Title: PSI 2018/2019 - Gravitational Physics - Lecture 5

Date: Feb 01, 2019 10:15 AM

URL: http://pirsa.org/19020012

Abstract:

LECTURE 5: BLACK HOLES, HORIZONS & CAUSAL STRUCTURE

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Causal structure is a key feature of black holes - how do we encode this?

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$\boxed{r_+ = 2GM}$$

Kruskals explicitly extend across horizon at r_+

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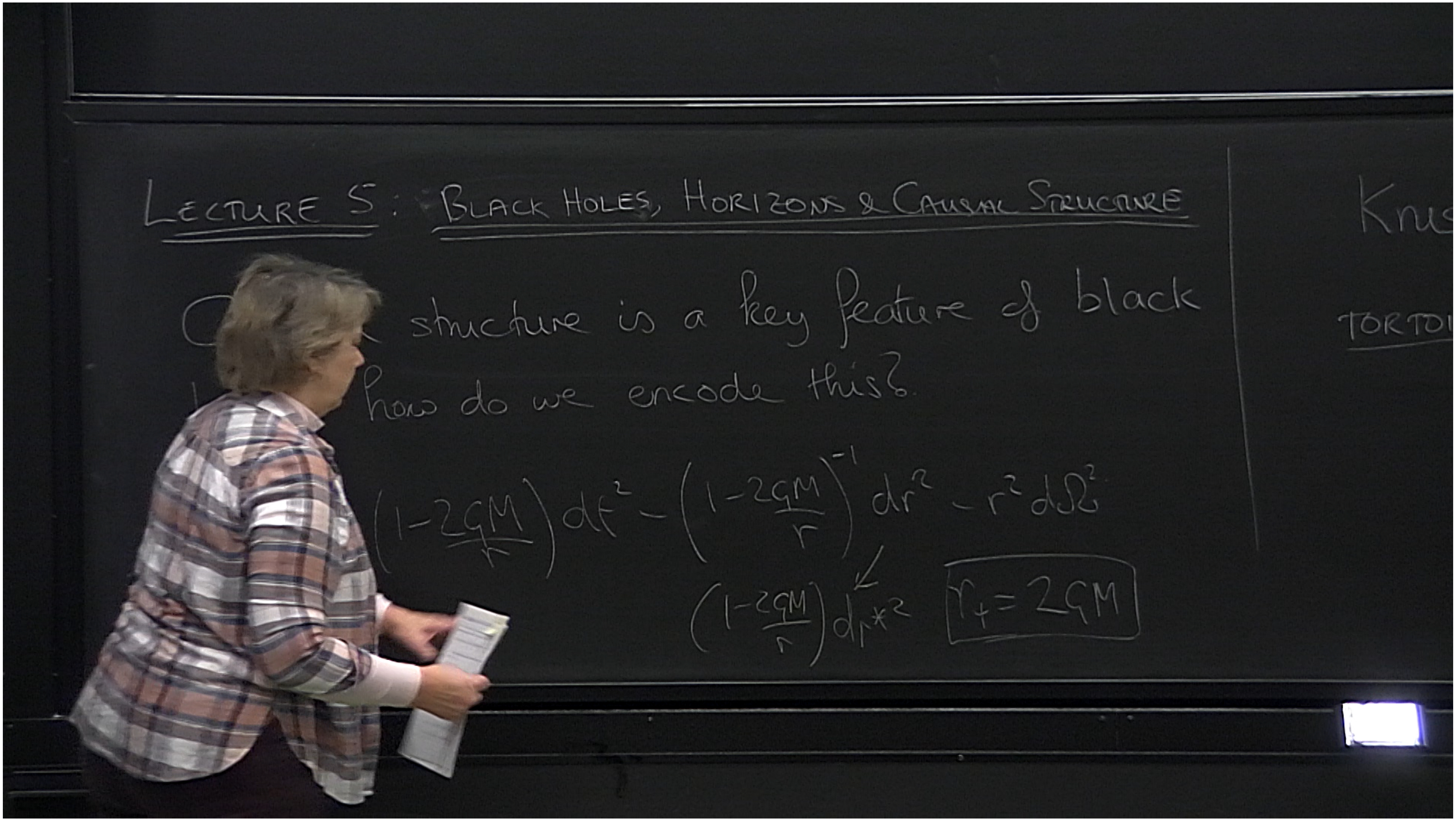
TORTOISE coord $\quad r^* = \int \frac{dr}{1-\frac{2GM}{r}} = r + 2GM \log \frac{r-2GM}{2GM}$

or $r + r_+ \log \frac{r-r_+}{r_+}$


LECTURE 5: BLACK HOLES, HORIZONS & CAUSAL STRUCTURE
structure is a key feature of black
how do we encode this?
$(1-\frac{2GM}{r})dt^2 - (1-\frac{2GM}{r})^{-1}dr^2 - r^2 d\Omega^2$
$(1-\frac{2GM}{r})dr^{*2}$
$\boxed{r_+ = 2GM}$

Kruskals explicitly extend across horizon at r_+

TORTOISE coord $\quad r^* = \int \frac{dr}{1-\frac{2GM}{r}} = r + 2GM \log \frac{r-2GM}{2GM}$

$\text{or } r + r_+ \log \frac{r-r_+}{r_+}$

$\rightarrow$ radial null geod. $t = \pm r^* + \text{const}$

Kruskals explicitly extend across horizon at r_+

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$\rightarrow$ radial null geod. $t = \pm r^* + \text{const}$

KRUSKAL: $\quad U = -r_+ \exp\left[\frac{-(t-r^*)}{2r_+}\right]$

$V = r_+ \exp\left[\frac{t+r^*}{2r_+}\right]$

$$dU\,dV = r_+^2 \exp\left[\frac{t+r^*}{2r_+} - \frac{t-r^*}{2r_+}\right]\left(\frac{dt^2 - dr^{*2}}{4r_+^2}\right)$$

$$=$$

$$dU\,dV = r_+^2 \exp\left[\frac{t+r^*}{2r_+} - \frac{t-r^*}{2r_+}\right]\left(\frac{dt^2 - dr^{*2}}{4r_+^2}\right)$$

$$= \frac{1}{4}\, e^{r^*/r_+}\left[dt^2 - \frac{dr^2}{\left(1-\frac{r_+}{r}\right)^2}\right]$$

$$e^{r^*/r_+} \downarrow e^{r/r_+}\left(1-\frac{r_+}{r}\right)$$

$$dU\,dV = r_+^2 \exp\left[\frac{t+r^*}{2r_+} - \frac{t-r^*}{2r_+}\right]\left(\frac{dt^2 - dr^{*2}}{4r_+^2}\right)$$

$$= \frac{1}{4}\, e^{r^*/r_+}\left[dt^2 - \frac{dr^2}{\left(1-\frac{r_+}{r}\right)^2}\right]$$

$$e^{r^*/r_+} \to e^{r/r_+}\left(1-\frac{r_+}{r}\right)$$

$$= \frac{e^{r/r_+}}{4}\left[\left(1-\frac{r_+}{r}\right)dt^2 - \frac{dr^2}{1-\frac{r_+}{r}}\right]$$

$$\to \boxed{ds^2 = 4\,dU\,dV\,e^{-r/r_+} - r^2 d\Omega_{II}^2}$$

Metric now regular at r_+

Metri

$$dU\,dV = r_+^2 \exp\left[\frac{t+r^*}{2r_+} - \frac{t-r^*}{2r_+}\right]\left(\frac{dt^2 - dr^{*2}}{4r_+^2}\right)$$

$$= \frac{1}{4}\,e^{r^*/r_+}\left[dt^2 - \frac{dr^2}{\left(1-\frac{r_+}{r}\right)^2}\right]$$

$$e^{r^*/r_+} \to e^{r/r_+}\left(1-\frac{r_+}{r}\right)\frac{r}{r_+}$$

$$= \frac{e^{r/r_+}}{4}\left[\left(1-\frac{r_+}{r}\right)dt^2 - \frac{dr^2}{1-\frac{r_+}{r}}\right]$$

$$\longrightarrow \boxed{ds^2 = 4\frac{r_+}{r}\,dU\,dV\,e^{-r/r_+} - r^2\,d\Omega_{II}^2}$$

$dr^{*2})$

Metric now regular at r_+

$r = \text{const} \longleftrightarrow UV = \text{const}$

$t = \text{const} \longleftrightarrow U/V = \text{const.}$

$r = r_+ \longleftrightarrow UV = 0$

Metric now regular at r_+

$r = \text{const} \longleftrightarrow UV = \text{const}$

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$r = 0 \longleftrightarrow UV = r_+^2$

Metric now regular at r_+

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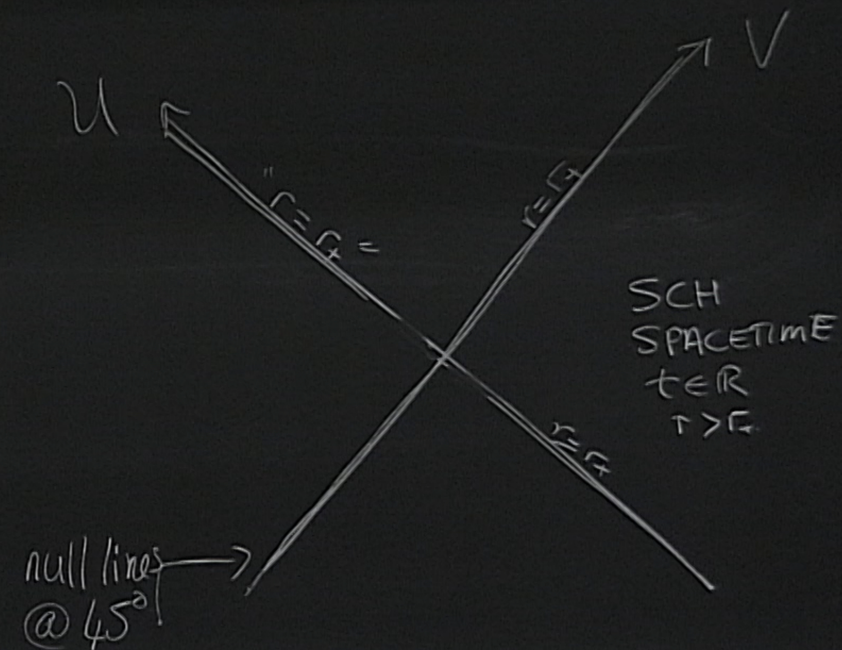

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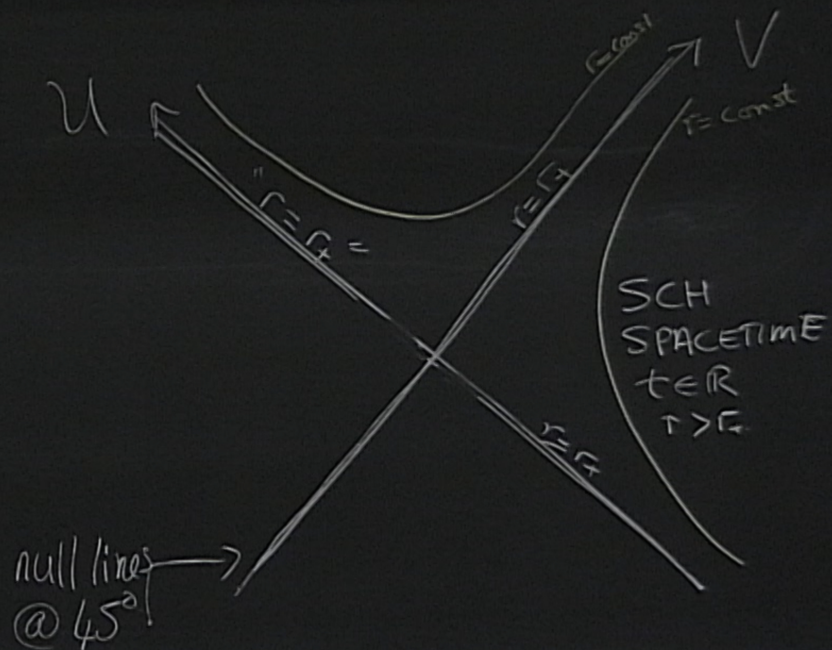

Metric now regular at r_+

$r = \text{const} \longleftrightarrow UV = \text{const}$

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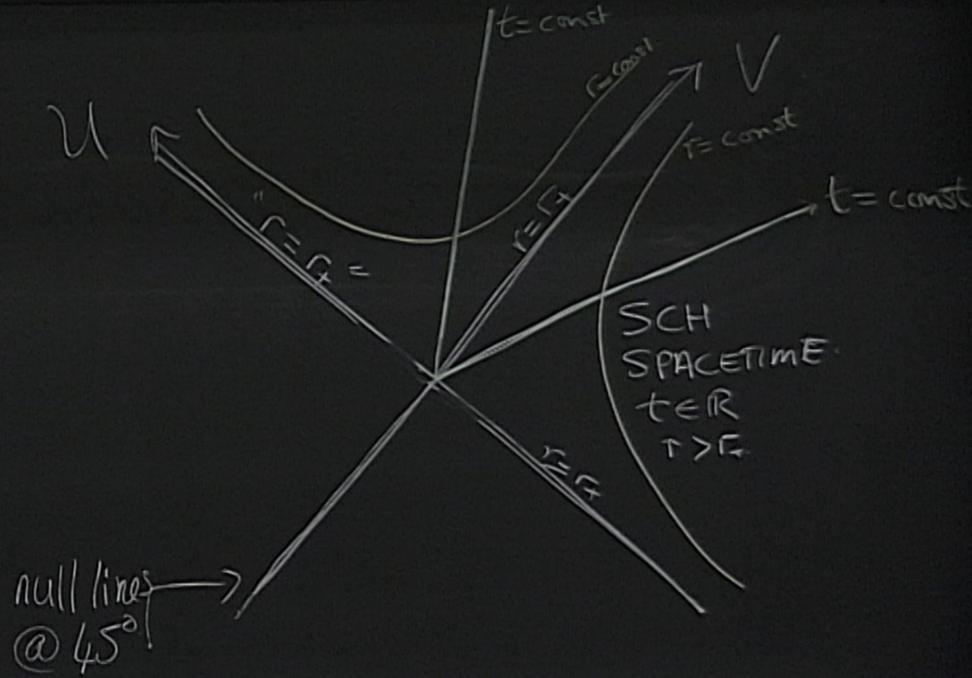

Metric now regular at r_+

$$r = \text{const} \longleftrightarrow UV = \text{const}$$

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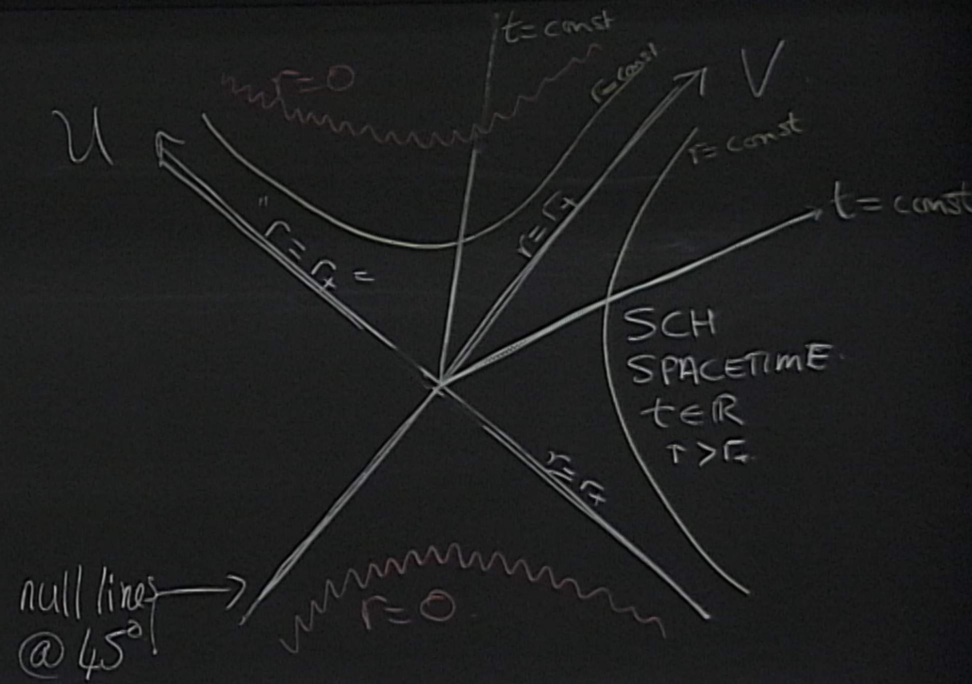

Metric now regular at r_+

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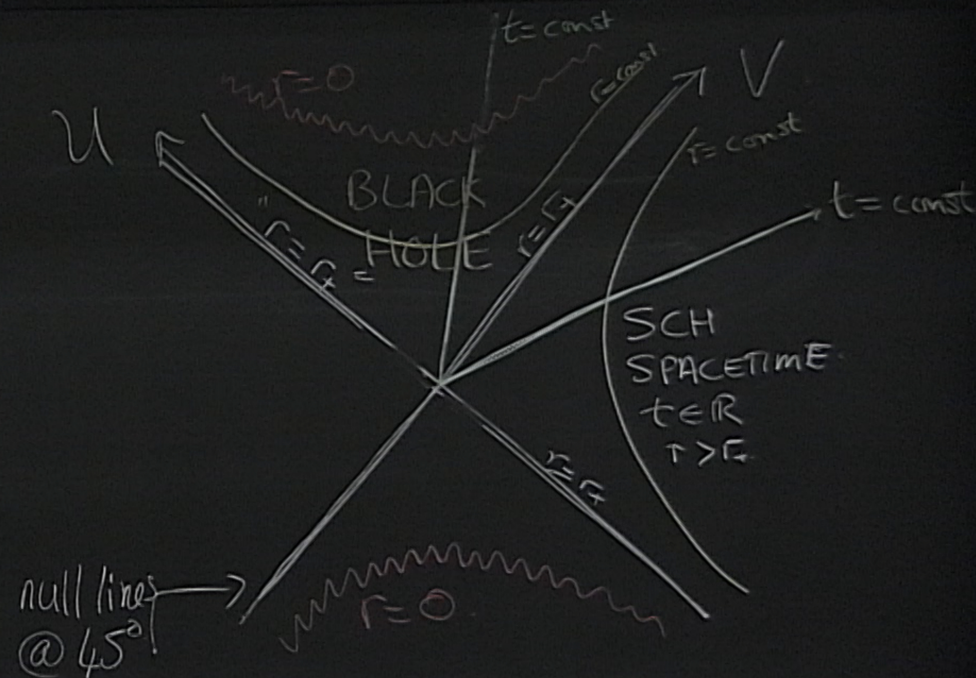

Metric now regular at r_+

$r = \text{const} \longleftrightarrow UV = \text{const}$

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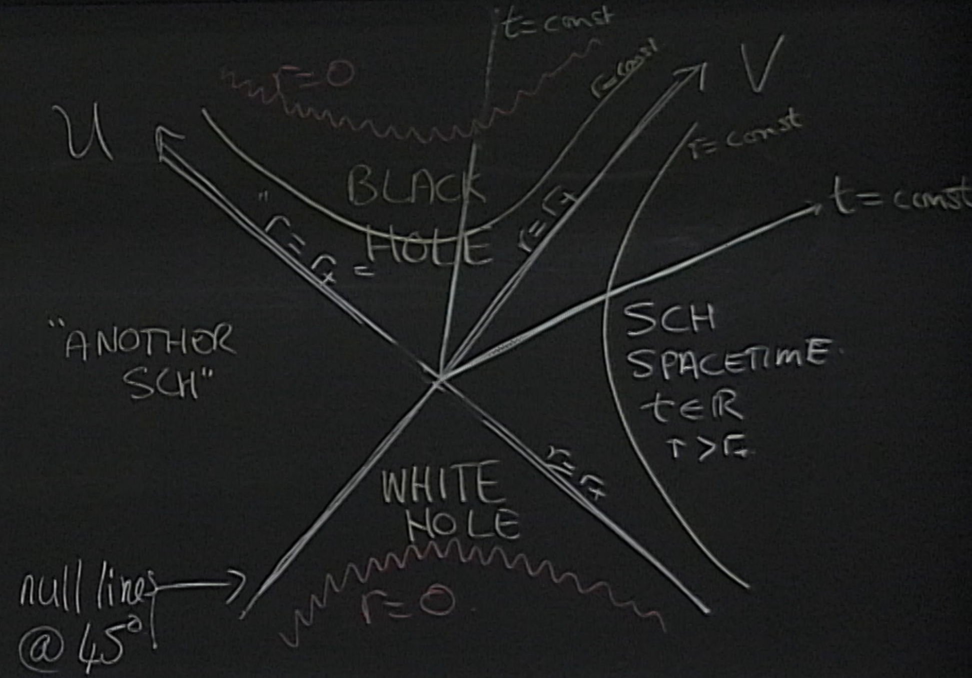
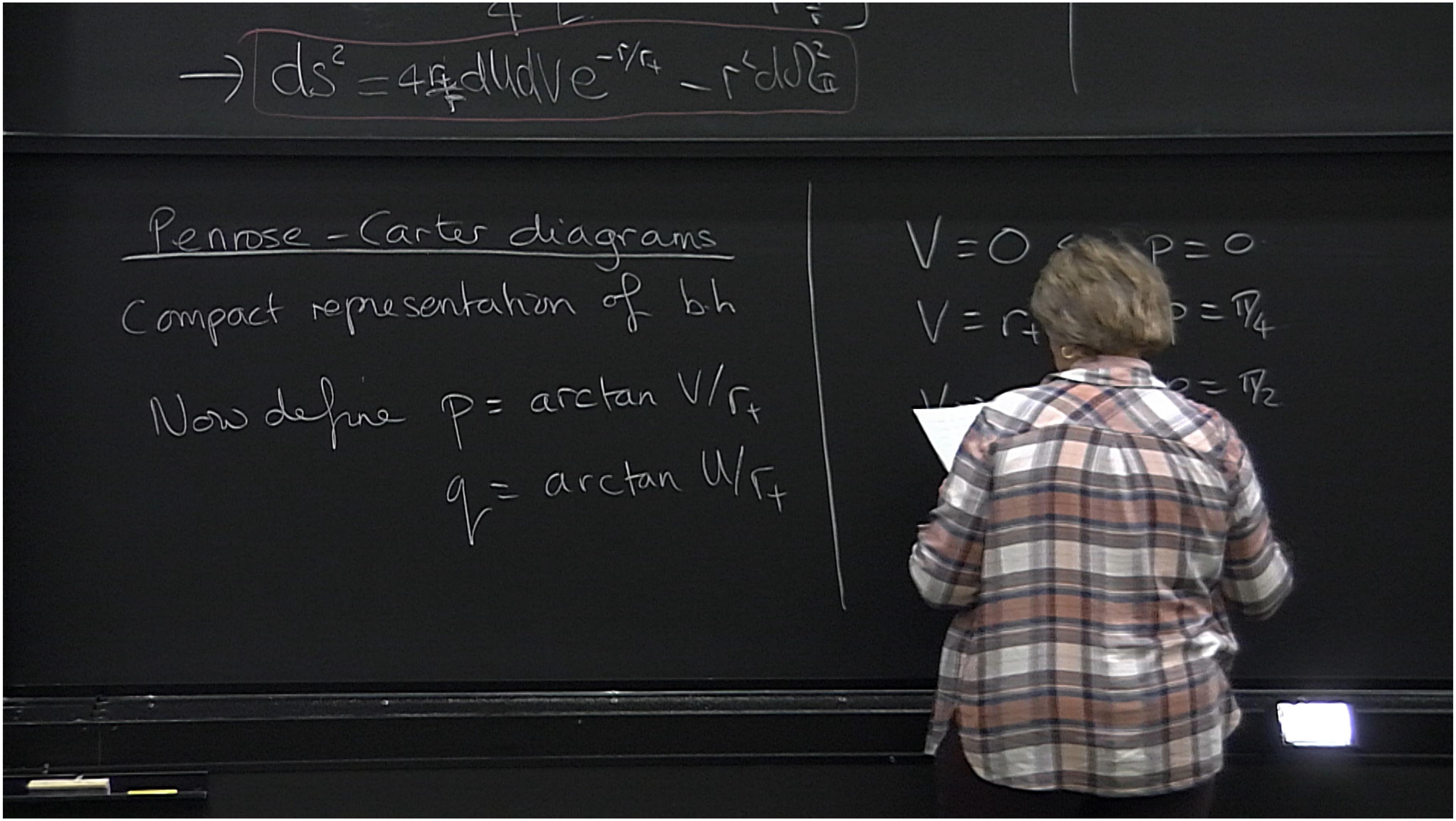
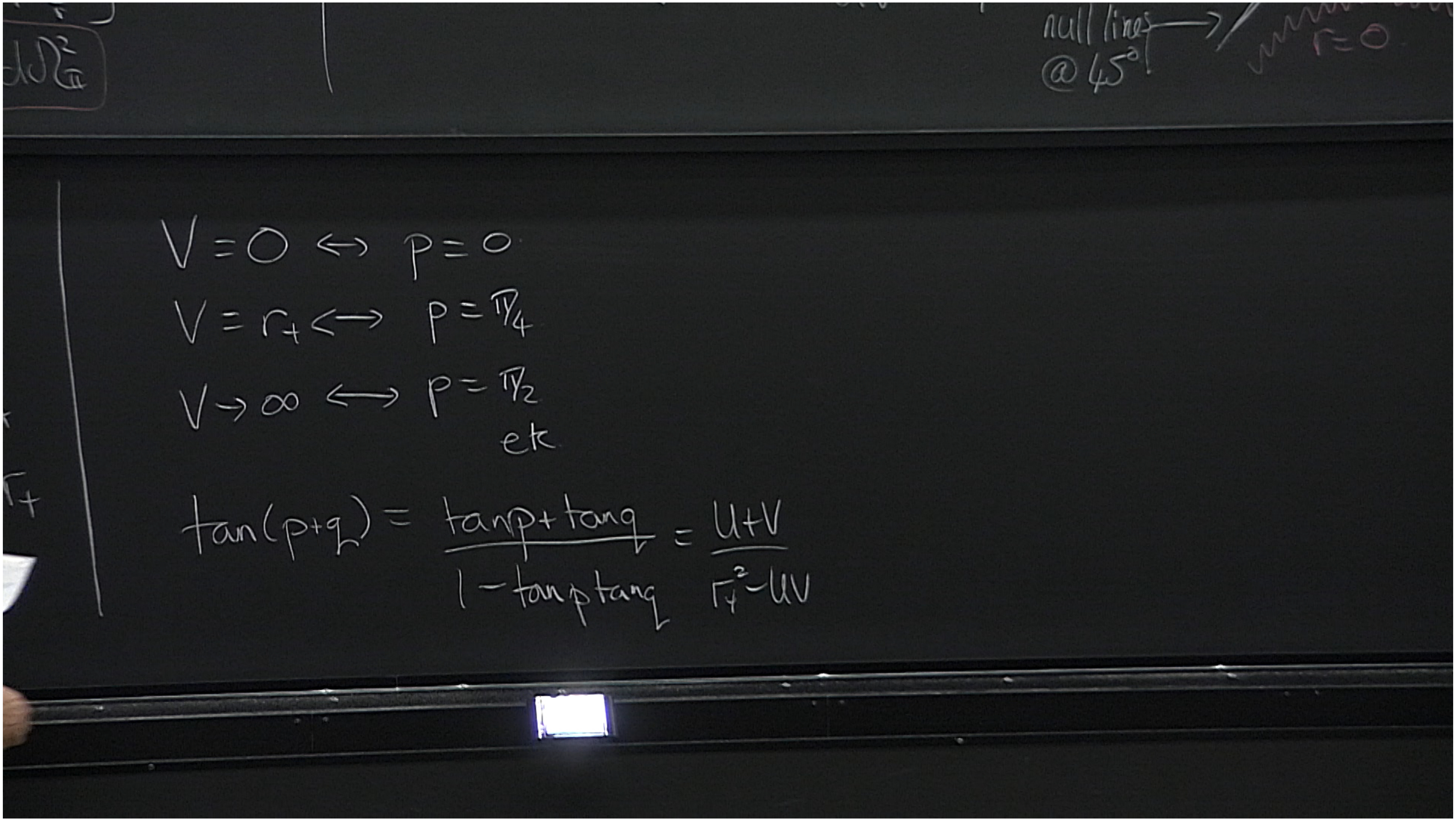
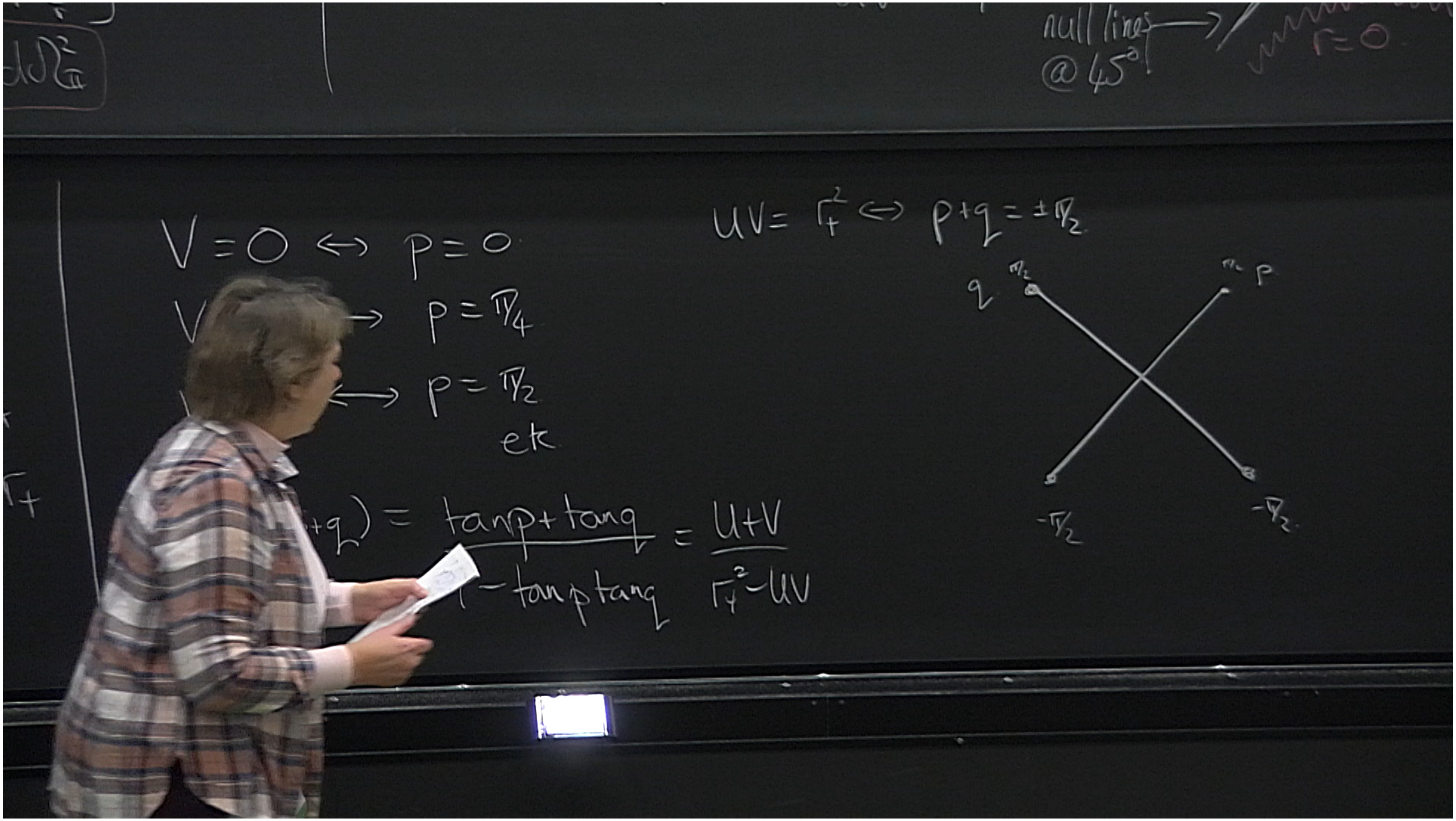
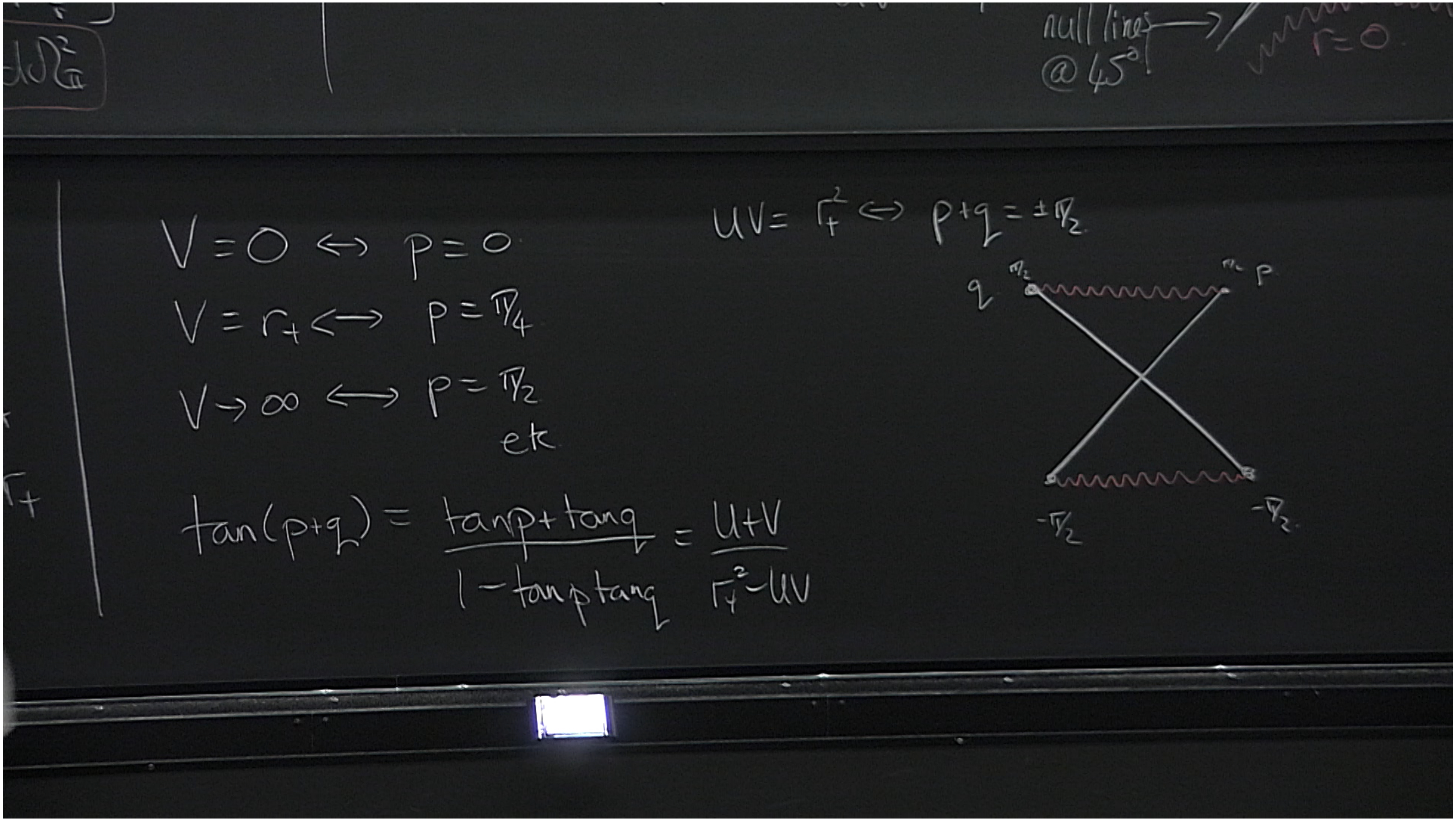
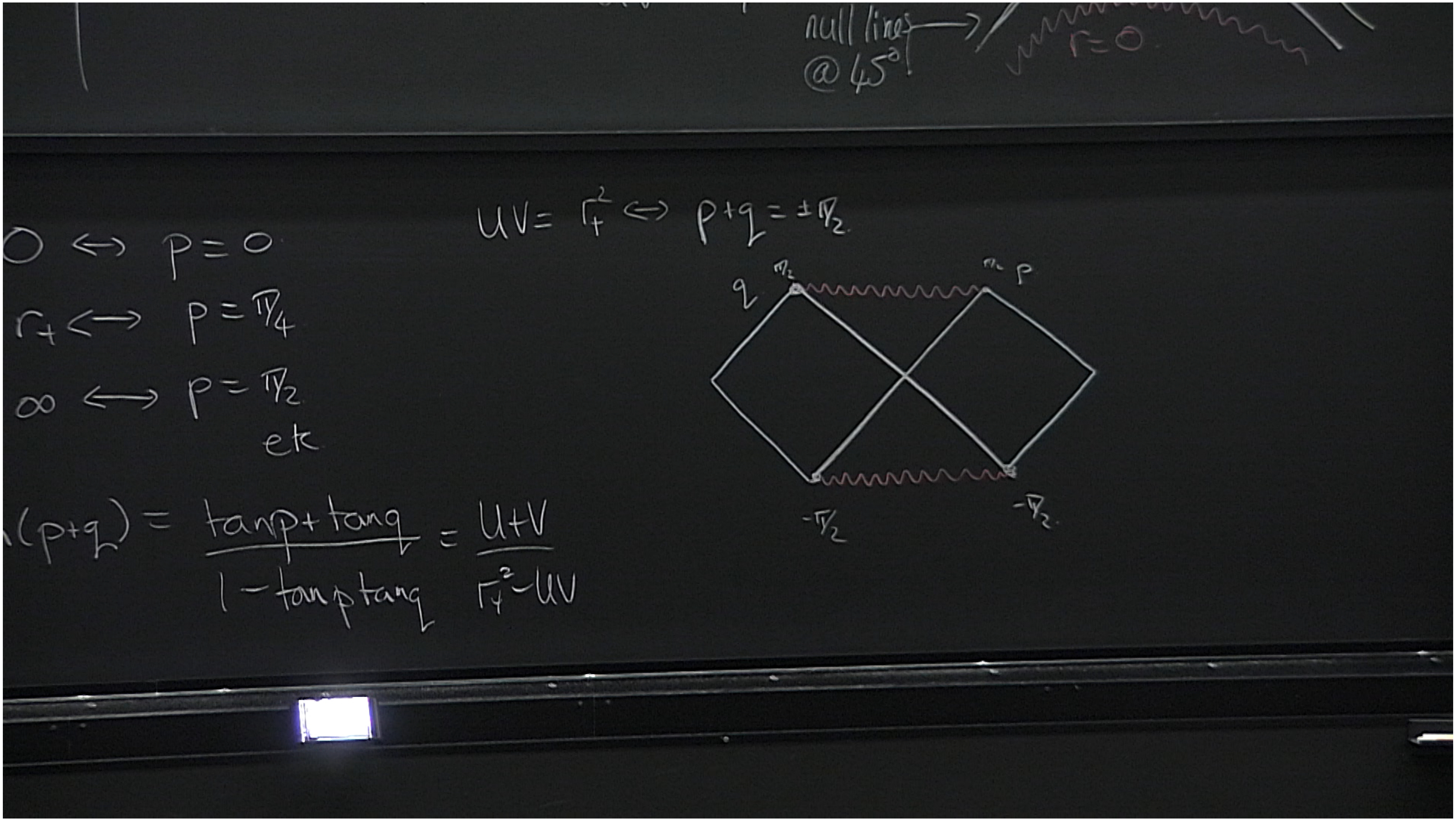


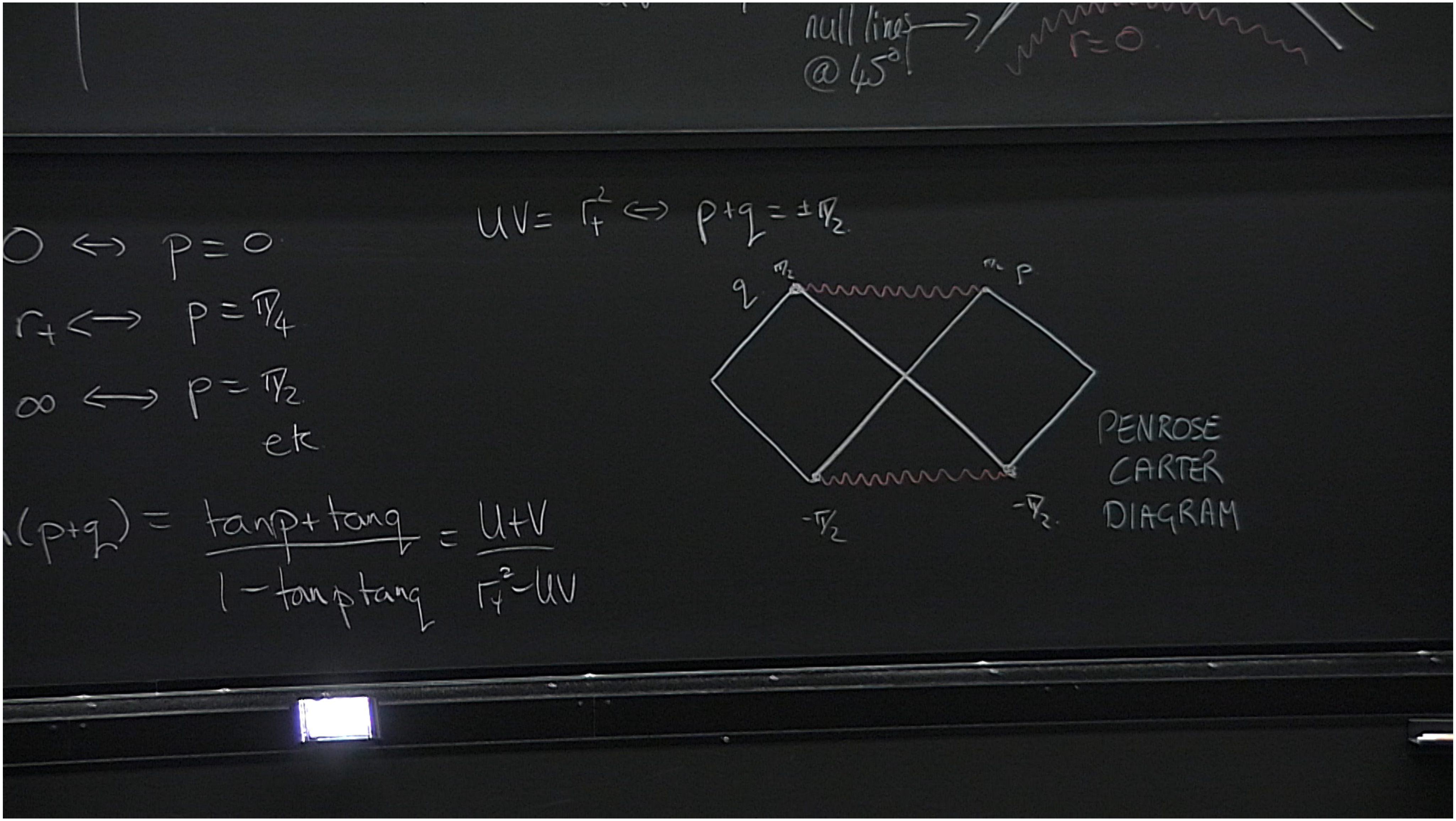

→ ds² = 4r₊ dU dV e^(−r/r₊) − r² dΩ²_II
Penrose - Carter diagrams
Compact representation of b.h
Now define p = arctan V/r₊
q = arctan U/r₊
V = 0
p = 0
V = r₊
= π/4
= π/2


null lines
@ 45°!
r = 0
V = 0 ⟷ p = 0
V = r+ ⟷ p = π/4
V → ∞ ⟷ p = π/2
etc
tan(p+q) = (tan p + tan q) / (1 − tan p tan q) = (U+V) / (r+² − UV)

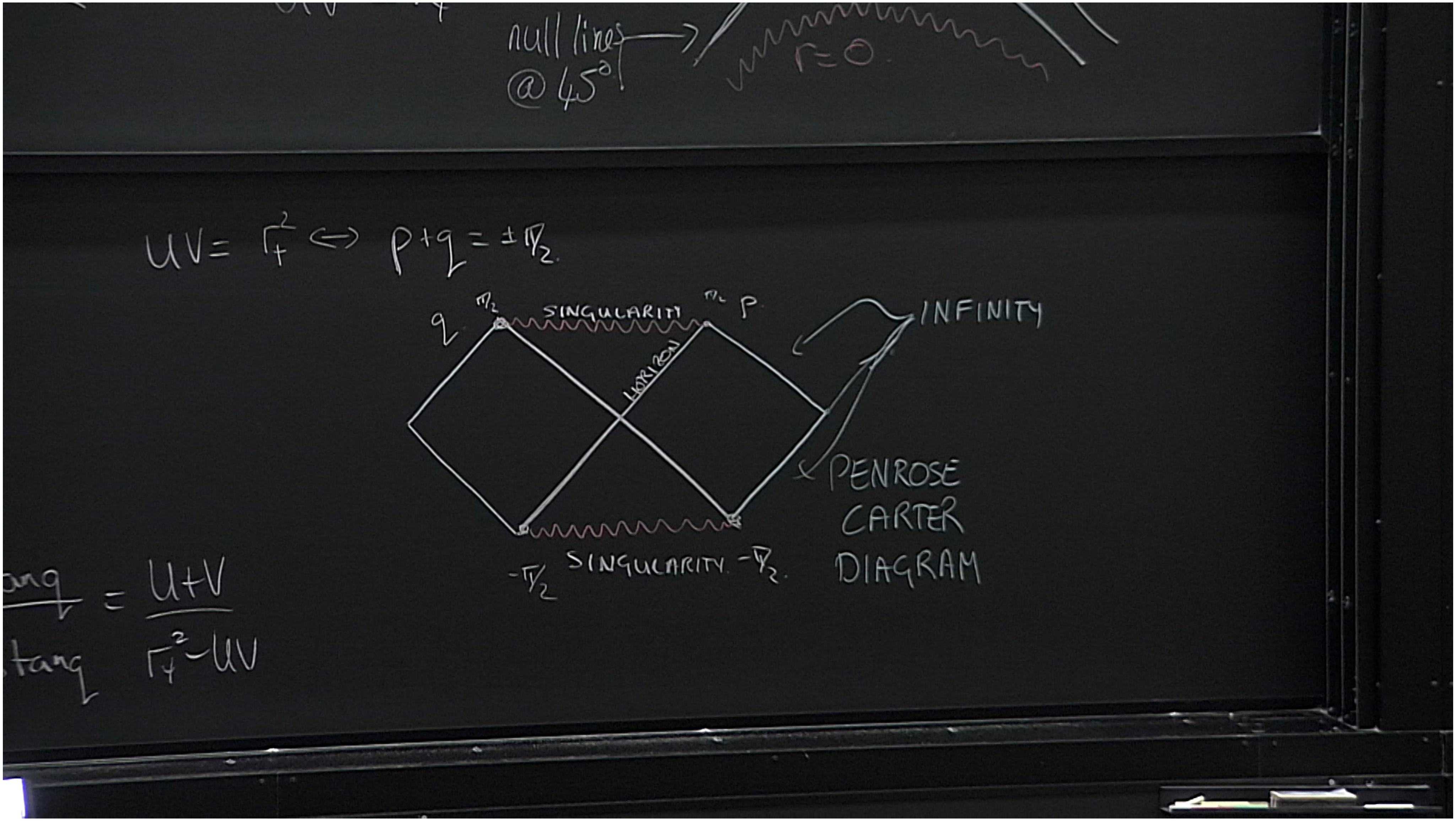

null lines
@ 45°!
r = 0
V = 0 ↔ p = 0
p = π/4
p = π/2
etc
UV = r_+^2 ↔ p+q = ±π/2
q
p
π/2
−π/2
tan p + tan q / 1 − tan p tan q = U+V / r_+^2 − UV

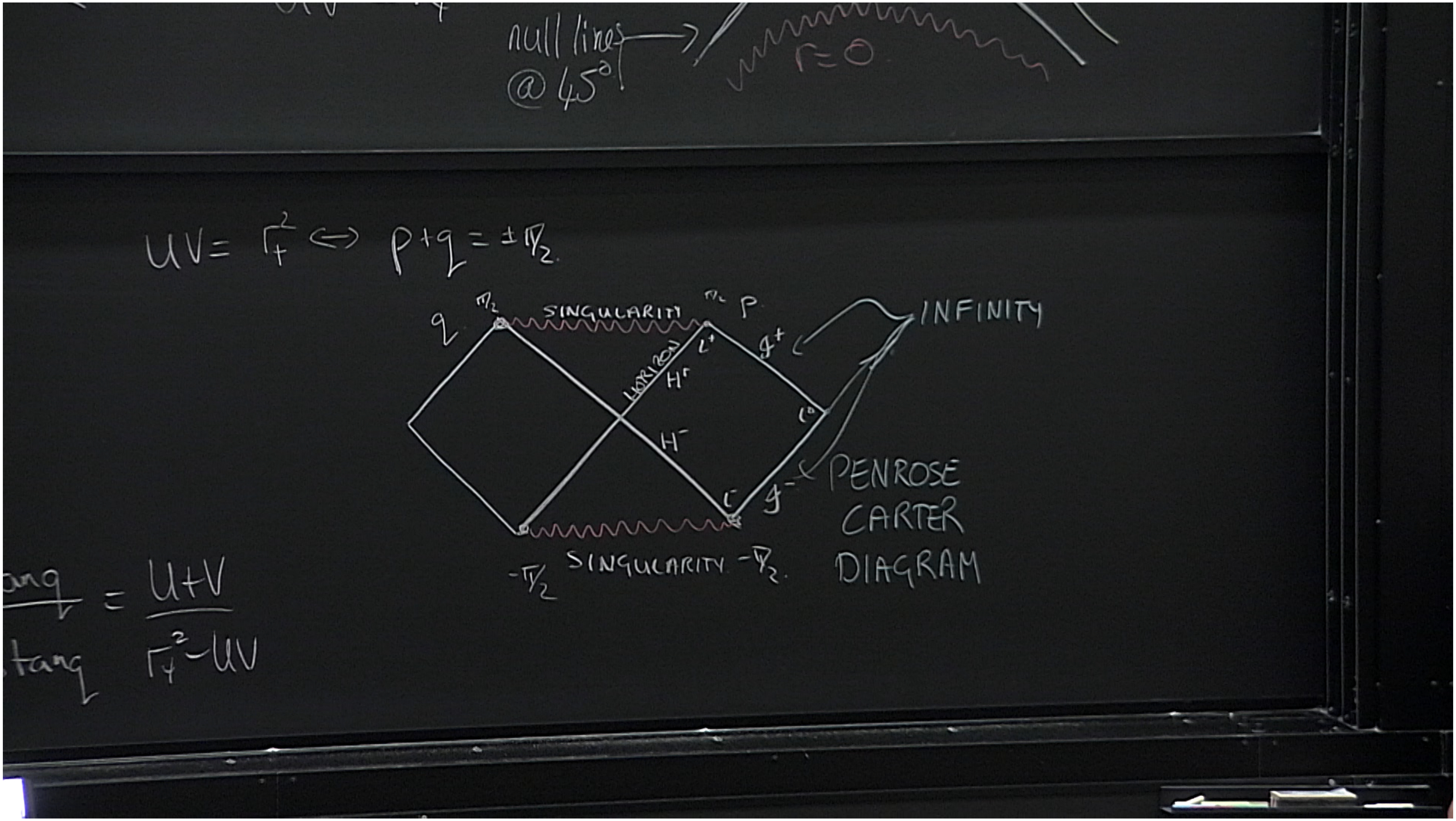

null lines
@ 45°!
$r=0$
$UV = r_+^2 \Leftrightarrow p+q = \pm \pi/2$
$V=0 \Leftrightarrow p=0$
$V=r_+ \Longleftrightarrow p=\pi/4$
$V\to\infty \Longleftrightarrow p=\pi/2$
etc
$\tan(p+q) = \dfrac{\tan p + \tan q}{1-\tan p \tan q} = \dfrac{U+V}{r_+^2 - UV}$
q
p
$\pi/2$
$-\pi/2$


null lines @ 45°!
r=0
UV = r_+^2 ⟺ p+q = ±π/2
0 ⟺ p = 0
r_+ ⟺ p = π/4
∞ ⟺ p = π/2
etc
(p+q) = (tan p + tan q)/(1 − tan p tan q) = (U+V)/(r_+^2 − UV)
q
π/2
p
−π/2



null lines @ 45°!
r=0
UV = r+^2 ⟺ p+q = ±π/2
0 ⟺ p = 0
r+ ⟺ p = π/4
∞ ⟺ p = π/2
etc
(p+q) = (tan p + tan q)/(1 − tan p tan q) = (U+V)/(r+^2 − UV)
q
π/2
π/2
p
−π/2
−π/2
PENROSE CARTER DIAGRAM


null lines @45°!
r=0
UV = r+² ⟺ p+q = ±π/2
q
π/2
SINGULARITY
π/2 p
INFINITY
HORIZON
PENROSE CARTER DIAGRAM
-π/2
SINGULARITY
-π/2
= U+V / r+² − UV


null lines
@45°!
r=0
UV= r+² ⟺ p+q = ±π/2
SINGULARITY
SINGULARITY
HORIZON
H⁺
H⁻
INFINITY
PENROSE
CARTER
DIAGRAM
U+V
r+²−UV

$H^{\pm}$ – event horizon

$\pm$ / future/past.

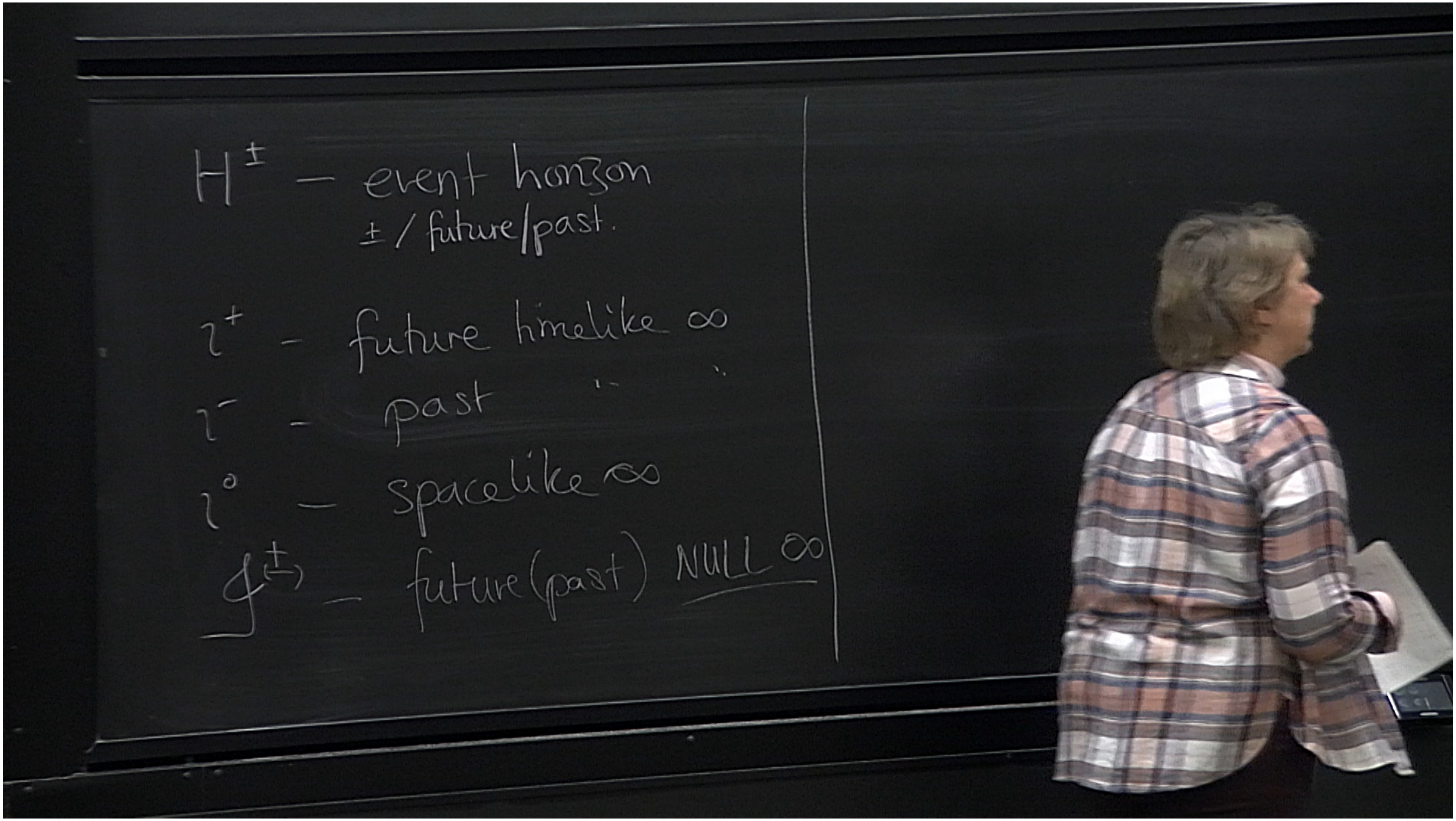
$H^{\pm}$ — event horizon

$\pm$ / future/past.

i^{+} — future timelike ∞

i^{-} — past " "

i^{0} — spacelike ∞


$H^{\pm}$ — event horizon
± / future/past.
i^{+} — future timelike ∞
i^{-} — past " "
i^{0} — spacelike ∞
$\mathscr{J}^{\pm}$ — future (past) NULL ∞

Event horizon is boundary
of the causal past of
future timelike infinity

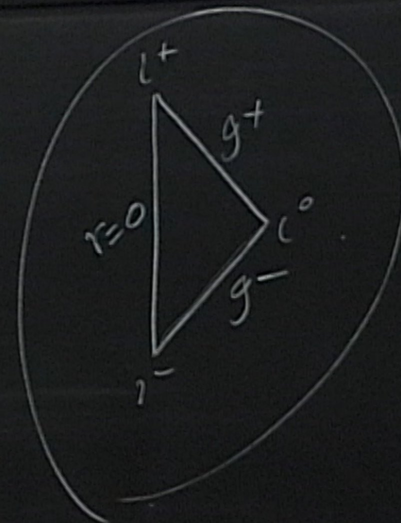
∞

NULL ∞

∞

NULL ∞

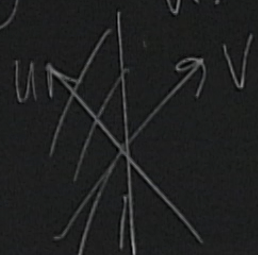
Event horizon is boundary of the causal past of future timelike infinity



MINKOWSKI $\quad dt^2 - dr^2 - r^2 d\Omega^2$

$u = t - r \qquad V = t + r$

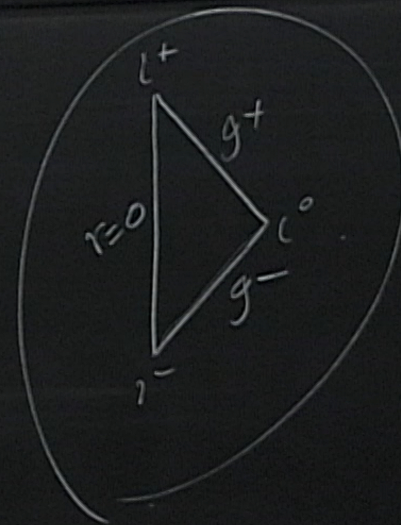
$r > 0 \qquad V > u$



∞

NULL ∞

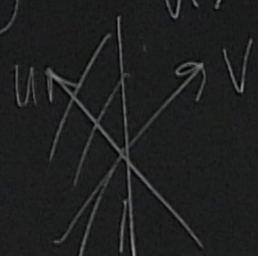
Event horizon is boundary of the causal past of future timelike infinity.



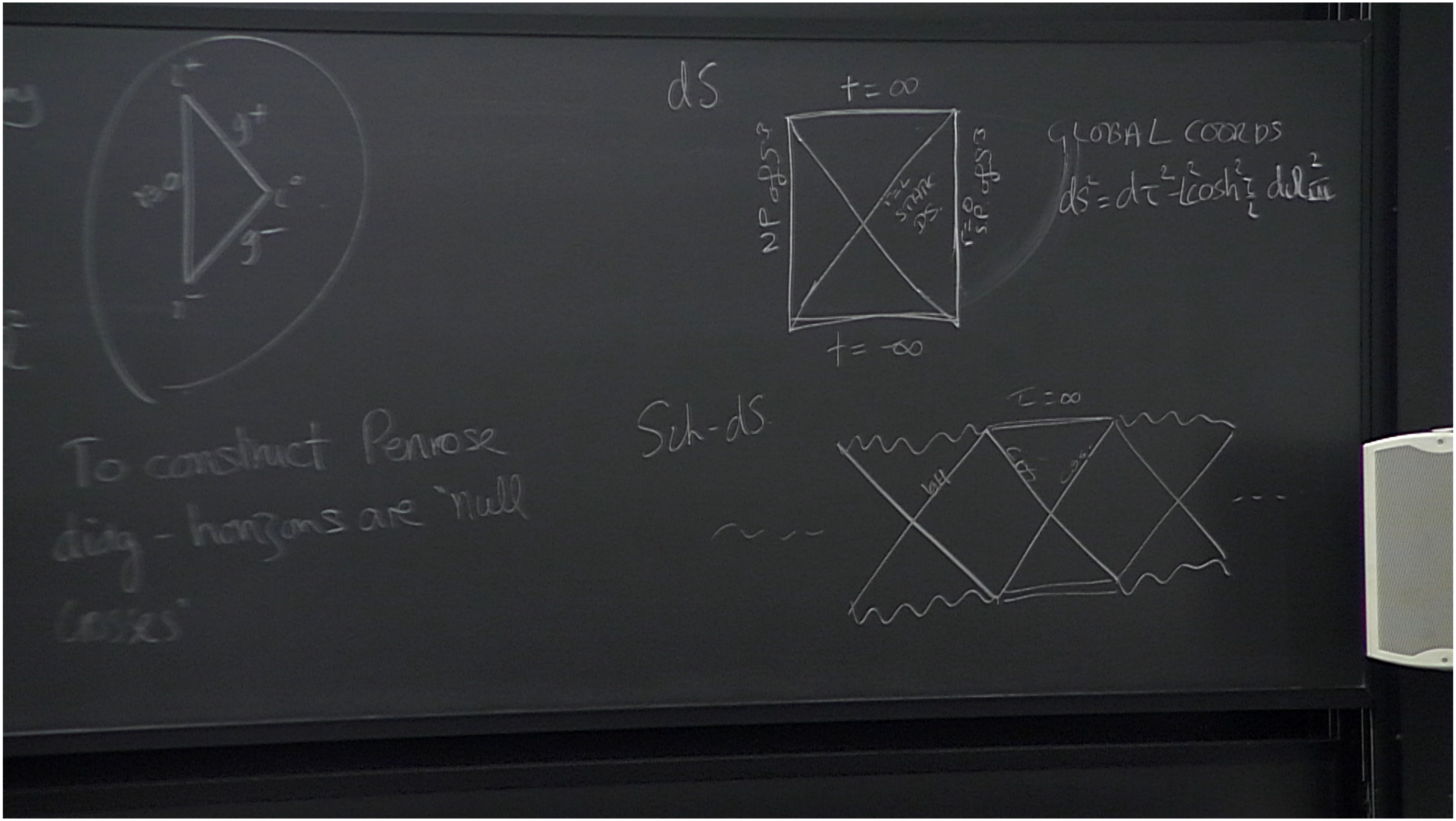
MINKOWSKI $dt^2 - dr^2 - r^2 d\Omega^2$

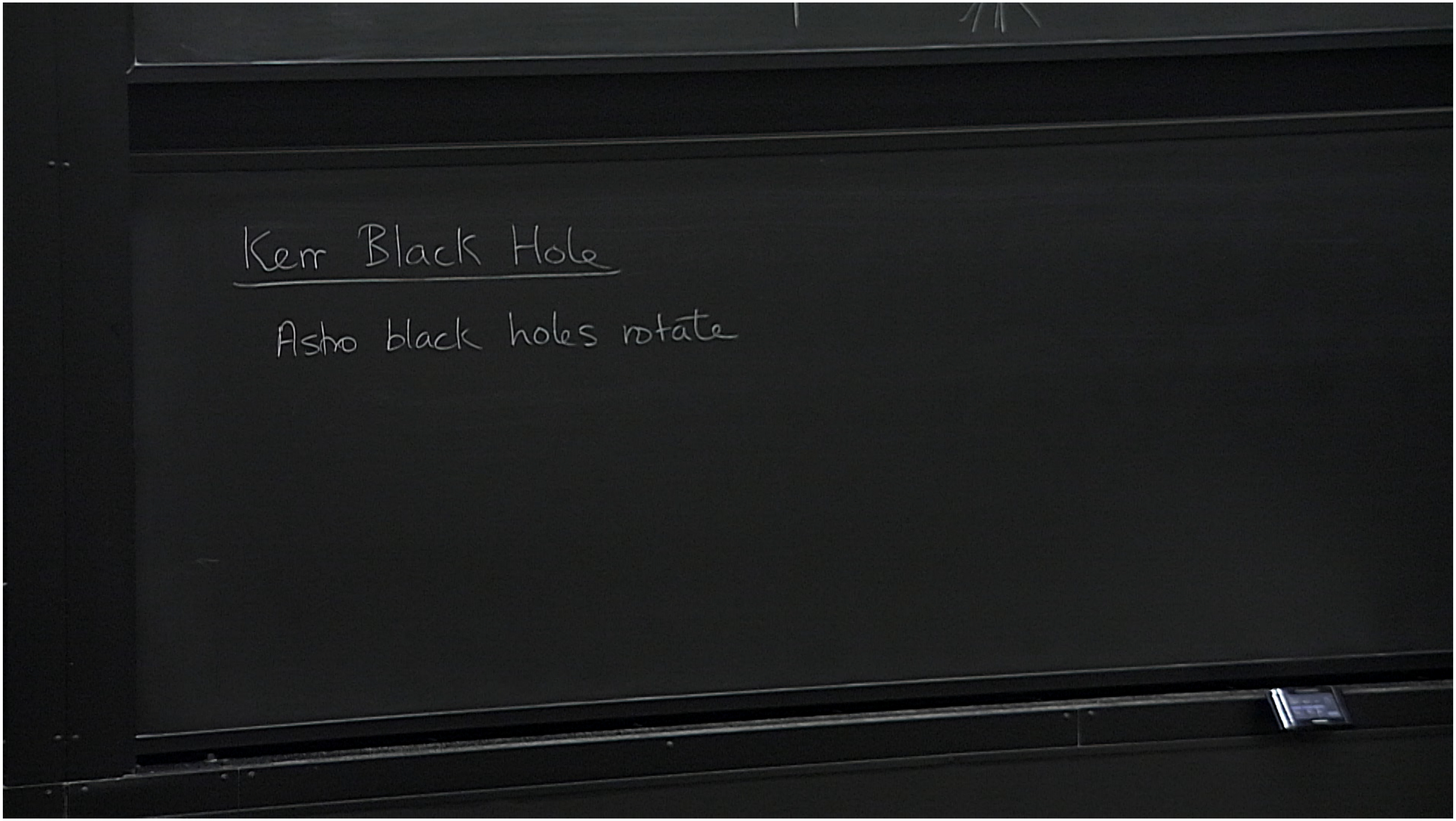
$U = t - r \qquad V = t + r$

$r > 0 \qquad V > U$



To construct Penrose diag - horizons are "null crosses"


dS
t = ∞
NP of S³
SP of S³
t = −∞
GLOBAL COORDS
ds² = dτ² − l² cosh² τ/l dΩ²
Sch-dS.
τ = ∞
To construct Penrose
diag - horizons are "null


Kerr Black Hole
Astro black holes rotate

$\mathscr{I}^{\pm}$ — future (past) NULL ∞

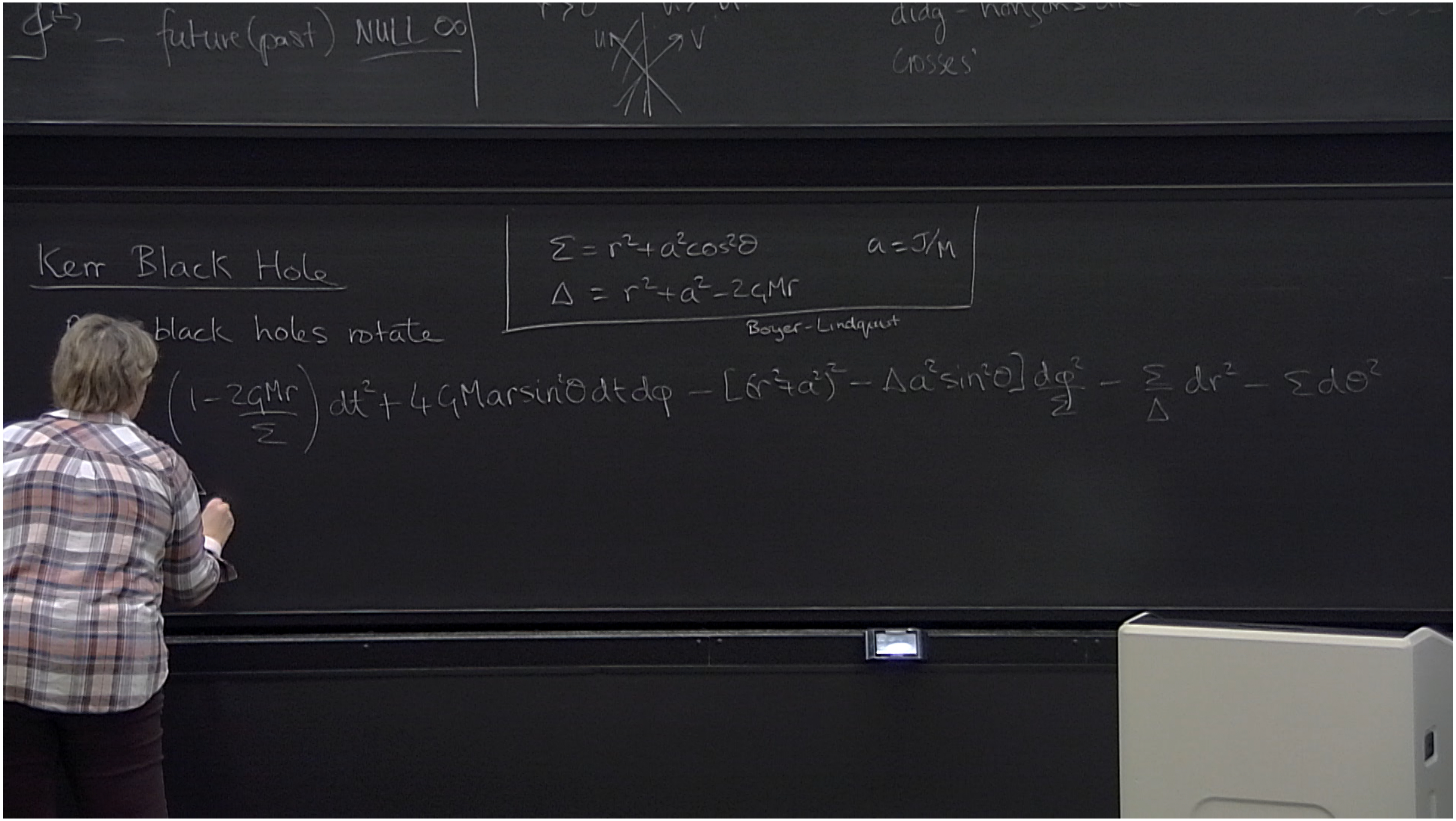
Crosses'

Kerr Black Hole

$$\Sigma = r^2 + a^2\cos^2\theta \qquad a = J/M$$
$$\Delta = r^2 + a^2 - 2GMr$$

Astro black holes rotate

$$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right)dt^2 + 4GMar\sin^2\theta\, dt\, d\varphi - \left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right]\frac{d\varphi^2}{\Sigma} - \frac{\Sigma}{\Delta}dr^2 - \Sigma\, d\theta^2$$



future (past) NULL ∞
u
v
crosses'
Kerr Black Hole
black holes rotate
$\Sigma = r^2 + a^2\cos^2\theta$
$a = J/M$
$\Delta = r^2 + a^2 - 2GMr$
Boyer-Lindquist
$\left(1 - \frac{2GMr}{\Sigma}\right) dt^2 + 4GMar\sin^2\theta\, dt\, d\varphi - [(r^2+a^2)^2 - \Delta a^2 \sin^2\theta] \frac{d\varphi^2}{\Sigma} - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2$

Kerr Black Hole

$$\Sigma = r^2 + a^2\cos^2\theta \qquad a = J/M$$

$$\Delta = r^2 + a^2 - 2GMr$$

Boyer-Lindquist

Also black holes rotate

$$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right)dt^2 + 4GMar\sin^2\theta\, dt\, d\varphi - \left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right]\frac{d\varphi^2}{\Sigma} - \frac{\Sigma}{\Delta}dr^2$$

$$= \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\, d\varphi\right]^2 - \frac{\sin^2\theta}{\Sigma}\left[a\,dt - (r^2+a^2)\,d\varphi\right]^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma\, d\theta^2.$$

$$\Delta = 0 \iff r = r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$$

$$r_+ < 2GM$$

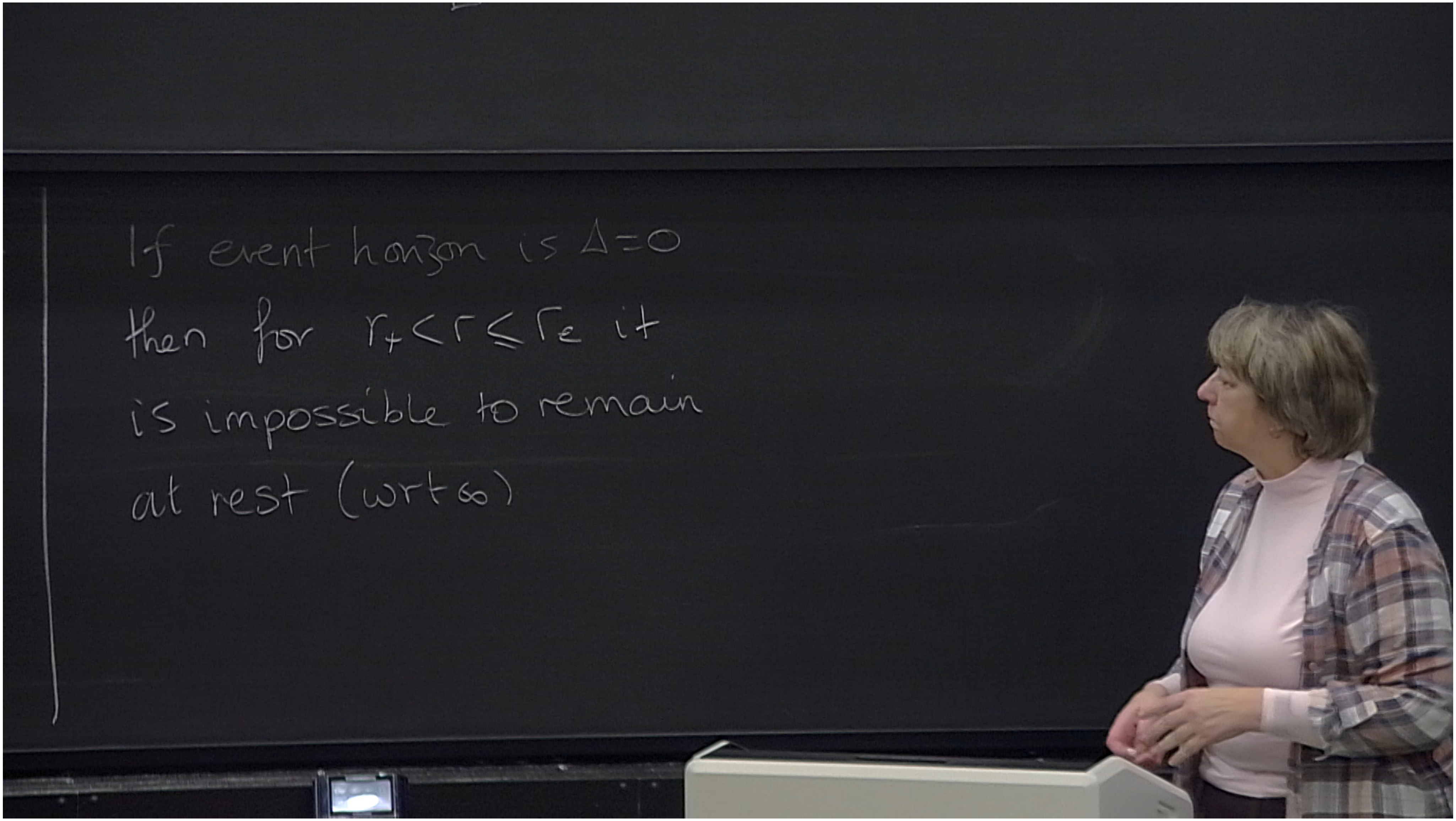
$$\Delta = 0 \longleftrightarrow r = r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$$

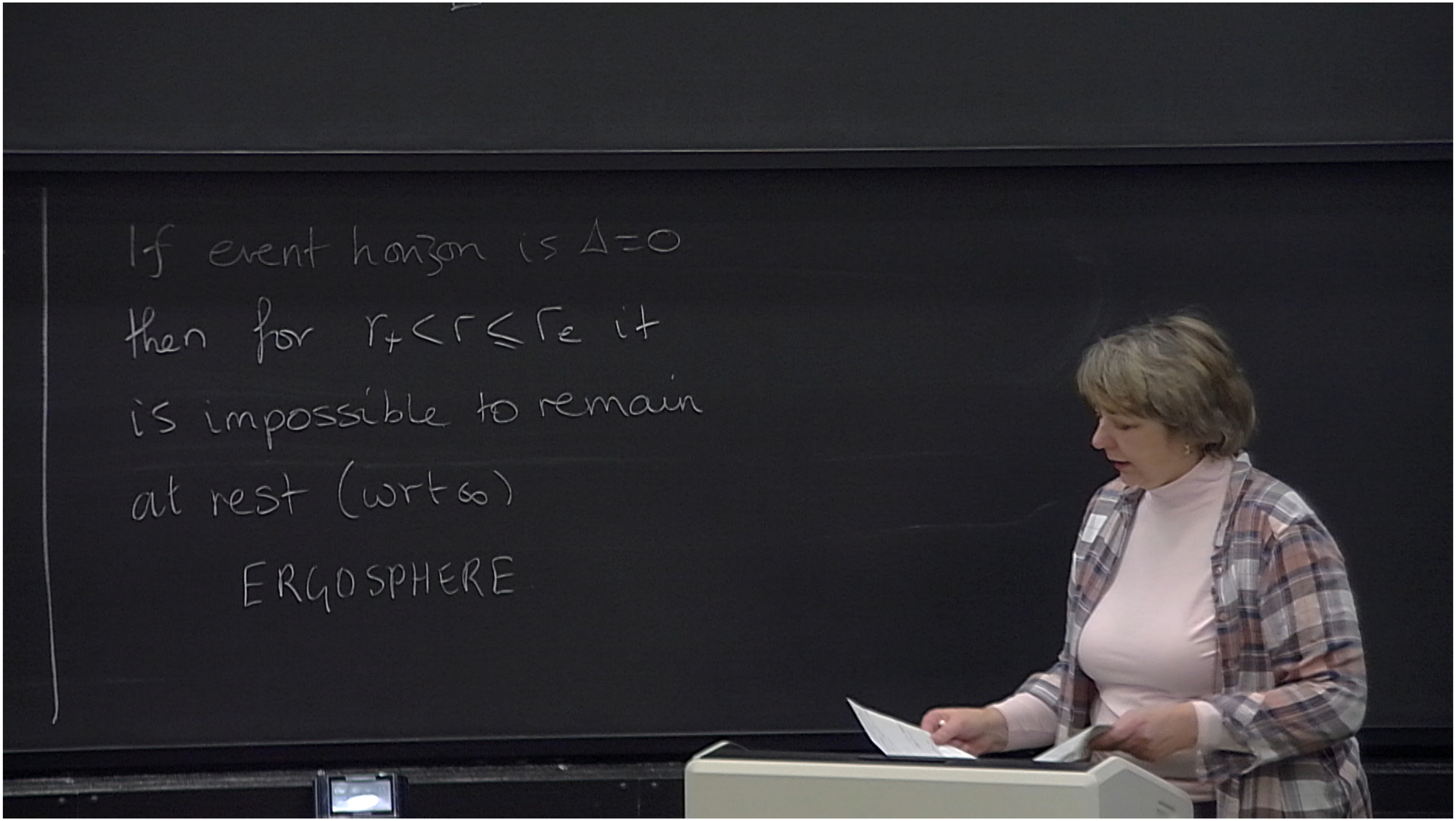
$$r_+ < 2GM$$

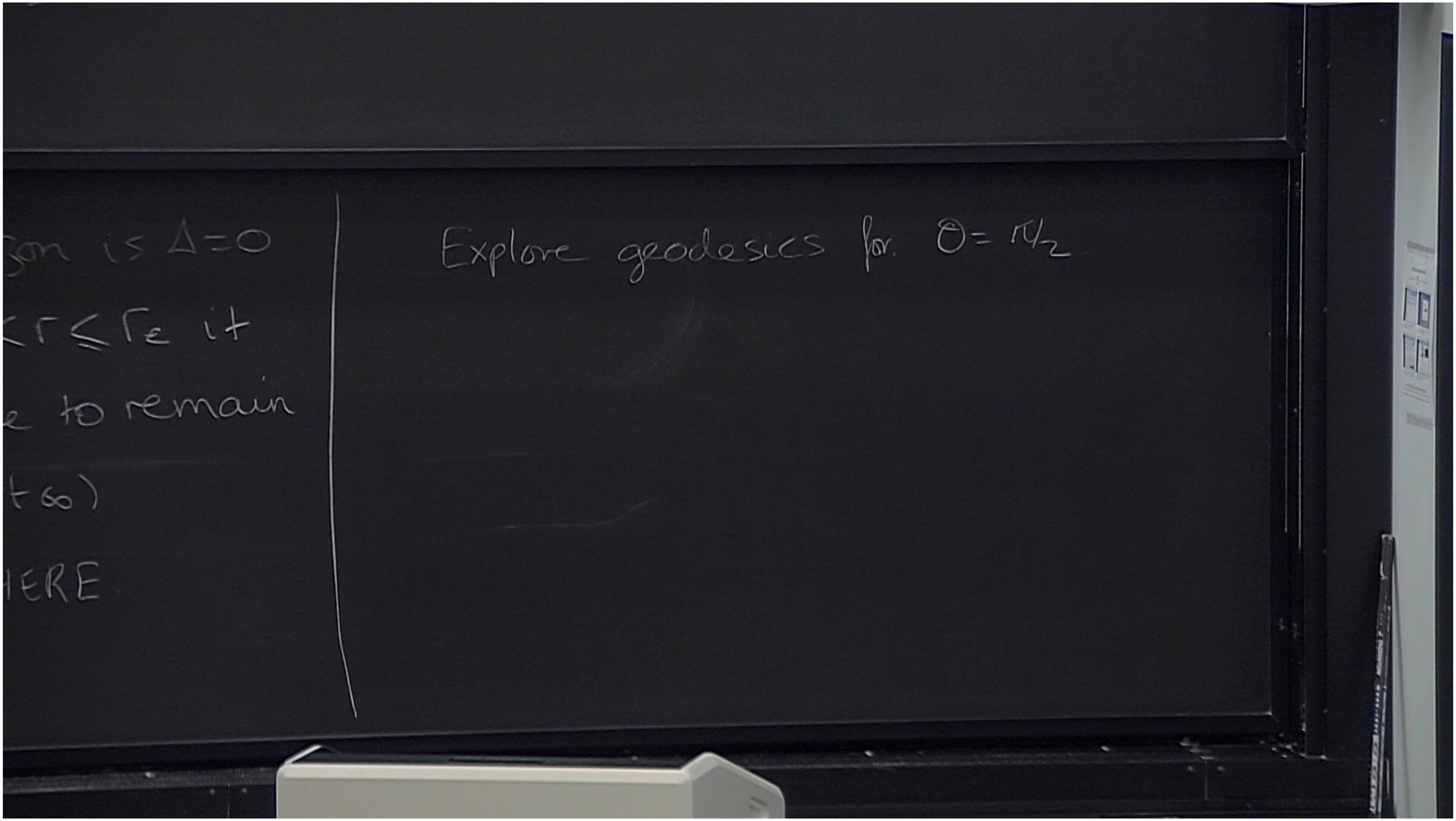
$$\left\| \frac{\partial}{\partial t} \right\|^2 \qquad r^2 + a^2\cos^2\theta - 2GMr = 0$$

$$r_e = GM \pm \sqrt{G^2M^2 - a^2\cos^2\theta}$$

$$\geqslant r_+$$


If event horizon is $\Delta = 0$
then for $r_+ < r \leq r_e$ it
is impossible to remain
at rest (wrt ∞)


If event horizon is $\Delta = 0$
then for $r_+ < r \leq r_E$ it
is impossible to remain
at rest (wrt ∞)
ERGOSPHERE


Explore geodesics for θ = π/2
is Δ = 0
r ≤ r_ε it
to remain
HERE

$$= \frac{e^{-r/r_+}}{4}\left[\left(1-\frac{r_+}{r}\right)dt^2 - \frac{dr^2}{1-\frac{r_+}{r}}\right]$$

$$\rightarrow \boxed{ds^2 = 4r_+\, dU\, dV\, e^{-r/r_+} - r^2 d\Omega_{II}^2}$$

$r = 0 \leftarrow$

Note, if k^μ is a Killing vector

$$\frac{d}{dt}\left[k_\mu \dot{X}^\mu\right] = \dot{X}^\nu \nabla_\nu \left(k_\mu \dot{X}^\mu\right)$$

$$\Downarrow \qquad = \cancel{\dot{X}^\nu \dot{X}^\mu \nabla_\nu k_\mu} + \dot{X}^\nu \cancel{k_\mu \nabla_\nu \dot{X}^\mu}$$

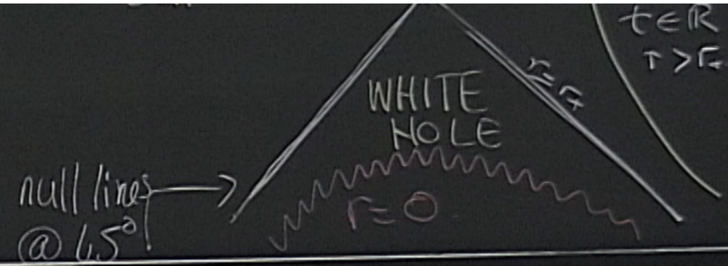
Conserved

Kerr has 2 Killing vectors $\partial_t \quad \partial_\varphi$.

$k_t = \frac{\partial}{\partial t}$

$$r = r_+ \longleftrightarrow UV = 0$$

$$r = 0 \longleftrightarrow UV = r_+^2$$



$$\nabla_\nu \dot{X}^\mu$$

$$\partial_\varphi$$

- $k_t = \dfrac{\partial}{\partial t} \longleftrightarrow E = g_{t\mu}\dot{X}^\mu$

$$= \left(1-\frac{2GM}{r}\right)\dot{t} + \frac{2GMa}{r}\dot{\varphi} \quad \text{at } \theta = \pi/2 \quad (\text{EX})$$

- $k_\varphi = \dfrac{\partial}{\partial \varphi} \longleftrightarrow h = -g_{\varphi\mu}\dot{X}^\mu$

$$= \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)\dot{\varphi} - \frac{2GMa}{r}\dot{t} \quad \text{at } \theta = \pi/2 \quad (\text{EX})$$

10) Wick
11) Fermionic rules
12) Yukawa diags

• $k_t = \frac{\partial}{\partial t} \leftrightarrow \quad E = g_{t\mu}\dot{X}^\mu$

$$= \left(1-\frac{2GM}{r}\right)\dot{t} + \frac{2GMa}{r}\dot{\varphi} \quad \text{at } \theta = \pi/2 \quad (\text{Ex})$$

• $k_\varphi = \frac{\partial}{\partial \varphi} \leftrightarrow \quad h = -g_{\phi\mu}\dot{X}^\mu$

$$= \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)\dot{\varphi} - \frac{2GMa}{r}\dot{t} \quad \text{at } \theta = \pi/2 \quad (\text{Ex})$$

Now define $P_\alpha = (E, -h) = g_{\alpha\beta}\dot{X}^\beta \quad (\alpha, \beta = t, \varphi)$

then $P_\alpha P_\beta g^{\alpha\beta} = g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta$

$$= E^2 g^{tt} - 2Eh g^{t\varphi} + h^2 g^{\varphi\varphi}$$

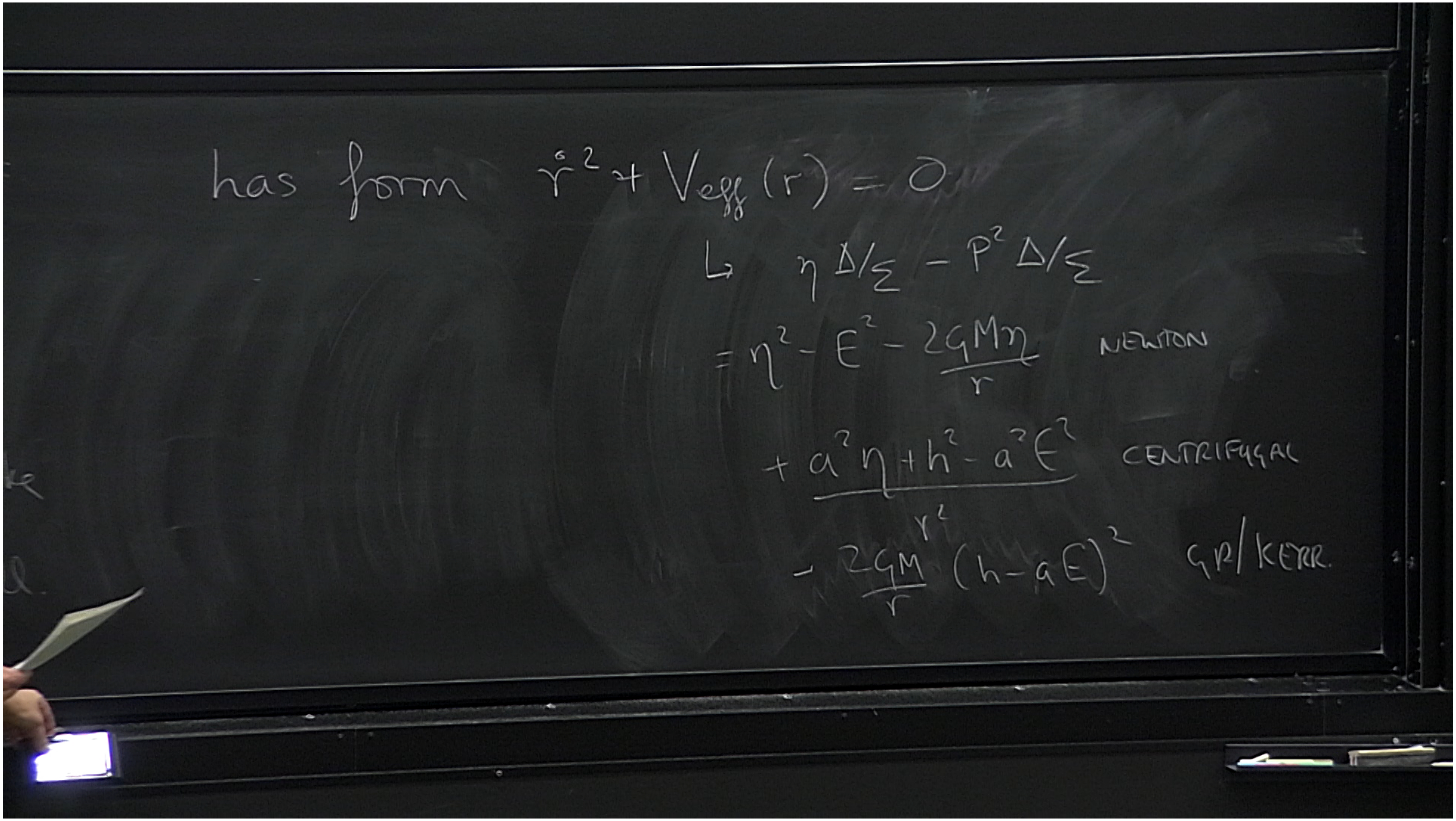
$$\longrightarrow \frac{1}{\Delta}\left[E^2(r^2+a^2) + \frac{2GM}{r}(h-aE)^2 - h^2 \right]$$

then $P_\alpha P_\beta g^{\alpha\beta} = g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta$

$$= E^2 g^{tt} - 2Eh g^{t\varphi} + h^2 g^{\varphi\varphi}$$

$$\longrightarrow \frac{1}{\Delta}\left[E^2(r^2+a^2) + \frac{2GM}{r}(h-aE)^2 - h^2 \right]$$

Geodesic eqn $\quad P^2 + g_{rr}\dot{r}^2 = \eta \begin{cases} 1 & \text{timelike} \\ 0 & \text{null.} \end{cases}$


has form $\dot{r}^2 + V_{eff}(r) = 0$
$\hookrightarrow \eta \Delta/\Sigma - P^2 \Delta/\Sigma$
$= \eta^2 - E^2 - \frac{2GM\eta}{r}$
NEWTON
$+ \frac{a^2\eta + h^2 - a^2 E^2}{r^2}$
CENTRIFUGAL
$- \frac{2GM}{r}(h - aE)^2$
GR/KERR.

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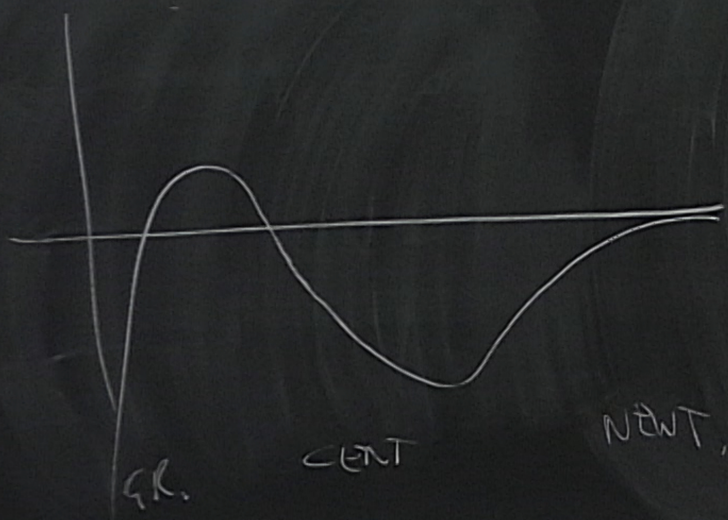

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