}_{\geq 0}$ $I \leftarrow$ set of vertices – dimension lattice

2nd motivation: mathemat

struct

underlying BPS

(Harvey

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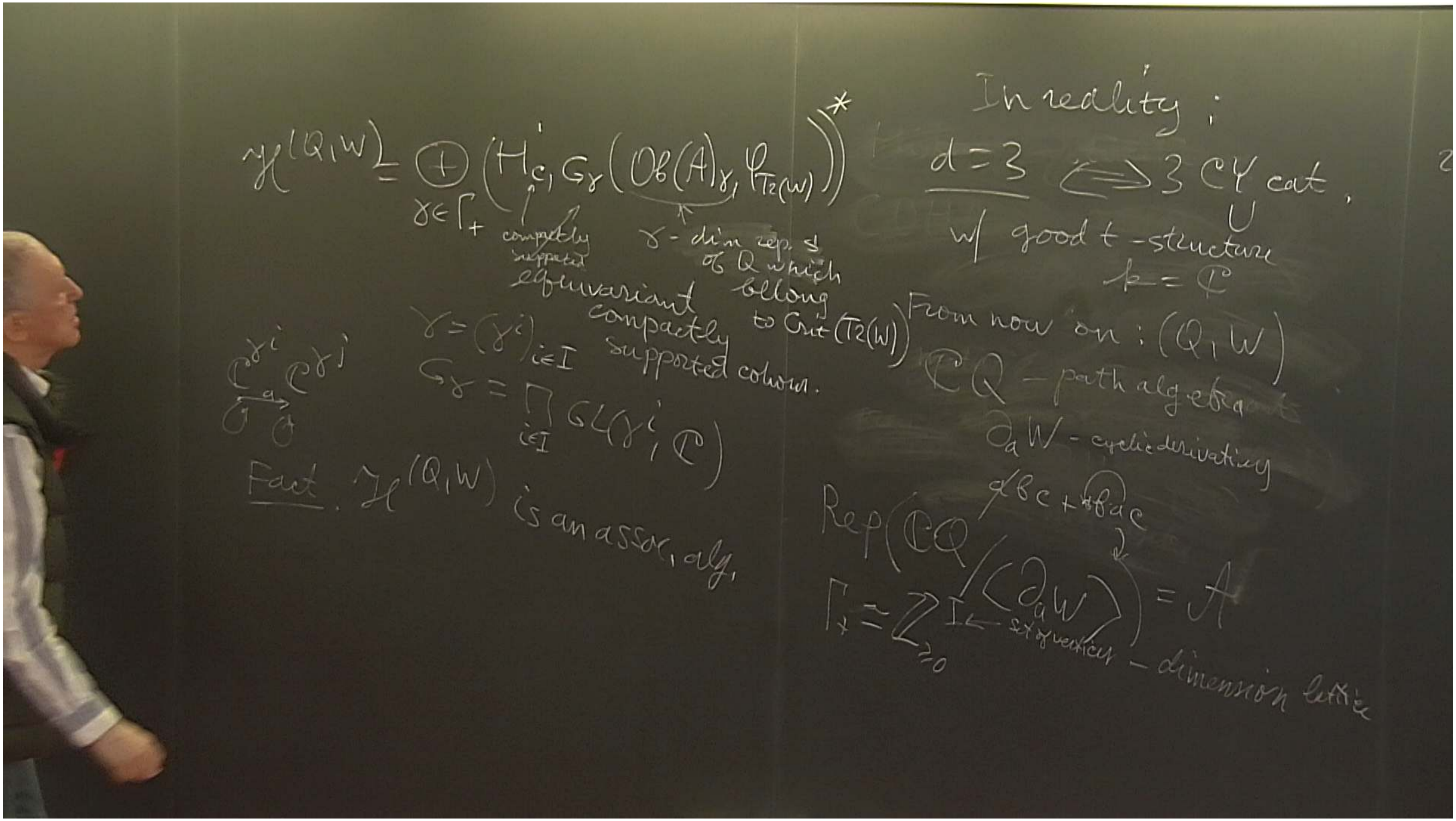


$$\mathcal{H}^{(Q,W)} = \bigoplus_{\gamma\in\Gamma_+}\left(H^{\cdot}_{c,G_\gamma}\left(Ob(A)_\gamma, \varphi_{Tr(W)}\right)\right)^*$$

compactly supported equivariant

compactly supported cohom.

γ - dim rep.s of Q which belong to Crit(Tr(W))

$$\gamma = (\gamma^i)_{i\in I}$$

$$G_\gamma = \prod_{i\in I} GL(\gamma^i, \mathbb{C})$$

$$\mathbb{C}^{\gamma^i} \xrightarrow{a} \mathbb{C}^{\gamma^j}$$

Fact. $\mathcal{H}^{(Q,W)}$ is an assoc. alg.

In reality:

$\underline{d=3}$ $\Longleftrightarrow$ 3 CY cat. $\cup$ w/ good t-structure

$k = \mathbb{C}$

From now on: (Q,W)

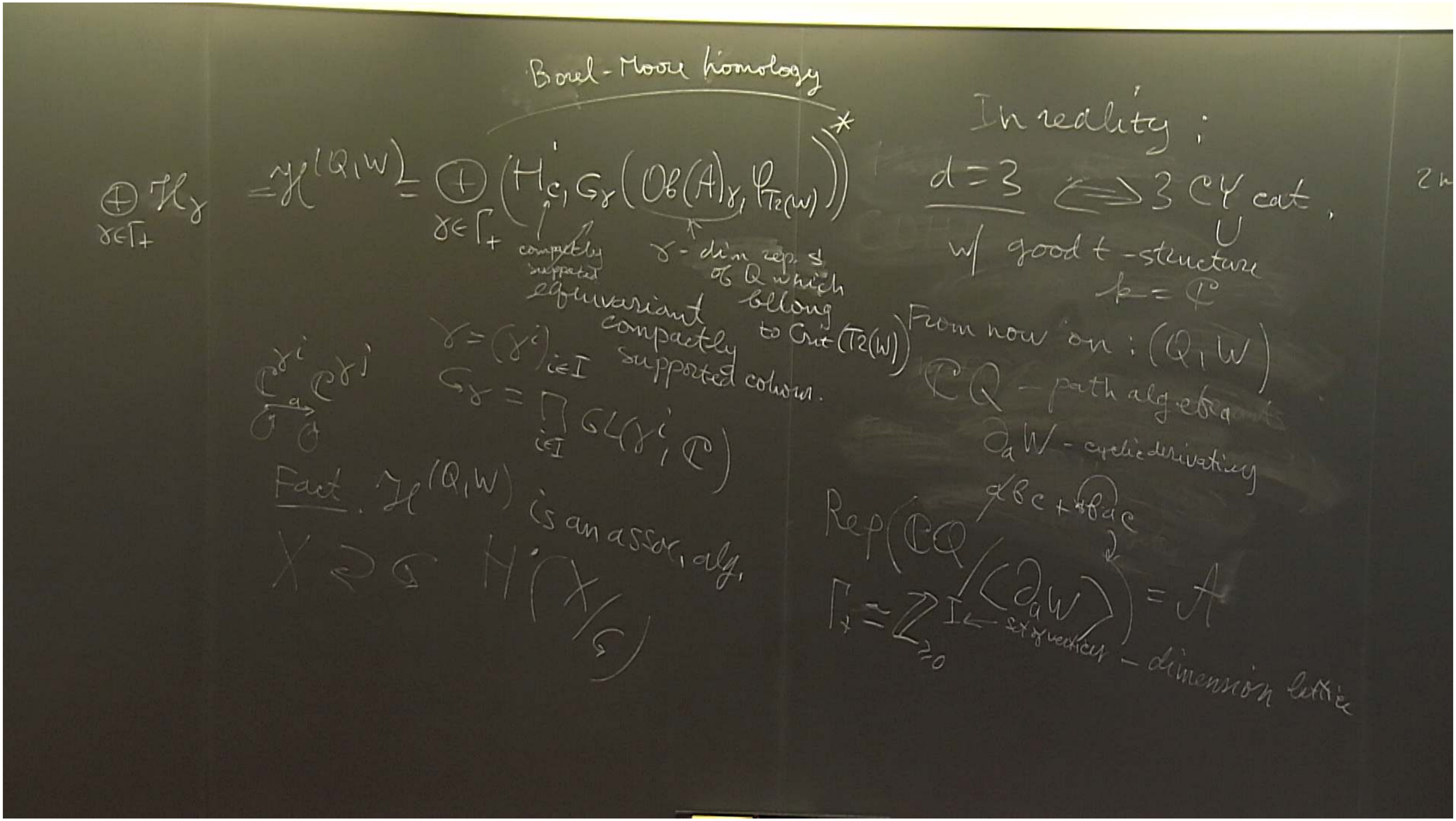
$\mathbb{C}Q$ – path algebra

$\partial_a W$ - cyclic derivatives

$\cancel{a}\beta c + \beta a c$

$$Rep\left(\mathbb{C}Q/\langle\partial_a W\rangle\right) = A$$

$\Gamma_+ \simeq \mathbb{Z}^I_{\geq 0}$ ← set of vertices – dimension lattice



Borel-Moore homology

$$\bigoplus_{\gamma\in\Gamma_+}\mathcal{H}_\gamma = \mathcal{H}^{(Q,W)} = \bigoplus_{\gamma\in\Gamma_+}\left(H^{\cdot}_{c,G_\gamma}\left(Ob(\mathcal{A})_\gamma,\ \varphi_{Tr(W)}\right)\right)^*$$

compactly supported equivariant compactly supported cohom.

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$X \circlearrowleft G \quad H^{\cdot}(X/G)$

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$abc + bac$

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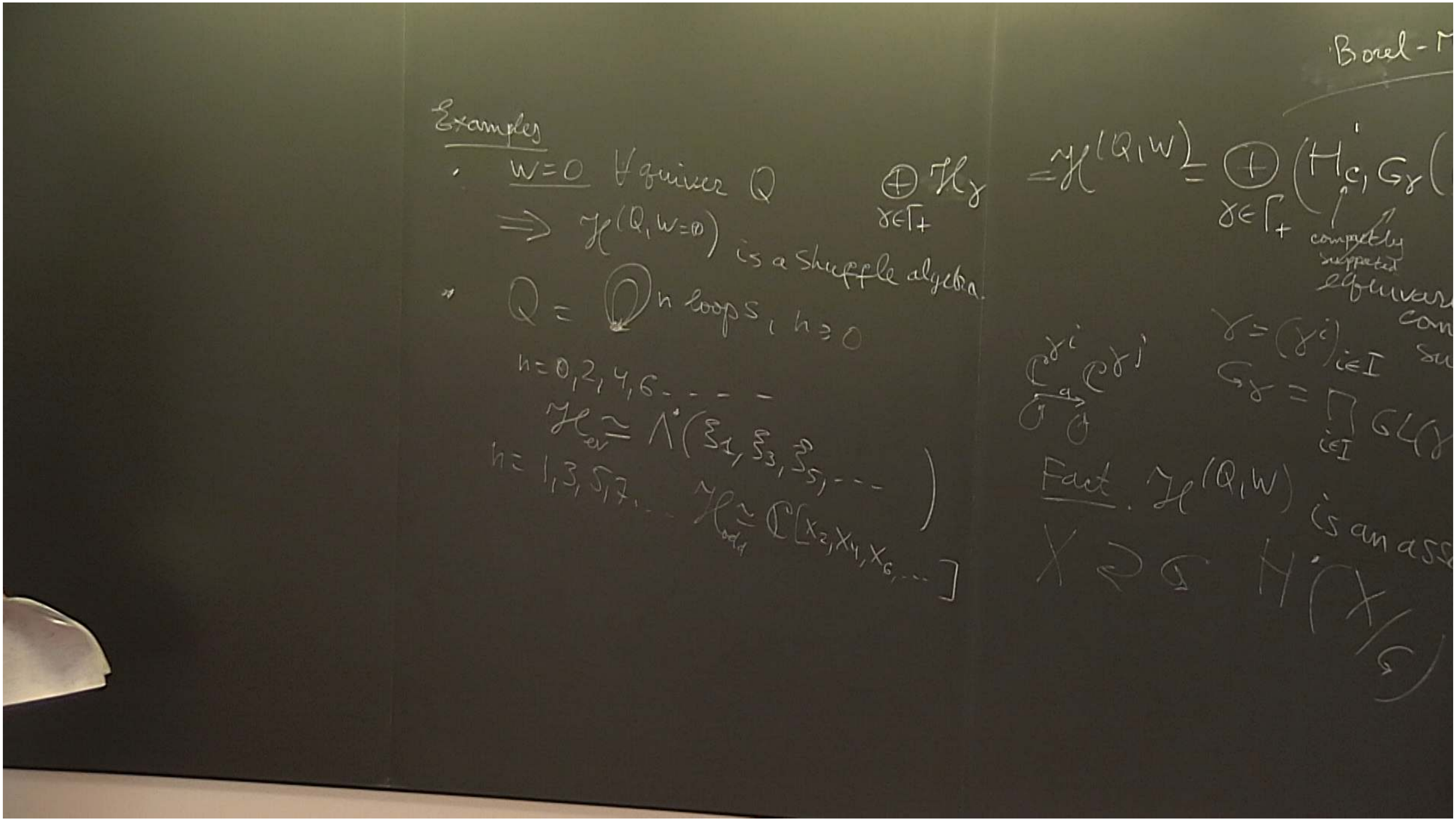
$\Gamma_+ \simeq \mathbb{Z}^{I}_{\geq 0} \leftarrow$ set of vertices - dimension lattice

2 n



Examples

- $W=0$ ∀ quiver Q $\qquad \bigoplus_{\gamma\in\Gamma_+} \mathcal{H}_\gamma$

$\Rightarrow \mathcal{H}^{(Q,W=0)}$ is a shuffle algebra.

- Q = n loops, $h \geq 0$

$h = 0, 2, 4, 6 \dots$

$\mathcal{H}_{ev} \simeq \Lambda^{\cdot}(\xi_1, \xi_3, \xi_5, \dots)$

$h = 1, 3, 5, 7, \dots \quad \mathcal{H}_{odd} \simeq \mathbb{C}[x_2, x_4, x_6, \dots]$

Borel-M

$= \mathcal{H}^{(Q,W)} = \bigoplus_{\gamma\in\Gamma_+} \left(H^{\cdot}_{c, G_\gamma}\right($

compactly supported

equivariant

com

su

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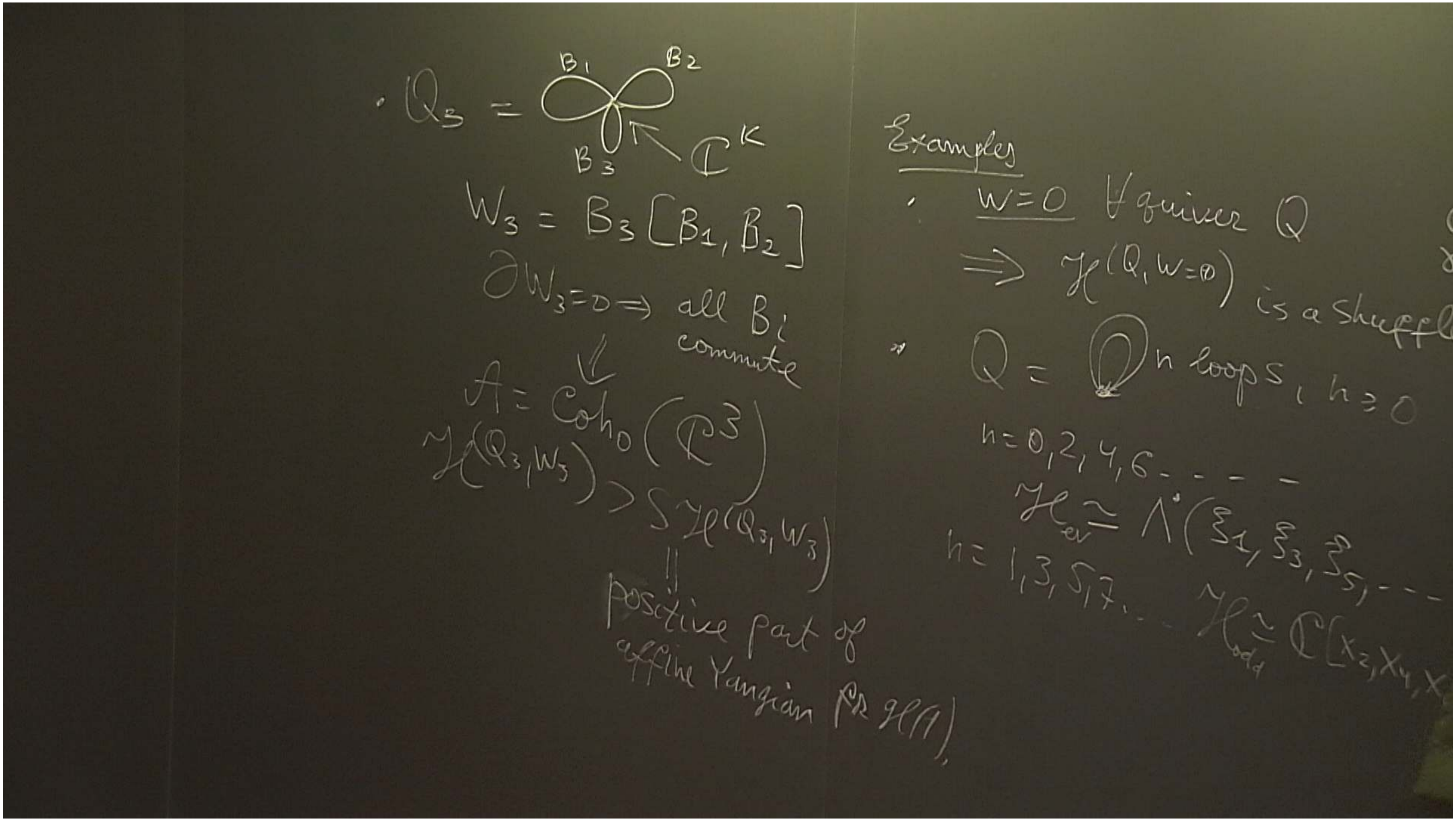

- $Q_3 =$ B_1 B_2 B_3 $\mathbb{C}^K$

$W_3 = B_3[B_1, B_2]$

$\partial W_3 = 0 \Rightarrow$ all B_i commute

$\Downarrow$

$A = \mathrm{Coh}_0(\mathbb{C}^3)$

$\mathcal{H}(Q_3, W_3) \supset S\mathcal{H}(Q_3, W_3)$

$\|$

positive part of affine Yangian for $\mathfrak{gl}(1)$.

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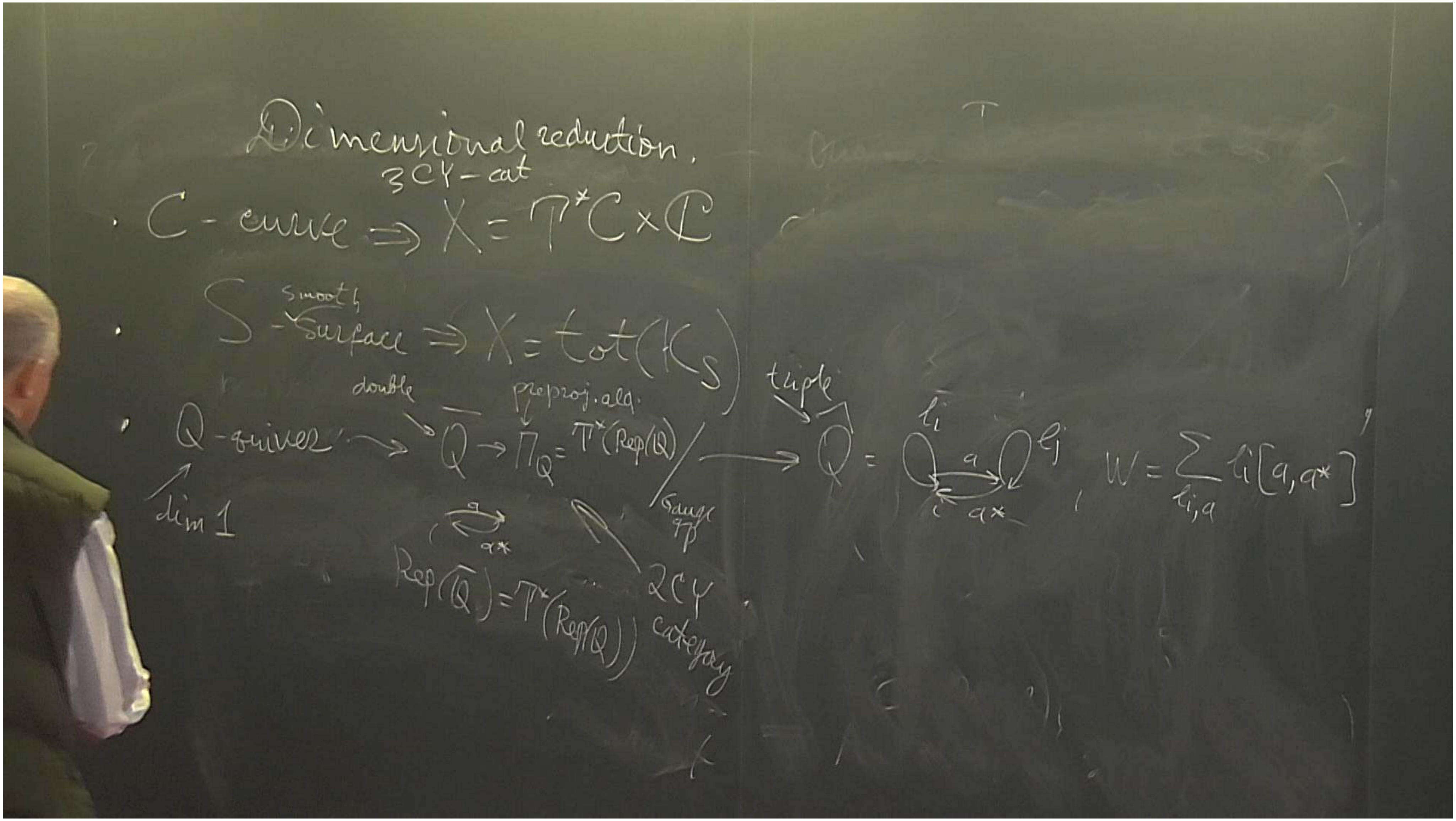


Dimensional reduction.

$\overset{\text{3CY-cat}}{}$

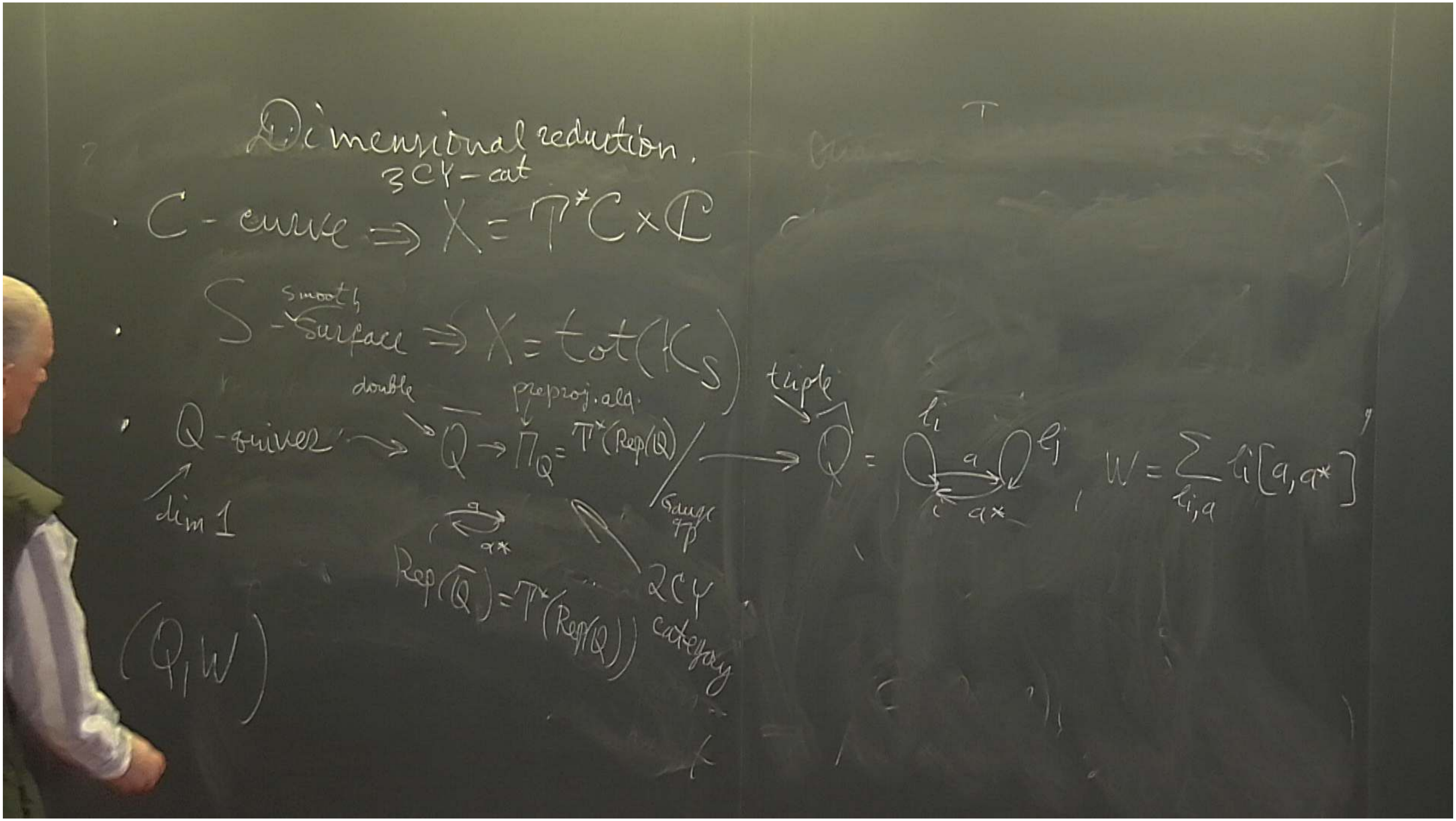
- C - curve $\Rightarrow X = T^*C \times \mathbb{C}$
- S - $\overset{\text{smooth}}{\text{surface}}$ $\Rightarrow X = \mathrm{tot}(K_S)$
- Q - quiver $\overset{\text{double}}{\rightsquigarrow} \overline{Q} \to \overset{\text{preproj. alg.}}{\Pi_Q} = T^*(\mathrm{Rep}(Q)) \big/ \text{Gauge gp} \xrightarrow{\text{triple}} \hat{Q} = $ (quiver with loops l_i, l_j and arrows a, a^*), $W = \sum_{l_i, a} l_i [a, a^*]$

dim 1

a, a^*

$\mathrm{Rep}(\overline{Q}) = T^*(\mathrm{Rep}(Q))$

2CY category



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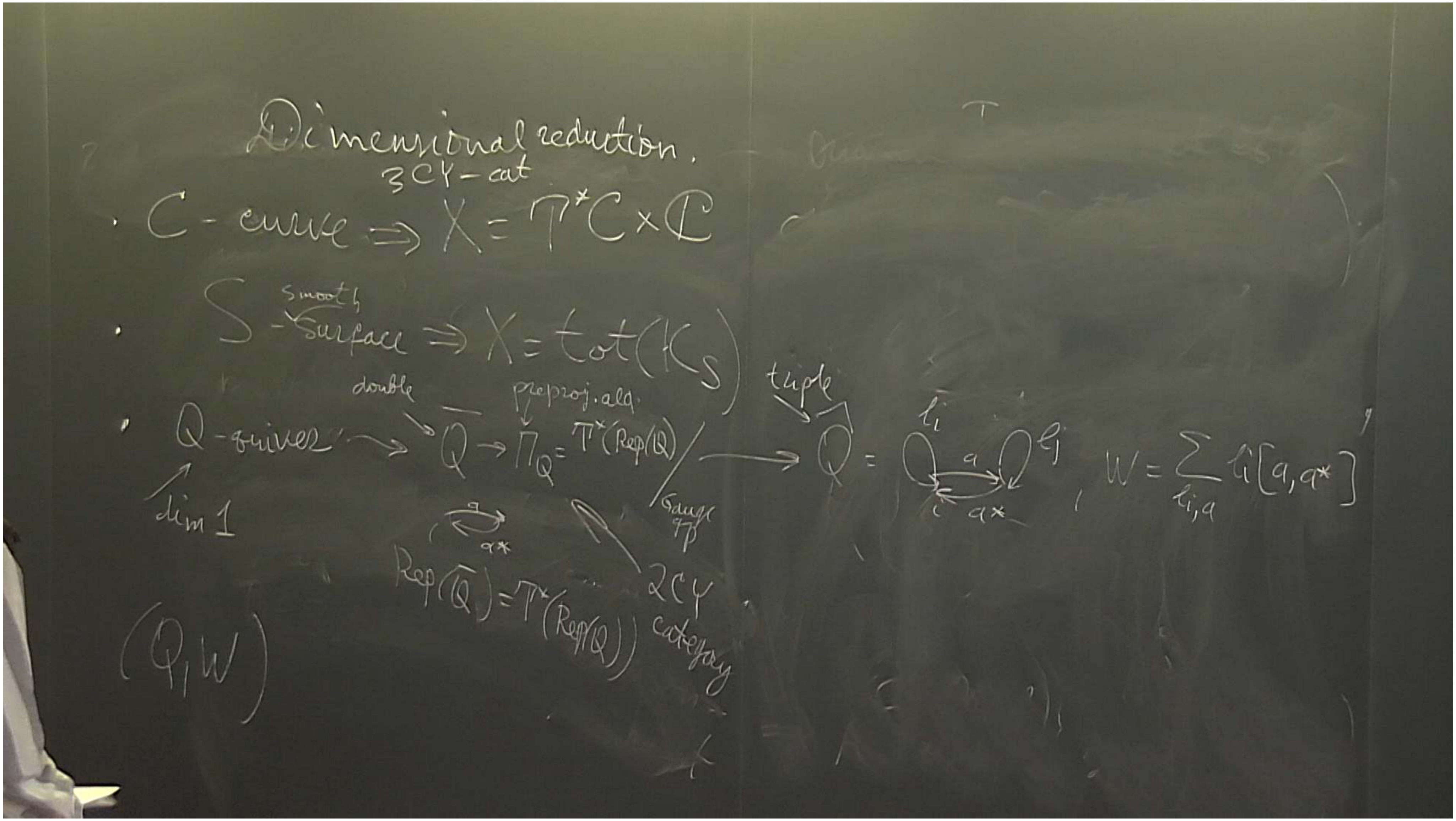
2CY category

(Q, W)



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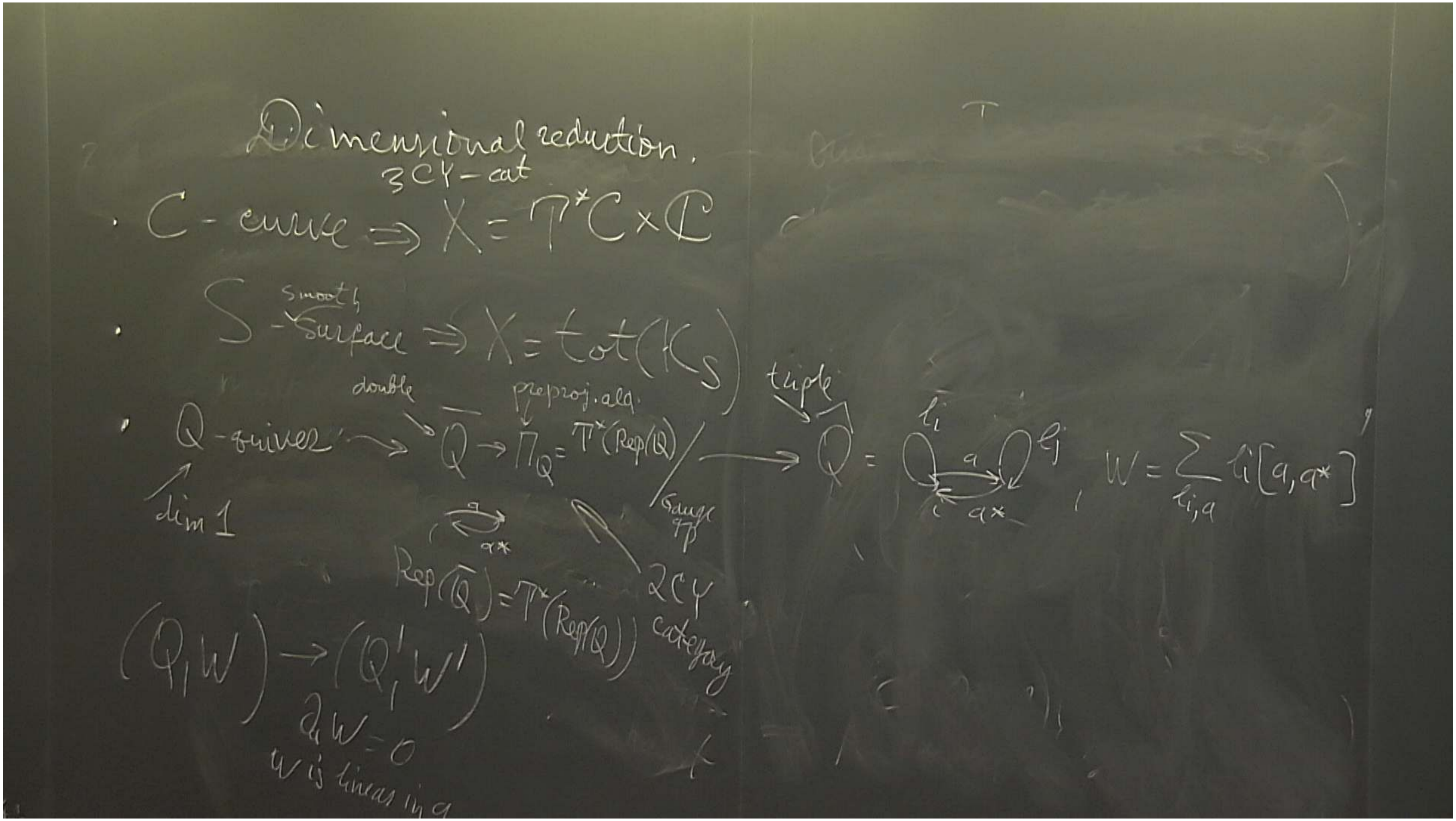

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a, a^*

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2CY category

$(Q, W) \to (Q', W')$

$\partial_a W = 0$

W is linear in a


CY categories
CY category of dim d is
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<,> Hom(E,F) ⊗ Hom(F,E) → k[-d]
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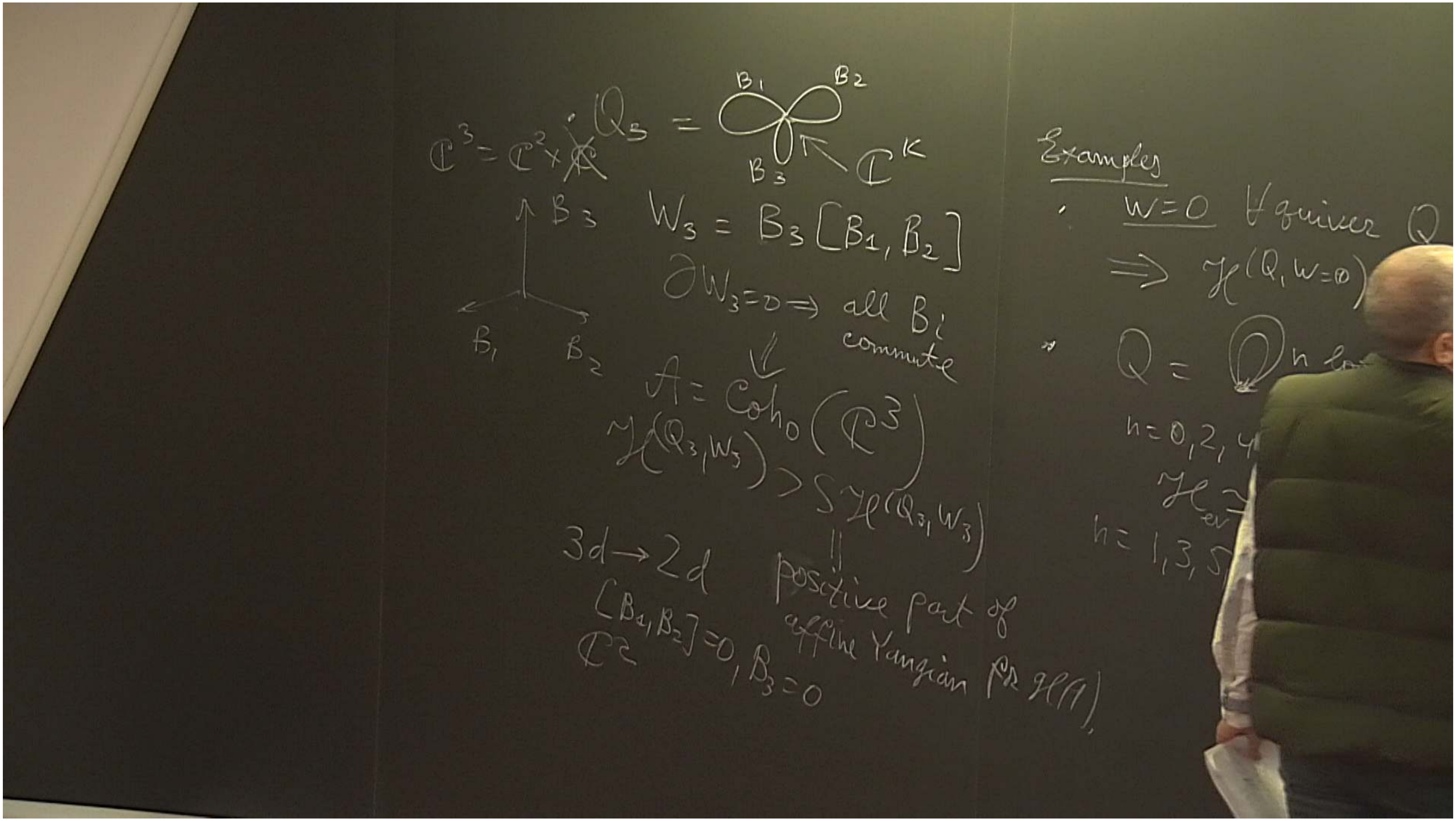
2CY ⟨S⟩



$\mathbb{C}^3 = \mathbb{C}^2 \times \cancel{\mathbb{C}}$

B_3

$B_1 \quad B_2$

$Q_3 =$

$B_1 \quad B_2$

$B_3 \quad \mathbb{C}^K$

$W_3 = B_3 [B_1, B_2]$

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$\Downarrow$

$\mathcal{A} = \mathrm{Coh}_0(\mathbb{C}^3)$

$\mathcal{H}(Q_3, W_3) \supset S\mathcal{H}(Q_3, W_3)$

$\|$

positive part of affine Yangian for $\mathfrak{gl}(1)$,

$3d \to 2d$

$[B_1, B_2] = 0, \; B_3 = 0$

$\mathbb{C}^2$

Examples

- $W = 0$ $\forall$ quiver Q

$\Longrightarrow \mathcal{H}(Q, W=0)$

- $Q =$ n lo

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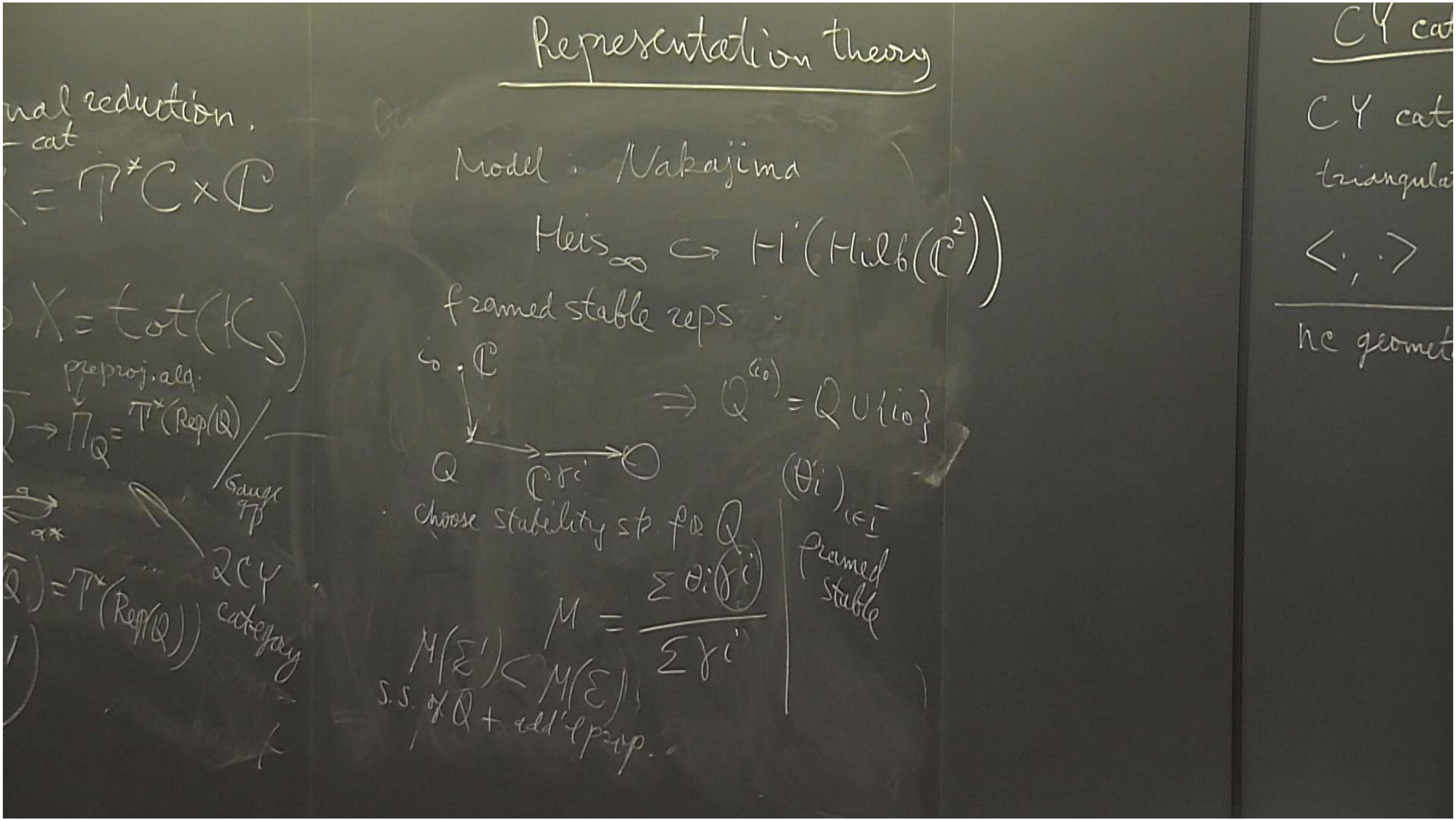
$\mathcal{H}_{ev} \cong$

$n = 1, 3, 5$



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preproj. alg.

$Q \to \Pi_Q = T^*(\mathrm{Rep}(Q)) /$ Gauge grp

a, a^*

$(\bar{Q}) = T^*(\mathrm{Rep}(Q))$

2CY category

Representation theory

Model: Nakajima

$\mathrm{Heis}_\infty \hookrightarrow H^*(\mathrm{Hilb}(\mathbb{C}^2))$

framed stable reps:

$$\begin{tikzcd} i_0 \bullet \mathbb{C} \arrow[d] & & \\ Q \bullet \arrow[r] & \mathbb{C}^{\gamma^i} \arrow[r] & \bigcirc \end{tikzcd}$$

$\Rightarrow Q^{(i_0)} = Q \cup \{i_0\}$

$(\theta^i)_{i \in I}$

framed stable

choose stability s.t. for Q

$$\mu = \frac{\sum \theta_i \gamma^i}{\sum \gamma^i}$$

$\mu(E') < \mu(E)$

s.s. of Q + add'l prop.

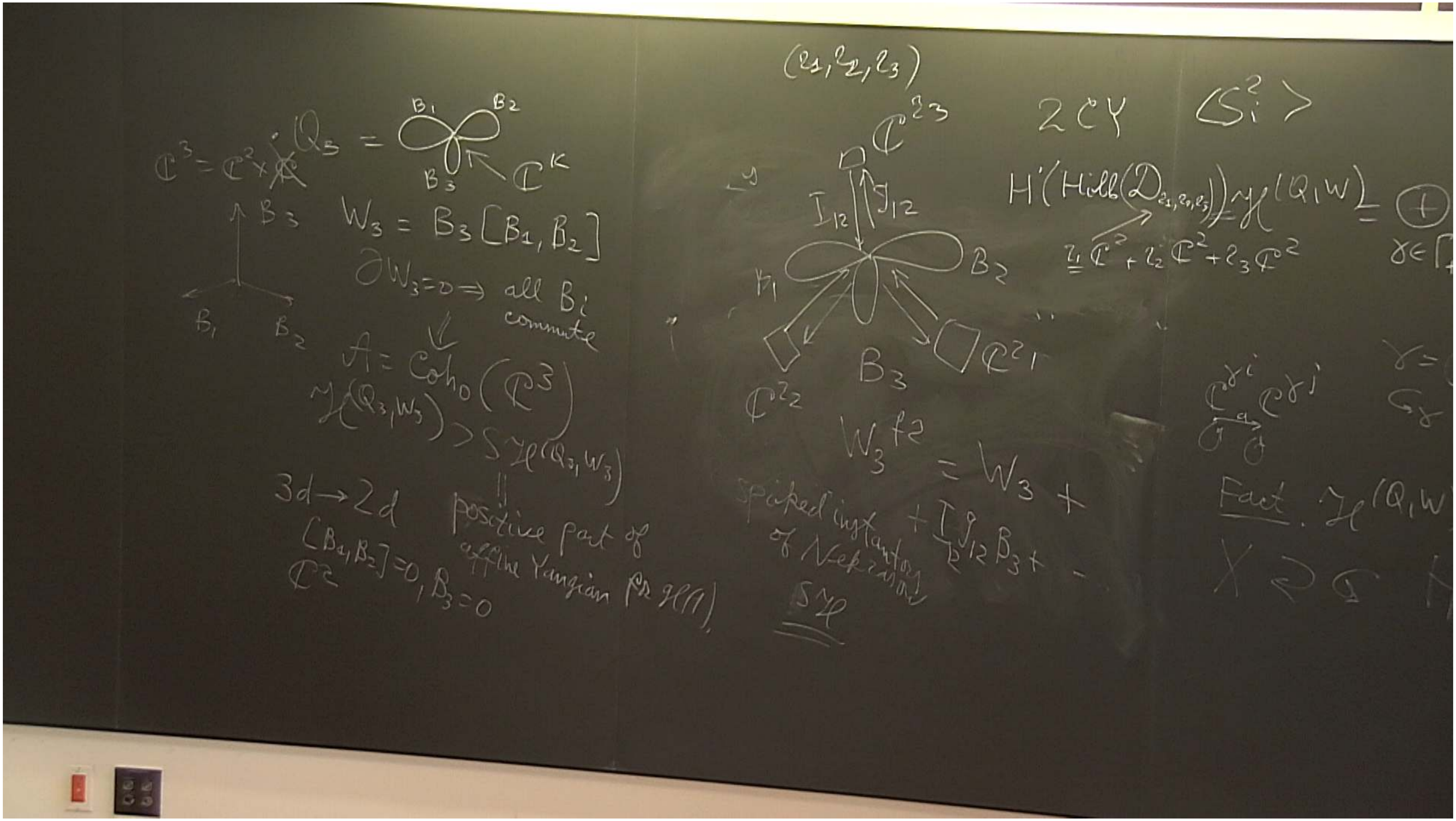
CY cat

CY cat

triangula

$\langle \cdot , \cdot \rangle$

nc geomet



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positive part of affine Yangian for $\mathfrak{gl}(1)$.

$3d \rightarrow 2d$

$[B_1,B_2]=0, B_3=0$

$\mathbb{C}^2$

(e_1, e_2, e_3)

$W_3^{fr} = W_3 + I_{12}J_{12}B_3 + \dots$

spiked instantons of Nekrasov

$\underline{\underline{S\mathcal{H}}}$

2CY $\langle S_i^2 \rangle$

$H^*(\mathrm{Hilb}(\mathcal{D}_{e_1,e_2,e_3})) \sim \mathcal{H}(Q,W) \stackrel{?}{=} \bigoplus$

$\stackrel{?}{=} e_1\mathbb{C}^2 + e_2\mathbb{C}^2 + e_3\mathbb{C}^2$

$\gamma \in \Gamma$

$\mathbb{C}^{\gamma^i} \xrightarrow{a} \mathbb{C}^{\gamma^j}$

$\gamma =$

G_γ

<u>Fact</u>. $\mathcal{H}(Q,W)$

$X \supset G$