Title: SO(7,7) structure of the SM fermions

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Series: Quantum Gravity

Date: February 28, 2019 - 2:30 PM

URL: http://pirsa.org/19020035

Abstract: I will describe the relevant representation theory that allows to think of all components of fermions of a single generation of the Standard Model as components of a single Weyl spinor of an orthogonal group whose complexification is SO(14,C). There are then only two real forms that do not lead to fermion doubling. One of these real forms is the split signature orthogonal group SO(7,7). I will describe some exceptional phenomena that occur for the orthogonal groups in 14 dimensions, and then specifically for this real form. The real form SO(7,7) suggests a link to generalised geometry, which I will describe. I will also describe why spinors in general are really differential forms in disguise.

SO(7,7) structure of Standard Model Fermions

Based on 1803.06160

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Nature accurately described by the SM of particle physics

Finalised in mid 70's, last ingredient - Higgs - observed in 2012

SM, extended with right-handed neutrinos to accommodate neutrino masses, together with GR and assumptions about the nature of inflation, dark matter and dark energy explains the overwhelming majority of observations we make about our Universe

However, the SM is seemingly a mess devoid of any structure that can give clues as to what is behind it. Low energy approximation to some other, likely more symmetric theory, rather than a fundamental theory.

Still, closer inspection reveals that “hidden” structure is there. Needless to say, it is very important to have a clear understanding of what this extra structure is, for one day it may lead to a discovery of a more fundamental description

Researchers were after “hidden” structure within the SM from the times the model itself was discovered. Several GUT schemes have been put forward

Two most famous GUT scenarios

SU(5) theory by Georgi and Glashow 1974

SU(2)xSU(2)xSU(4) theory by Pati and Salam 1974

Both are “unified” within the SO(10) GUT, which is the most elegant proposal that describes all fermions of a single generation of the SM as components of a single complex Weyl irrep $\mathbf{16}_{\mathbb{C}}$

The SO(10) theory is arguably the most convincing GUT scenario. One can go down to the SM by a sequence of symmetry breakings, carefully arranged not to conflict the experiment (proton decay). The biggest question is what causes this and not that pattern of symmetry breaking. Why SM? There is no convincing explanation

Other attempts to “understand” the structure of the SM, notably non-commutative geometry. Not subject of this talk

Still, it seems that the SO(10) pattern on the SM fermions holds some important truth in it

Let me remind you the relevant representation theory, to set the stage for the later developments

Spinor Lagrangian

The structure of the SM is more transparent in the 2-component spinor formalism

2-component spinors are of two types

$$\chi_A \qquad \chi^{\dagger}_{A'}$$

unprimed (undotted) spinor

primed (dotted) spinor

Both are irreducible representations of Lorentz

Complex (Hermitian) conjugates of each other

Weyl Lagrangian

$$L = i\,(\chi^{\dagger})^{A'}\partial_{A'}{}^{A}\chi_A \equiv i\,\chi^{\dagger}\partial\chi$$

Real (Hermitian) modulo a surface term

Fermions of the SM

Two-component fermion fields	SU(3)	$SU(2)_L$	Y	T_3	$Q = T_3 + Y$
$\boldsymbol{Q}_i \equiv \begin{pmatrix} u_i \\ d_i \end{pmatrix}$	triplet	doublet	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{2}{3}$
	triplet		$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{3}$
$\bar{u}^i$	anti-triplet	singlet	$-\frac{2}{3}$	0	$-\frac{2}{3}$
$\bar{d}^i$	anti-triplet	singlet	$\frac{1}{3}$	0	$\frac{1}{3}$
$\boldsymbol{L}_i \equiv \begin{pmatrix} \nu_i \\ \ell_i \end{pmatrix}$	singlet	doublet	$-\frac{1}{2}$	$\frac{1}{2}$	0
	singlet		$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$\bar{\ell}^i$	singlet	singlet	1	0	1

All fields are 2-component spinors, transforming under SU(3) x SU(2) x U(1) as indicated

The generation indices i=1,2,3   Colour indices suppressed

Bar over a symbol is a part of the name, not to be confused with complex conjugation

SM Lagrangian

We describe it in words instead of writing a long expression

Every of the 2-component spinors in the table will have its Weyl kinetic term. Spinors are coupled to the SU(3) x SU(2) x U(1) gauge fields, and the Higgs field, which is a complex valued SU(2) doublet, of hypercharge Y=1/2. All terms of mass dimension four that are compatible with the gauge and Lorentz symmetry are written down, together with their Hermitian conjugates.

Plus there are kinetic terms for the gauge fields - usual F^2

Plus there is the kinetic plus potential term for the Higgs. Potential is quartic and makes Higgs acquire a non-trivial VEV.

Right-handed sterile neutrinos $\bar{\nu}_i$ can be added for free

If add Majorana mass terms for them, gets see-saw mechanism

SO(10) structure of SM fermions

To see all fermions of a single generation inside a single irreducible representation of SO(10) need to think of leptons as the fourth colour of quarks

$$\begin{pmatrix} \nu \\ l \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix}^{lepton}$$

Then have SU(4) mixing the four colours of quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}^{red} \quad \begin{pmatrix} u \\ d \end{pmatrix}^{green} \quad \begin{pmatrix} u \\ d \end{pmatrix}^{blue} \quad \begin{pmatrix} u \\ d \end{pmatrix}^{lepton}$$

Overall, we have fields transforming under Pati-Salam group

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4)$$

in the following representations

$$Q = \begin{pmatrix} u \\ d \end{pmatrix} \quad (\mathbf{2}, \mathbf{1}, \mathbf{4}) \qquad \bar{Q} = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \quad (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}})$$

One then notes $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2 = \mathrm{SO}(4)$

$$\mathrm{SU}(4)/\mathbb{Z}_2 = \mathrm{SO}(6)$$

Cartan's isomorphisms

And $\mathrm{SO}(4) \times \mathrm{SO}(6) \subset \mathrm{SO}(10)$

We also have the following representation theory fact:

A Weyl spinor representation of SO(2n), when restricted to SO(2k) x SO(2(n-k)) embedded into SO(2n) in the standard way, will split as a Weyl spinor of both SO(2k) and SO(2(n-k)), plus another Weyl spinor of both SO(2k) and SO(2(n-k)), of opposite chiralities

This shows that all spinors of a single generation of SM arise as components of a single Weyl spinor of SO(10), with Pati-Salam group embedded into SO(10) in the standard way

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4) \sim \mathrm{SO}(4) \times \mathrm{SO}(6) \subset \mathrm{SO}(10)$$

2-component spinors of single generation are components of $\mathbf{16}_{\mathbb{C}}$ irreducible Weyl representation of SO(10)

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Back to the main story:

In the description of SO(10) GUT Lorentz spinor indices played no role. The GUT fermion is an object

$$\Psi_{\alpha A}$$

plus complex conjugate

SO(10) spinor index

$$\alpha = 1, \ldots, 16$$

SO(1,3) 2-component spinor index

$$A = 1, 2$$

Overall, single generation of fermions is described by 16 x 2 complex functions or

64 real valued functions

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Can “unify” the Lorentz and GUT spinor indices by repeating

$$\mathrm{SO}(2k) \times \mathrm{SO}(2(n-k)) \subset \mathrm{SO}(2n)$$

Should put the Lorentz SO(1,3) and GUT SO(10) groups together

Some real form of

$$\mathrm{SO}(4, \mathbb{C}) \times \mathrm{SO}(10, \mathbb{C}) \subset \mathrm{SO}(14, \mathbb{C})$$

Weyl spinor of SO(14,C) is 64 dimensional (complex), and splits

$$\mathbf{2}_{\mathbb{C}} \otimes \mathbf{16}_{\mathbb{C}} + \mathbf{2}_{\mathbb{C}} \otimes \mathbf{16}_{\mathbb{C}}$$

into a sum of two Weyl representations of opposite chiralities

This is as we want, should just select an appropriate real form

SO(14, C) real form

Standard representation theory of Clifford algebras shows that there are only two real forms that give a real 64-dimensional Weyl representation

Have $\mathrm{SO}(s, r) \qquad s + r = 14$

To have Weyl representation being real need $s - r = 0 \mod 8$

The two possibilities are

$$\mathrm{SO}(7,7) \qquad s - r = 0$$
$$\mathrm{SO}(11,3) \qquad s - r = 8$$

Both contain Lorentz SO(1,3) and Pati-Salam groups as subgroups

$$\mathrm{SO}(1,3) \times \mathrm{SO}(6,4) \subset \mathrm{SO}(7,7)$$

$$\mathrm{SO}(1,3) \times \mathrm{SO}(10) \subset \mathrm{SO}(11,3)$$

This talk is an advertisement of the first option

So, both SO(7,7) and SO(11,3) contain Lorentz group, and the Pati-Salam GUT gauge group inside the subgroup that commutes with Lorentz

Representation theory works out correctly, fermions with correct quantum numbers get produced

Moreover, will get also the kinetic terms for all the fermions by dimensional reduction

$$\int_{\mathbb{R}^{14}} \Psi \partial\!\!\!/ \, \Psi$$

The dimensional reduction of the 14D Weyl Lagrangian to 4D will produce all the correct fermion kinetic terms

(this exercise is carried out in my paper)

Everything we described is representation theory of the type very familiar to people. Precisely these ideas were considered by people in the past, but they never became popular. Why?

Percacci SO(11,3) scheme

Chamseddine-Mukhanov SO(13,1) scheme, but then fermion doubling

Can easily couple fermions to gauge fields

SO(11,3) or SO(7,7) connection

$$\int_{\mathbb{R}^{14}} \Psi \gamma^I (\partial_I + \frac{1}{2}[\gamma^J, \gamma^K]\omega_I{}^{JK})\Psi$$

What kind of kinetic terms can we write for these fields? If write YM will get negative sign terms for some components. But also should not write YM because spin (Lorentz) gravitational connection is involved

The difficulty here is that of unifying gravity with other interactions!

But SM fermions suggest that this unification takes place!

The biggest puzzle of this scheme is what breaks the SO(14) symmetry down to Lorentz plus SM gauge group

Thus, to make any progress on any idea of this sort, and to convert it into a theory, one has to solve a very hard problem of what makes gravity a special force

Thus, in spite of representation theory working out beautifully, it seems that much more than representation theory is needed to make the next step

The talk could end here. But I will add few more remarks of representation theoretic nature that show that SO(14) is very special. It may be that the clues for the next step can be found this way

Representation theory Fact #1

Consider the action of SO(2n) on its Weyl spinor representation

$$\dim(\mathrm{SO}(2n)) = \frac{2n(2n-1)}{2}$$

Dimension of the group

$$\dim(W_{2n}) = 2^{n-1}$$

Dimension of the Weyl representation

The dimension of the spinor representation grows with n much faster than dimension of the group

While for small n we have

$$\dim(\mathrm{SO}(2n)) > \dim(W_{2n})$$

The last n when this is true is n=7 giving SO(14)

This will not be true for sufficiently large n

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Indeed, for n=7

$$\boxed{\begin{aligned}\dim(\mathrm{SO}(14)) &= 91 \\ \dim(W_{14}) &= 64\end{aligned}}$$

For n=8

$$\dim(\mathrm{SO}(16)) = 120$$

$$\dim(W_{16}) = 128$$

Last dimension when

$$\dim(\mathrm{SO}(2n)) > \dim(W_{2n})$$

Why is this interesting?

When dimension of the group is bigger than dimension of the space it acts on, generically, there is a non-trivial subgroup stabilising a point - symmetry breaking

Again generically, this stabiliser subgroup will define some geometric structure. All this is very interesting for SO(7,7)!

Representation theory Fact #2

The dimension of the generic orbit for SO(7,7) acting in its Weyl representation is 63 - the “scale” of the spinor can not be changed

$$\dim(\mathrm{SO}(7,7)) - \dim(\mathrm{orbit}) = 91 - 63 = 28$$

This suggests that the stabiliser is related to G_2 $\quad \dim(G_2) = 14$

(could also be SO(8) but this is not what happens)

There are three possible generic orbits, with stabilisers being

Cases 1,1’	$G_2 \times G_2$	Compact real form
	$G_2' \times G_2'$	Split real form
Case 2	$G_2^{\mathbb{C}}$	

Representation theory Fact #3

With the stabilisers related to $G_2 \subset \mathrm{SO}(7)$

Generic Weyl spinor of SO(7,7) defines the second metric in $\mathbb{R}^{7,7}$

The way this arises is that the stabiliser subgroup is one consisting of matrices that commute with $I \in \mathrm{End}(\mathbb{R}^{7,7})$

$$I^2 = 1, \qquad I^2 = -1$$

Cases 1,1'

Case 2

I preserves the split signature metric

$$G(I\cdot, I\cdot) = G(\cdot, \cdot)$$

$$I = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}$$

Lowering the index of I with the metric in $\mathbb{R}^{7,7}$

we get the second metric (in seven dimensions) plus 2-form

Setup familiar from generalised geometry with its two metrics

- one kinematical, one dynamical

Representation theory summary

Generic fermion of the Standard Model defines a metric in seven dimensions

Extremely rare phenomenon when a spinor defines a metric
And breaks symmetry in a non-trivial pattern as well

Speculation: is gravity about dynamics of this spinor defined metric?

I do not know if this mathematics is related to reality, but there are highly exceptional geometrical phenomena happening in 7+7 D, with the SM fermions pointing at them as relevant

Is something important hidden here?

More representation theory: Weyl Lagrangian in 14D

Spinors of SO(n,n) admit a beautiful explicit description in terms of differential forms

Clifford algebra in n+n dimensions can be realised by operators acting on differential forms in n dimensions

$$(a^i)^\dagger := dx^i \qquad\qquad a_i := i_{\partial/\partial x^i}$$

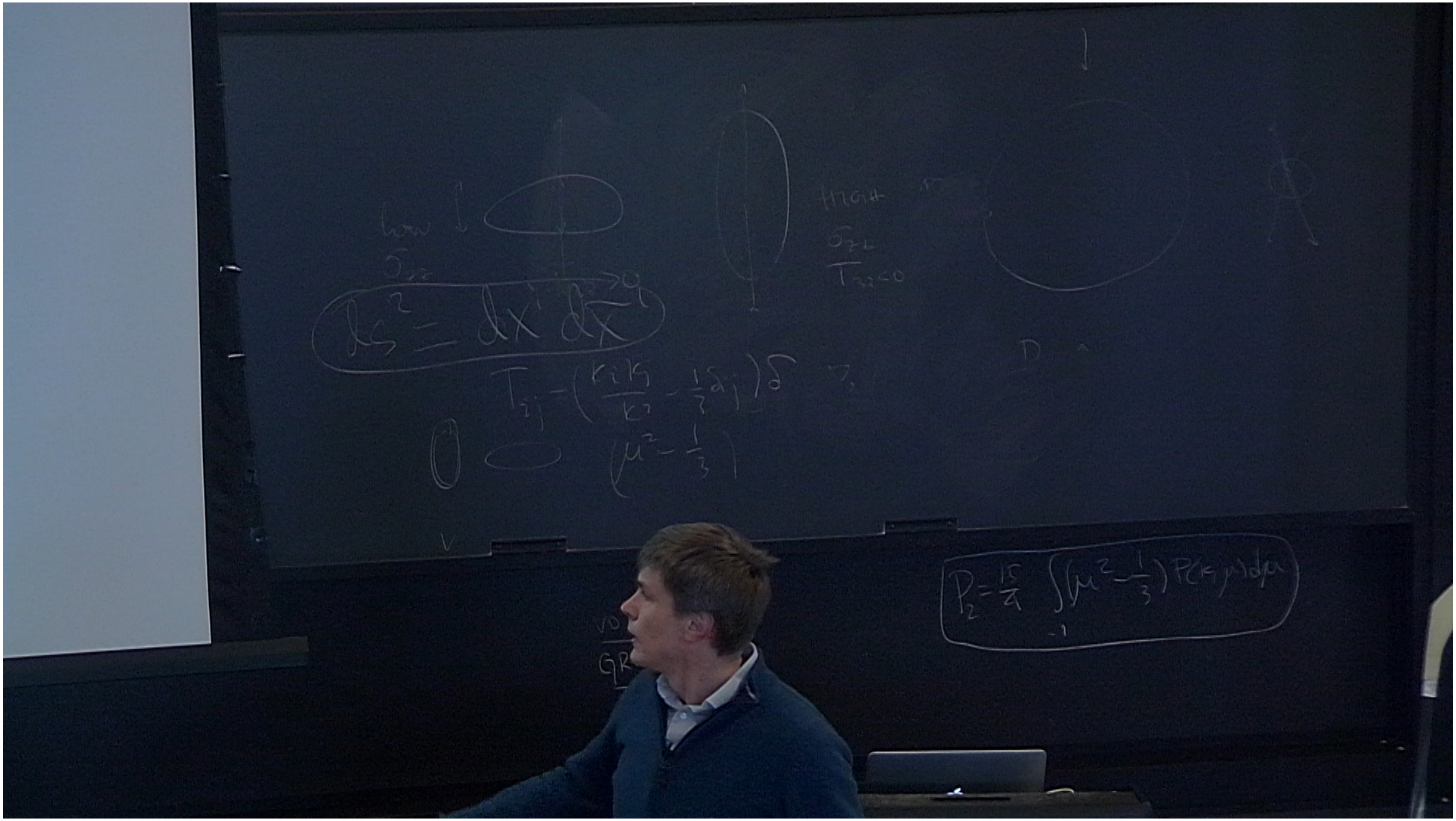
They satisfy the following anti-commutation relations

$$(a^i)^\dagger a_j + a_j (a^i)^\dagger = \delta^i_j$$

All others anti-commute

This gives a realisation of Cliff(n,n)

Weyl representations are those in fixed parity (even or odd) differential forms


low
HIGH
$ds^2 = dx^i dx_i$
$P_2 = \frac{15}{4} \int_{-1} (\mu^2 - \frac{1}{3}) P(k,\mu) d\mu$

Example of SO(2,2)

$$(a^1)^\dagger, (a^2)^\dagger, a_1, a_2$$

SO(2,2) Lie algebra is realised by all quadratic operators

Get two commuting copies of SL(2) Lie algebra

$$H = a_1 a_1^\dagger - a_2 a_2^\dagger, \qquad E_+ = a_1 a_2^\dagger, \qquad E_- = a_2 a_1^\dagger.$$

$$[E_+, E_-] = H, \qquad [H, E_\pm] = \pm 2E_\pm.$$

$$\bar{H} = a_1 a_1^\dagger + a_2 a_2^\dagger - 1 \equiv a_1 a_1^\dagger - a_2^\dagger a_2, \qquad \bar{E}_+ = a_1 a_2, \qquad \bar{E}_- = a_2^\dagger a_1^\dagger.$$

$$[\bar{E}_+, \bar{E}_-] = \bar{H}, \qquad [\bar{H}, \bar{E}_\pm] = \pm 2\bar{E}_\pm.$$

The action on odd forms

$$Hdx^2 = (a_1 a_1^\dagger - a_2 a_2^\dagger) dx^2 = dx^2, \qquad Hdx^1 = (a_1 a_1^\dagger - a_2 a_2^\dagger) dx^1 = -dx^1,$$

$$E_- \circ dx^2 = a_2 a_1^\dagger dx^2 = -dx^1, \qquad E_+ dx^1 = a_1 a_2^\dagger dx^1 = -dx^2,$$

The action on even forms

$$\bar{H}\,1 = (a_1 a_1^\dagger - a_2^\dagger a_2)\,1 = 1, \qquad \bar{H}dx^1dx^2 = (a_1a_1^\dagger - a_2^\dagger a_2)dx^1dx^2 = -dx^1dx^2,$$
$$\bar{E}_-\,1 = a_2^\dagger a_1^\dagger\,1 = -dx^1dx^2, \qquad \bar{E}_+ \circ dx^1dx^2 = a_1a_2dx^1dx^2 = -1.$$

Overall, get SO(2,2) = SL(2) x SL(2)

Weyl spinors transforming non-trivially with respect to the first SL(2) are odd forms, and non-trivially with respect to the second SL(2) are even forms

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\alpha + \beta dx^1dx^2 \qquad \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = -\bar{\alpha}dx^2 + \bar{\beta}dx^1.$$

Two types of 2-component spinors of SO(2,2)

Dirac operator

To describe Dirac operator in $\mathbb{R}^{n,n}$

will describe spinors as differential forms in $\mathbb{R}^n$

with coefficient functions depending on both $x^i, \tilde{x}_i$

The Dirac operator is

$$D\psi = c(dx^i)\frac{\partial}{\partial x^i}\psi + c(d\tilde{x}_i)\frac{\partial}{\partial \tilde{x}_i}\psi$$

where c is Clifford multiplication

Explicitly $\quad c(dx^i) = dx^i = (a^i)^\dagger$

$$c(d\tilde{x}_i) = i_{\partial/\partial x^i} = a_i$$

Dirac operator as a version of the exterior derivative operator

Dirac operator in $\mathbb{R}^{2,2}$

$$ds^2 = dx^1 d\tilde{x}_1 + dx^2 d\tilde{x}_2$$ Off-diagonal form of the metric

$$x^{1,2} = u^{1,2} + \tilde{u}^{1,2}, \qquad \tilde{x}_{1,2} = u^{1,2} - \tilde{u}^{1,2}$$

$$ds^2 = (du^1)^2 + (du^2)^2 - (d\tilde{u}^1)^2 - (d\tilde{u}^2)^2$$ Diagonal form of the metric

Two chiral Dirac operators $\partial^T : S_- \to S_+, \qquad \partial : S_+ \to S_-$

$$\partial_A{}^{A'} \equiv \partial^T = \begin{pmatrix} \partial/\partial u^2 - \partial/\partial\tilde{u}^2 & -\partial/\partial u^1 + \partial/\partial\tilde{u}^1 \\ -\partial/\partial u^1 - \partial/\partial\tilde{u}^1 & -\partial/\partial u^2 - \partial/\partial\tilde{u}^2 \end{pmatrix} = 2\begin{pmatrix} \partial/\partial\tilde{x}_2 & -\partial/\partial\tilde{x}_1 \\ -\partial/\partial x^1 & -\partial/\partial x^2 \end{pmatrix}$$

$$\partial_{A'}{}^{A} \equiv \partial = \begin{pmatrix} \partial/\partial u^2 + \partial/\partial\tilde{u}^2 & -\partial/\partial u^1 + \partial/\partial\tilde{u}^1 \\ -\partial/\partial u^1 - \partial/\partial\tilde{u}^1 & -\partial/\partial u^2 + \partial/\partial\tilde{u}^2 \end{pmatrix} = 2\begin{pmatrix} \partial/\partial x^2 & -\partial/\partial\tilde{x}_1 \\ -\partial/\partial x_1 & -\partial/\partial\tilde{x}_2 \end{pmatrix}$$

More compact notation $\partial^T = 2\begin{pmatrix} \tilde{\partial}^2 & -\tilde{\partial}^1 \\ -\partial_1 & -\partial_2 \end{pmatrix}, \qquad \partial = 2\begin{pmatrix} \partial_2 & -\tilde{\partial}^1 \\ -\partial_1 & -\tilde{\partial}^2 \end{pmatrix}$

$$\partial\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 2\begin{pmatrix} \partial_2\alpha - \tilde{\partial}^1\beta \\ -\partial_1\alpha - \tilde{\partial}^2\beta \end{pmatrix} \qquad \partial^T\begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = 2\begin{pmatrix} \tilde{\partial}^2\bar{\alpha} - \tilde{\partial}^1\bar{\beta} \\ -\partial_1\bar{\alpha} - \partial_2\bar{\beta} \end{pmatrix}$$

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Two types of 2-component spinors of SO(2,2)

Weyl Lagrangian in $\mathbb{R}^{n,n}$

Weyl Lagrangian exists only for SO(n,n) with n odd

SO(n,n) invariant inner product

$$(\Psi_1, \Psi_2) = \sigma(\Psi_1)\Psi_2\Big|$$

canonical involution

$$v_1 \otimes \ldots \otimes v_k \to v_k \otimes \ldots \otimes v_1$$

restriction to top form

$$S[\Psi] = \int_{\mathbb{R}^{n,n}} (\Psi, D\Psi)$$

Vanishes by integration by parts for n=1 mod 4

Summary

SM fermions point in the direction of a real form of SO(14,C)

SO(14,C) is the largest orthogonal group that acts densely in its irreducible Weyl representation (of “unit” spinors)

A generic SM fermion breaks SO(14,C) to the subgroup G2xG2, and the stabiliser defines a metric in seven dimensions. The fact that a spinor defines a metric is exceptional

Of the two possible real forms SO(7,7) is much more beautiful

Spinors of SO(7,7) are differential forms in 7D. Dirac operator is a version of the exterior derivative operator. Non-trivial Weyl Lagrangian exists only for SO(3,3) and SO(7,7). Dimensional reduction to 3+1 gives the SM fermion kinetic terms.

Weyl Lagrangian exists for SO(3,3)

Dimensional reduction to 3+1 gives

$$\mathrm{SO}(3,1) \times \mathrm{SO}(2) \subset \mathrm{SO}(3,3)$$

single electrically charged Weyl fermion in 3+1

The next non-trivial case is for SO(7,7)

Dimensional reduction to 3+1 gives

$$\mathrm{SO}(3,1) \times \mathrm{SO}(4) \times \mathrm{SO}(6) \subset \mathrm{SO}(7,7)$$

the fermion content is that of the Pati-Salam version of the SM

Split signature pseudo-orthogonal groups allow for a very nice explicit description of spinors and the Dirac operator

Outlook

Non-zero SM fermion field defines a metric in seven dimensions

Could it be that gravity is an effective field theory describing fluctuations of this metric? This would answer the question of why metric is non-zero, and also why gravity is a special force

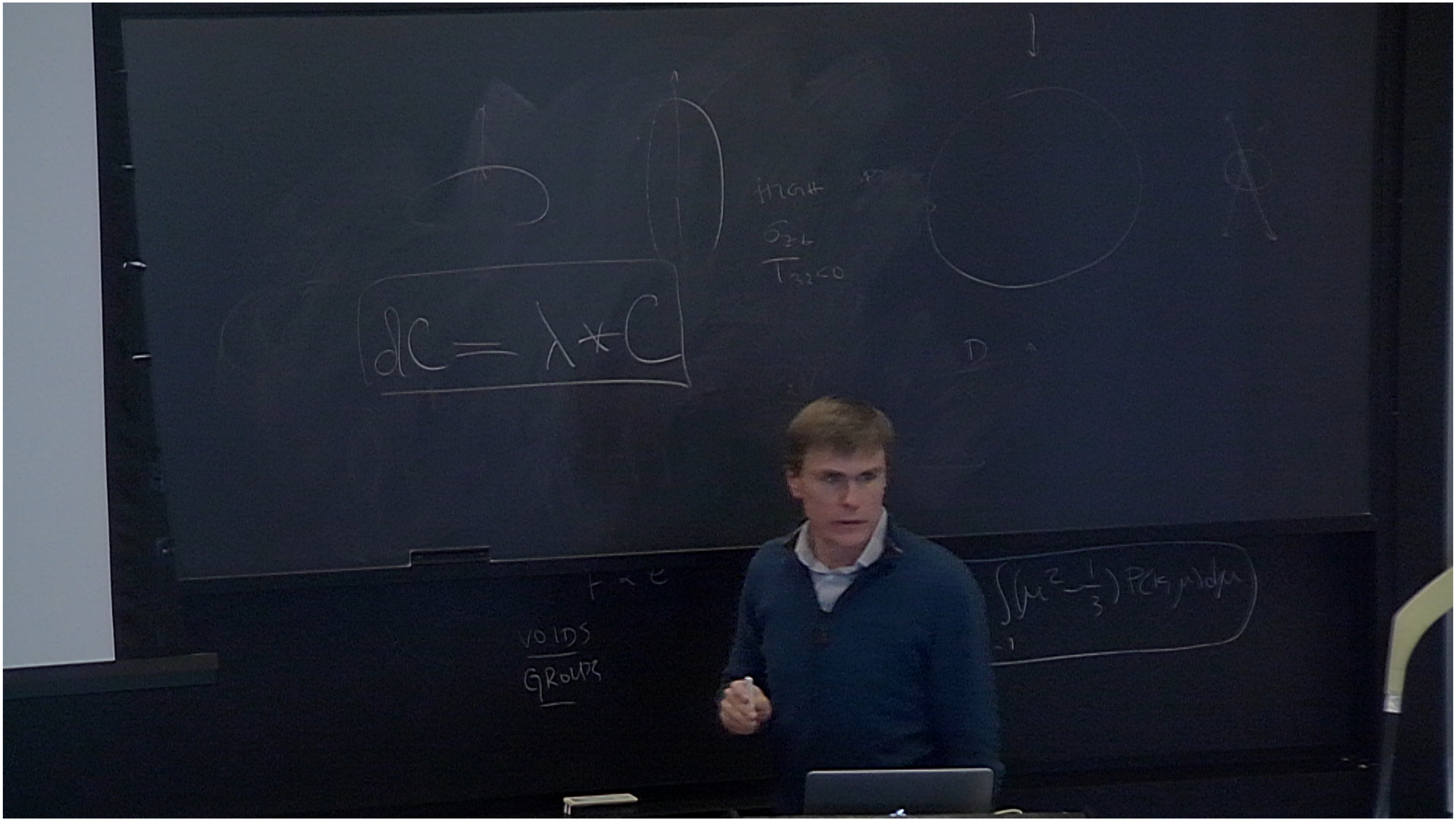
Question that can guide further developments:

$$S[\Psi] = \int_{\mathbb{R}^{7,7}} (\Psi, D\Psi) + V(\Psi)$$

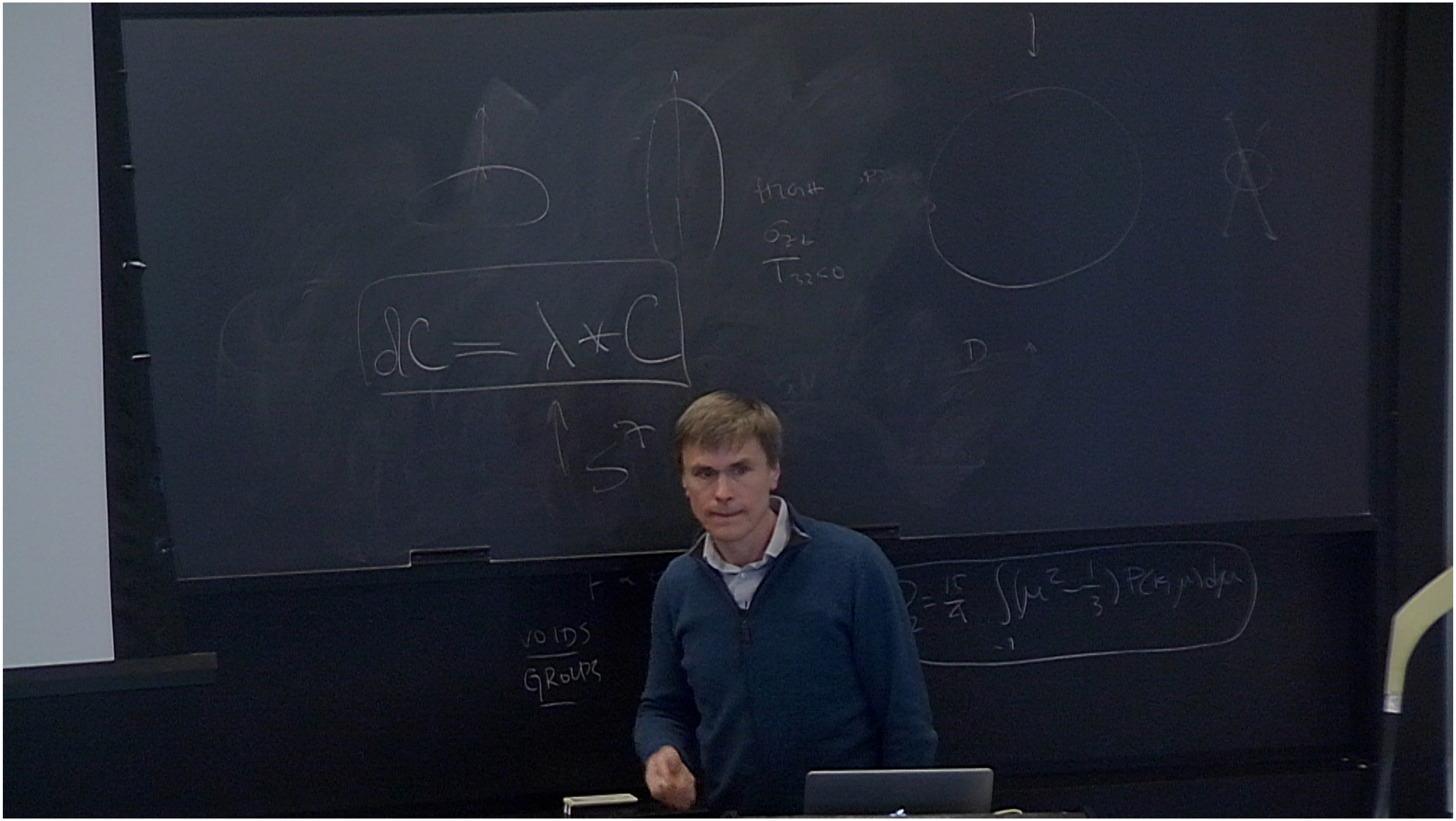
Order 8 invariant

Is there a solution of this theory that "spontaneously compactifies" to 4D and breaks the symmetry to the SM gauge group?

Such Lagrangian only exists in 7+7 dimensions!



$\boxed{dC = \lambda * C}$
VOIDS
GROUPS



$dC = \lambda * C$
σ_{zz}
$T_{zz}<0$
VOIDS
GROUPS