Title: Boundary Chiral Algebras for 3d N=4 Theories

Date: Jan 21, 2019 02:00 PM

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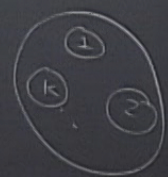
Abstract: <p>I will describe some general mathematical structures expected to arise from field theories with boundary conditions in terms of factorization algebras, and outline some results and future directions in the study of boundary chiral algebras for 3d N=4 theories following the work of Costello and Gaiotto.</p>

TQFTs = loc const fact algs

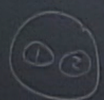
$\underline{\text{Th}^{m}}$ [Lurie] Trans invt " . . . .

$\simeq E_n$-algs

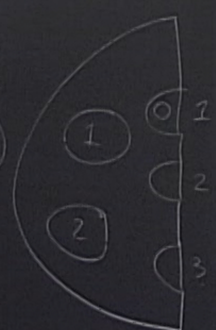
$\underline{\text{Def}^{c}}$: The E_n-op is



$$\mathcal{O}_n(k) = C_*(\mathrm{Conf}_k(D^n))$$

 $\rightsquigarrow \mu\ V^{\otimes 2} \longrightarrow V$

$\underline{\text{Def}^{c}}$: The $E_{n,n-1}$-op is



$$\mathcal{O}_n(k,l) = C_*(\mathrm{Conf}_{k,l}(H^n))$$

" " -algs are

- E_n-alg A
- E_{n-1} alg B
- $\rho: A \longrightarrow CH^*(B)$ of E_n-alg

$\underline{\text{Dunn-Additivity}}$:

$$E_{n+m} = E_n\text{-alg}(E_m\text{-alg})$$

$E_n = E_1$-obj in E_{n-1}-alg's, A

$B = E_0$ (module) obj for A, " "

X hol^{m}/alg^{c} curve, Fact_X, $\otimes$-cat

E_2-alg $\hookrightarrow \mathrm{Fact}_X$ monoidal cat

$\Rightarrow E_3 = E_1$ obj in $E_2 \overset{(*)}{\hookrightarrow} E_1$ obj's in Fact_X

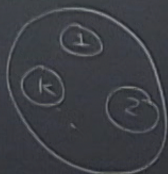
$\underline{\text{Def}^{c}}$ A boundary chiral FT for 3d TFT

a module obj for its image under $(*)$

st fact algs

nvt " . . ..

is

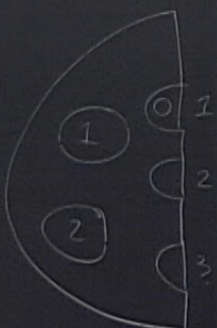


$)))$

$\to V \quad +$

is

$(H^*))$



B) of E_n-alg

Dunn-Additivity:

$E_{n+m} = E_n\text{-alg}(E_m\text{-alg})$

$E_n = E_1\text{-obj in } E_{n-1}\text{-alg's}, \ A$

$B = E_0 \text{ (module) obj for } A$, " "

X hol^c/alg^c curve, fact_X, $\otimes$-cat

$E_2\text{-alg} \hookrightarrow \mathrm{fact}_X$ monoidal emb

$\Rightarrow E_3 = E_1 \text{ obj in } E_2 \overset{(*)}{\hookrightarrow} E_1 \text{ obj's in } \mathrm{fact}_X$

Def^n A boundary chiral FT for $\underline{\text{3d TFT}}$

a module obj for its image under (*)

Hol^c/Alg^c FTs on X = fact algs$/X$

$X = \tilde{X}$ a smooth var or $\tilde{X}_{dR}$

Ran_X = "moduli space of (non-empty) finite subsets of X"

$= \mathrm{colim}\left[X \xrightarrow{\Delta} X^2 \rightrightarrows X^3 \cdots \right]$ (S_2, S_3)

$= \mathrm{colim}_{I \in \mathrm{fSet}^{op}_{surj}} X^I \in \mathrm{PreStk} = [\mathrm{AffSch}, \mathrm{Grpd}]$ (Cat)

$QC(\mathrm{Ran}_X) = \{ \mathrm{Ran}_X \Rightarrow QC(-) : \mathrm{AffSch} \to \mathrm{Cat} \}$

(Factoriz) $\cong V_{x_1} \otimes V_{x_2}$

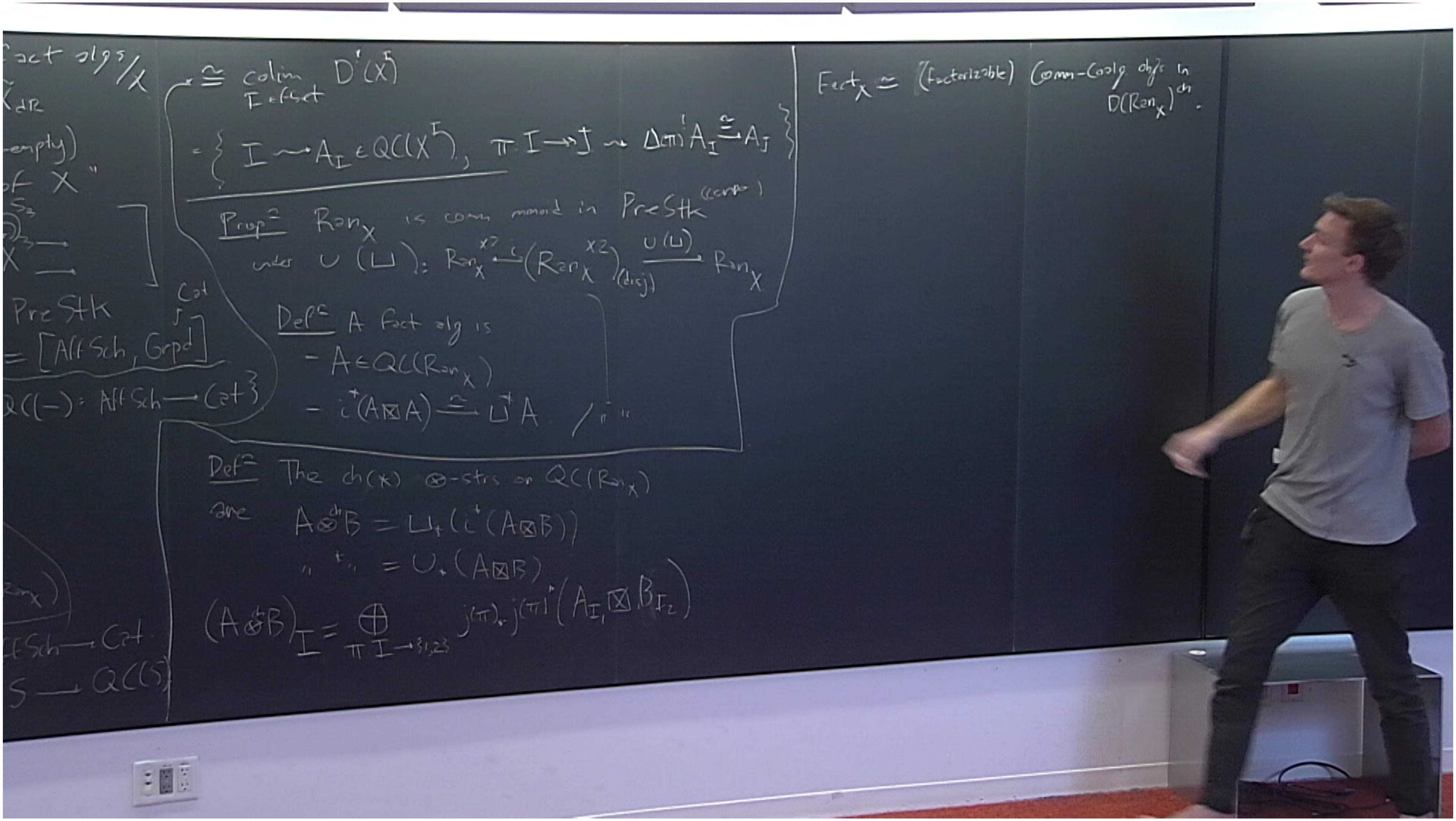
V_{x_1,x_2} = qcoh sheaf to $\{x_1,x_2\}$ $\xrightarrow{x_2 \to x_1}$ V_{x_1}

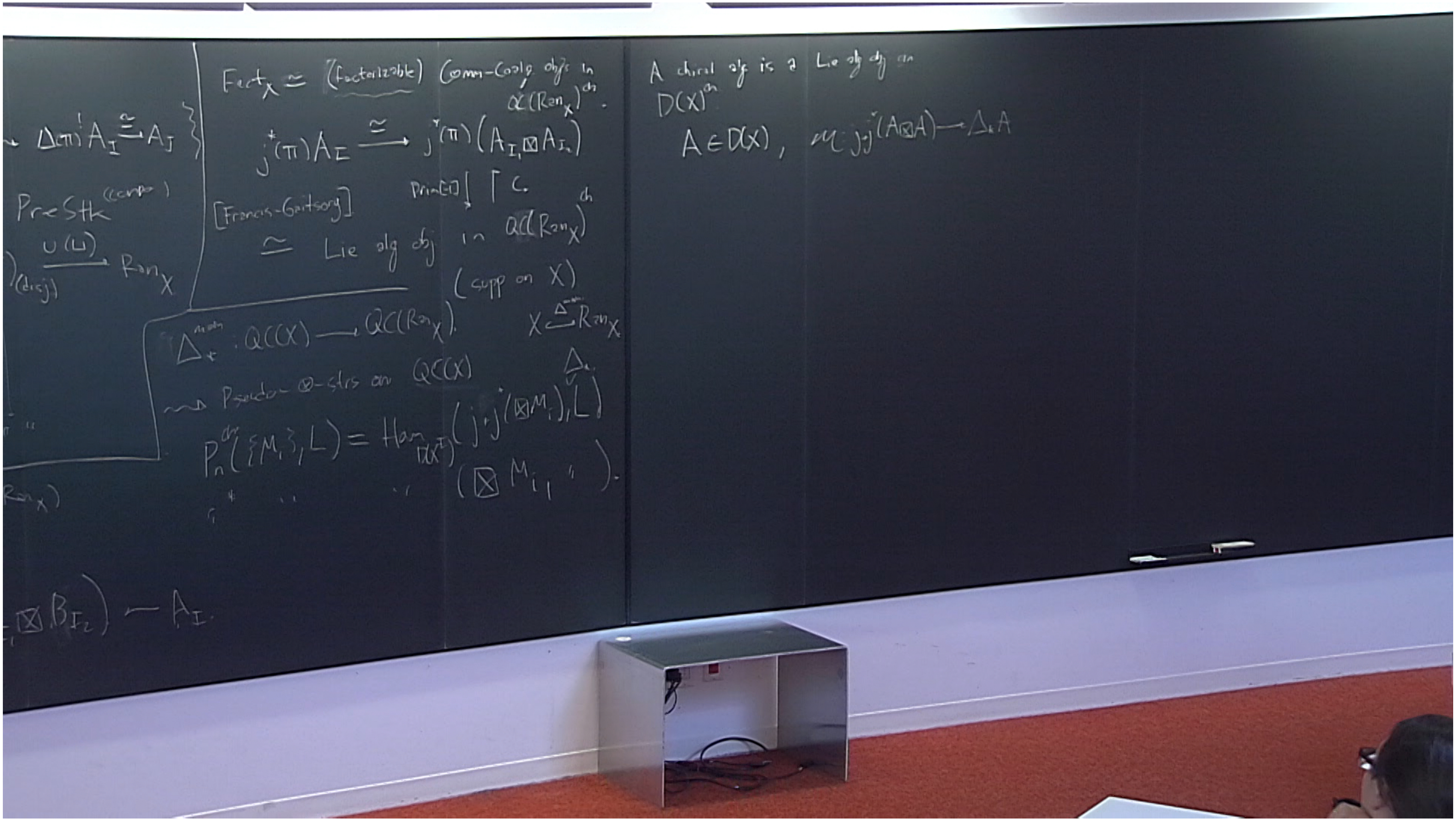
(QC/Ran_X)

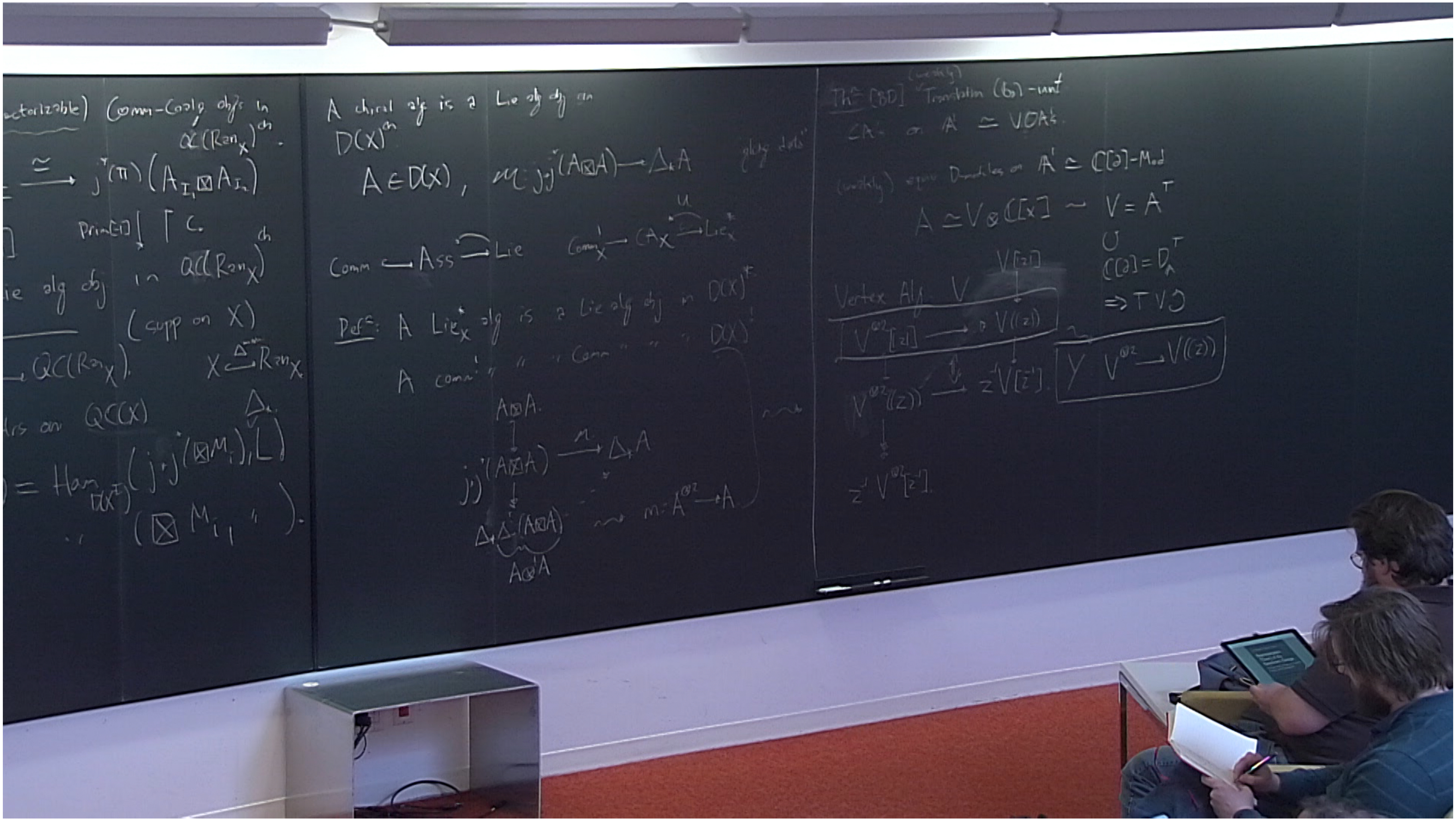
x_1 x_2

$QC : \mathrm{AffSch} \to \mathrm{Cat}$

$S \mapsto QC(S)$


Fact algs / X
X_dR
(non-empty)
of X
PreStk
= [AffSch, Grpd]
QC(-): AffSch → Cat
AffSch → Cat
S ↦ QC(S)
≅ colim_{I ∈ fSet} D^!(X^I)
= { I ⇝ A_I ∈ QC(X^I), π: I ↠ J ⇝ Δ(π)^! A_I ≅ A_J }
Prop^n Ran_X is comm monoid in PreStk^(corr)
under ∪ (⊔): Ran_X^{×2} ← (Ran_X^{×2})_(disj) → Ran_X
Def^n A fact alg is
- A ∈ QC(Ran_X)
- i^*(A ⊠ A) ≅ ∪^* A
Def^n The ch(*) ⊗-strs on QC(Ran_X) are
A ⊗^ch B = ∪_*(i^*(A ⊠ B))
" ⊗^* " = ∪_*(A ⊠ B)
(A ⊗^* B)_I = ⊕_{π: I ↠ {1,2}} j(π)_* j(π)^*(A_{I_1} ⊠ B_{I_2})
Fact_X ≃ (factorizable) Comm-Coalg objs in D(Ran_X)^ch.

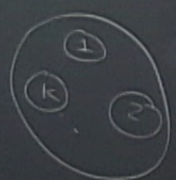

Fact_X ≃ (factorizable) Comm-Coalg obj's in QC(Ran_X)^ch.
Δ(π)^! A_I ≃ A_J
j^*(π) A_I ≃ j^*(π)(A_{I_1} ⊠ A_{I_2})
PreStk^(compn)
U(⊔)
Ran_X
Prim[-1] ↓ ↑ C.
[Francis-Gaitsgory]
≃ Lie alg obj in QC(Ran_X)^ch
(supp on X)
Δ_*^{main} : QC(X) → QC(Ran_X).
X ↪ Ran_X
~~> Pseudo-⊗-strs on QC(X)
P_n^ch({M_i}, L) = Hom_{D(X^I)}(j_*j^*(⊠M_i), Δ_* L)
(⊠ M_i,).
(Ran_X)
⊠ B_{I_2}) → A_I.
A chiral alg is a Lie alg obj in D(X)^ch.
A ∈ D(X), μ: j_*j^*(A⊠A) → Δ_* A


(Comm-Coalg obj's in $QC(Ran_X)^{ch}$.
$\xrightarrow{\simeq} j^*(\pi)(A_{I_1} \boxtimes A_{I_2})$
Prim[-1]
Lie alg obj in $QC(Ran_X)^{ch}$
(supp on X)
$QC(Ran_X)$. $X \xrightarrow{\Delta} Ran_X$
QC(X)
$= Hom_{D(X^I)}(j_* j^*(\boxtimes M_i), L)$
$(\boxtimes M_i, \ldots)$.
A chiral alg is a Lie alg obj on $D(X)^{ch}$
$A \in D(X)$, $m: j_* j^*(A \boxtimes A) \to \Delta_* A$
gluing data
Comm $\hookrightarrow$ Ass $\rightleftarrows$ Lie
$Comm^!_X \to CA_X \rightleftarrows Lie^*_X$
Def$^{\underline{n}}$: A Lie^*_X alg is a Lie alg obj in $D(X)^*$
A comm$^!$ " " " Comm " " " $D(X)^!$
$A \boxtimes A$
$j_* j^*(A \boxtimes A) \xrightarrow{m} \Delta_* A$
$\Delta_* \Delta^!(A \boxtimes A)$ $\rightsquigarrow$ $m: A^{\otimes 2} \to A$
$A \otimes^! A$
Th$^{\underline{m}}$ [BD] (weakly) Translation (G_a)-inv^t
CA's on $\mathbb{A}^1$ $\simeq$ VOA's.
(weakly) equiv D-modules on $\mathbb{A}^1 \simeq \mathbb{C}[\partial]$-Mod
$A \simeq V \otimes \mathbb{C}[x] \sim V = A^T$
$\mathbb{C}[\partial] = D^T_{\mathbb{A}}$
$\Rightarrow T \; V$
Vertex Alg. V
$V^{\otimes 2}[z] \to V((z))$
$V^{\otimes 2}((z)) \to z^{-1}V[z^{-1}]$
$z^{-1}V^{\otimes 2}[z^{-1}]$
$Y: V^{\otimes 2} \to V((z))$

nst fact algs

invt " . . .

is

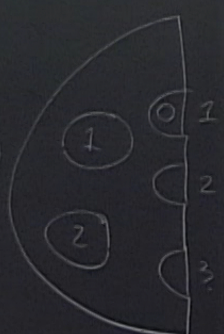


$D^\circ))$

$\longrightarrow V$

+

pp is



$_{\mathbb{R}}(H^\circ))$

(B) of E_n-alg

Def^{n}:

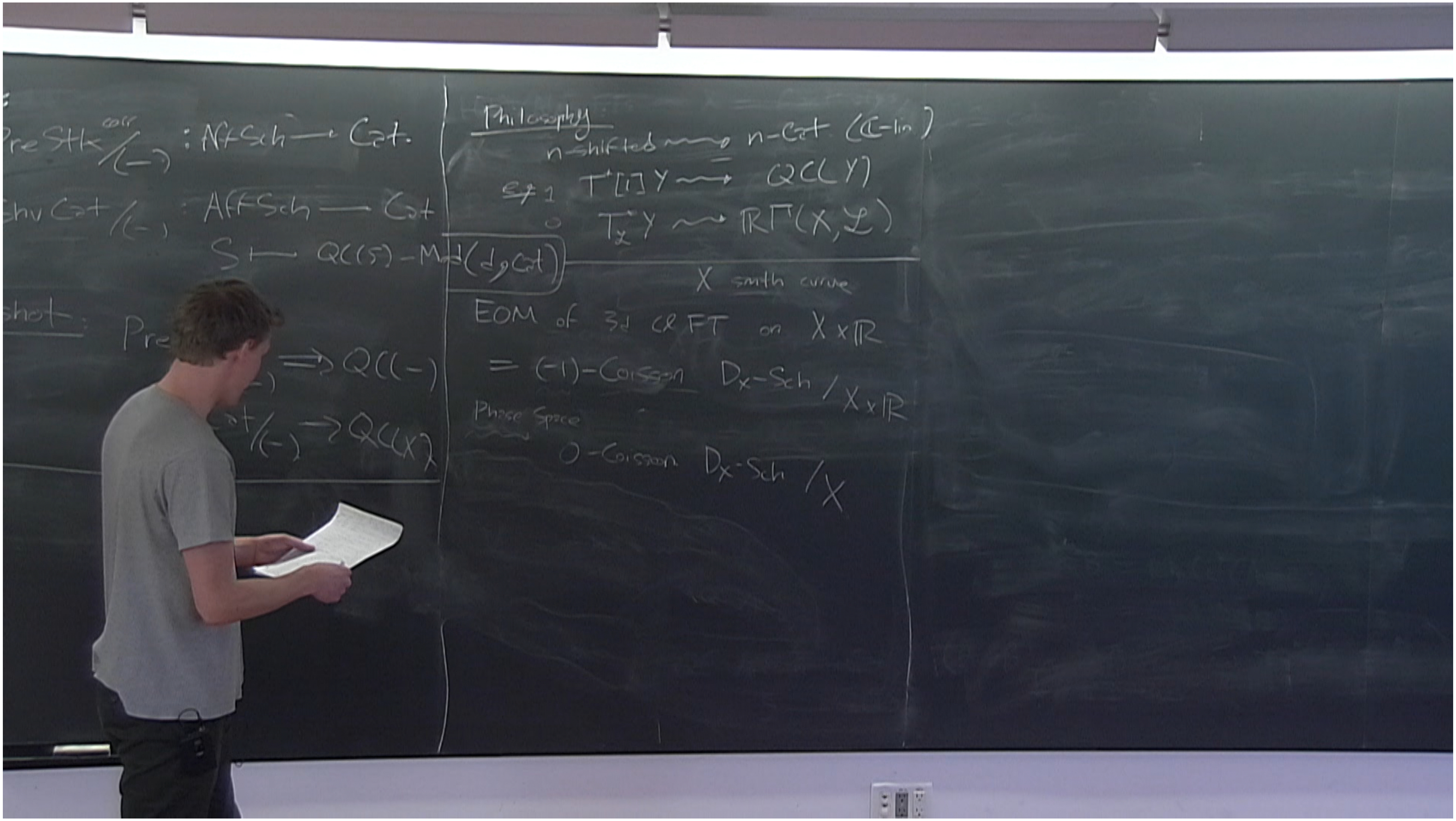
$$\mathrm{PreStk}^{corr}_{/(-)} : \mathrm{AffSch} \longrightarrow \mathrm{Cat}.$$

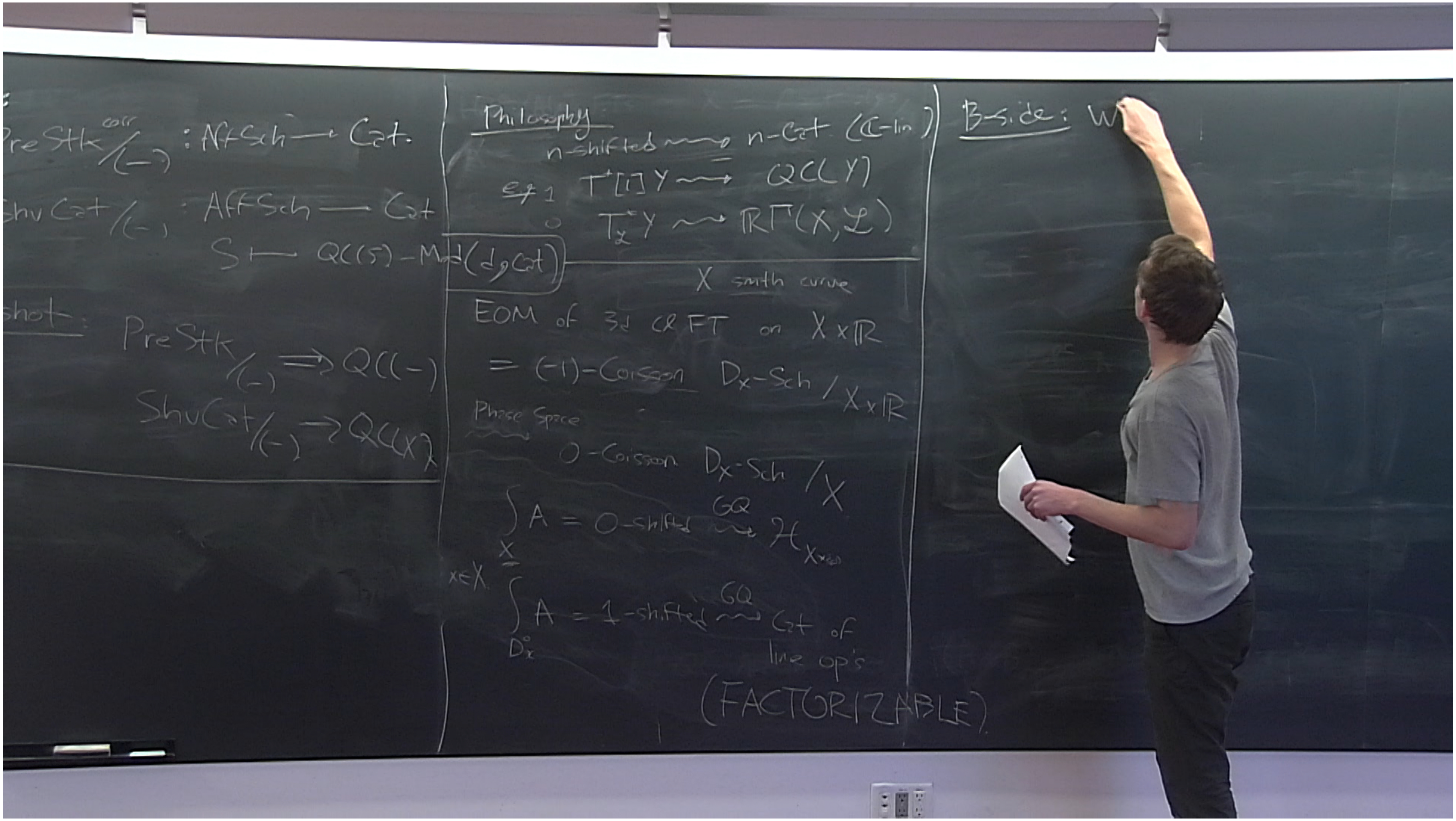
$$\mathrm{ShvCat}_{/(-)} : \mathrm{AffSch} \longrightarrow \mathrm{Cat}$$

$$S \longmapsto \mathrm{QC}(S)\text{-}\mathrm{Mod}(\mathrm{dgCat})$$

<u>Upshot</u>: $\mathrm{PreStk}_{/(-)} \Longrightarrow \mathrm{QC}(-)$

$$\mathrm{ShvCat}_{/(-)} \Rightarrow \mathrm{QC}(X)$$


PreStk / (−) : NfSch → Cat.
ShvCat / (−) : AffSch → Cat
S ↦ QC(S)−Mod(dg Cat)
Philosophy
n-shifted ⇝ n-Cat (ℂ-lin)
T*[1]Y ⇝ QC(Y)
T*_X Y ⇝ RΓ(X, L)
X smth curve
EOM of 3d CℓFT on X×ℝ
= (−1)-Poisson D_X-Sch / X×ℝ
Phase Space
0-Poisson D_X-Sch / X


PreStk^corr/(−) : AffSch → Cat.
ShvCat/(−) : AffSch → Cat
S ↦ QC(S)−Mod(dgCat)
shot: PreStk/(−) ⇒ QC(−)
ShvCat/(−) ⇒ QC(X)
Philosophy
n-shifted ⇝ n-Cat (ℂ-lin)
eg 1 T*[1]Y ⟶ QC(Y)
0 T*_x Y ⇝ ℝΓ(X, ℒ)
X smth curve
EOM of 3d CℓFT on X×ℝ
= (−1)-Coisson D_X-Sch / X×ℝ
Phase Space
0-Coisson D_X-Sch / X
∫_X A = 0-shifted ⇝ GQ
x∈X
∫_{D°_x} A = 1-shifted ⇝ GQ Cat of line op's
(FACTORIZABLE)
B-side: W

$\rightsquigarrow$ n-Cat ($\mathbb{C}$-lin)

$\rightsquigarrow QC(Y)$

$\rightsquigarrow \mathbb{R}\Gamma(X, \mathcal{L})$

<u>B-side:</u> W a 2-shifted sympl stack

eg: $T^*[2]Y$, BG, $[V/\!/G] = [V/G] \times_{T^*[3]BG} BG$ (Safronov)

EOM for RW thy to W
$$\text{"}=\text{"}\ \mathrm{Maps}(\mathbb{R}_{dR} \times X_{dR}, W)$$

Phase Space = "$\mathrm{Maps}(X_{dR}, W)$"

$$\int_{\mathring{D}_x} A = \mathrm{Maps}(\mathring{D}_{x,dR}, W) \quad \text{1-shifted}$$
$$\simeq T^*[1]\,\mathrm{Maps}(\mathring{D}_{x,dR}, Y)$$

$$\xrightarrow{GQ} \mathcal{C}_B \simeq QC(\text{"}\quad\text{"})$$

(...TORIZABLE)

A chiral alg

$D(X)^{ch}$

$A \in D(X$

Comm $\hookrightarrow$ A

<u>Def^c:</u> A L

A c

$\rightsquigarrow$ n-Cat ($\mathbb{C}$-lin)
$\rightsquigarrow$ $QC(Y)$
$\rightsquigarrow$ $\mathbb{R}\Gamma(X,\mathcal{L})$

X smth curve
FT on $X \times \mathbb{R}$
D_X-Sch $/ X \times \mathbb{R}$

D_X-Sch $/ X$
$\overset{GQ}{\rightsquigarrow}$ $\mathcal{H}_{X \times \mathbb{R}}$

$\overset{GQ}{\rightsquigarrow}$ Cat of line op's
(FACTORIZABLE)

B-side: W a 2-shifted sympl stack

eg $T^*[2]Y$, BG, $[V/\!/G] = [V/G] \times_{T^*[3]BG} BG$

EOM for RW thy to W
$=$ "$\mathrm{Maps}(\mathbb{R}_{dR} \times X_{dR}, W)$"

Phase Space $=$ "$\mathrm{Maps}(X_{dR}, W)$"

$$\int_{\mathring{D}_x} A = \mathrm{Maps}(\mathring{D}_{x,dR}, W) \quad \text{1-shifted}$$
$$\simeq T^*[1]\,\mathrm{Maps}(\mathring{D}_{x,dR}, Y)$$

$\xrightarrow{GQ} \mathcal{C}_B \simeq QC($" "$)$

A-side: W 0-shifted

eg T^*Y, $[V/\!/G]$

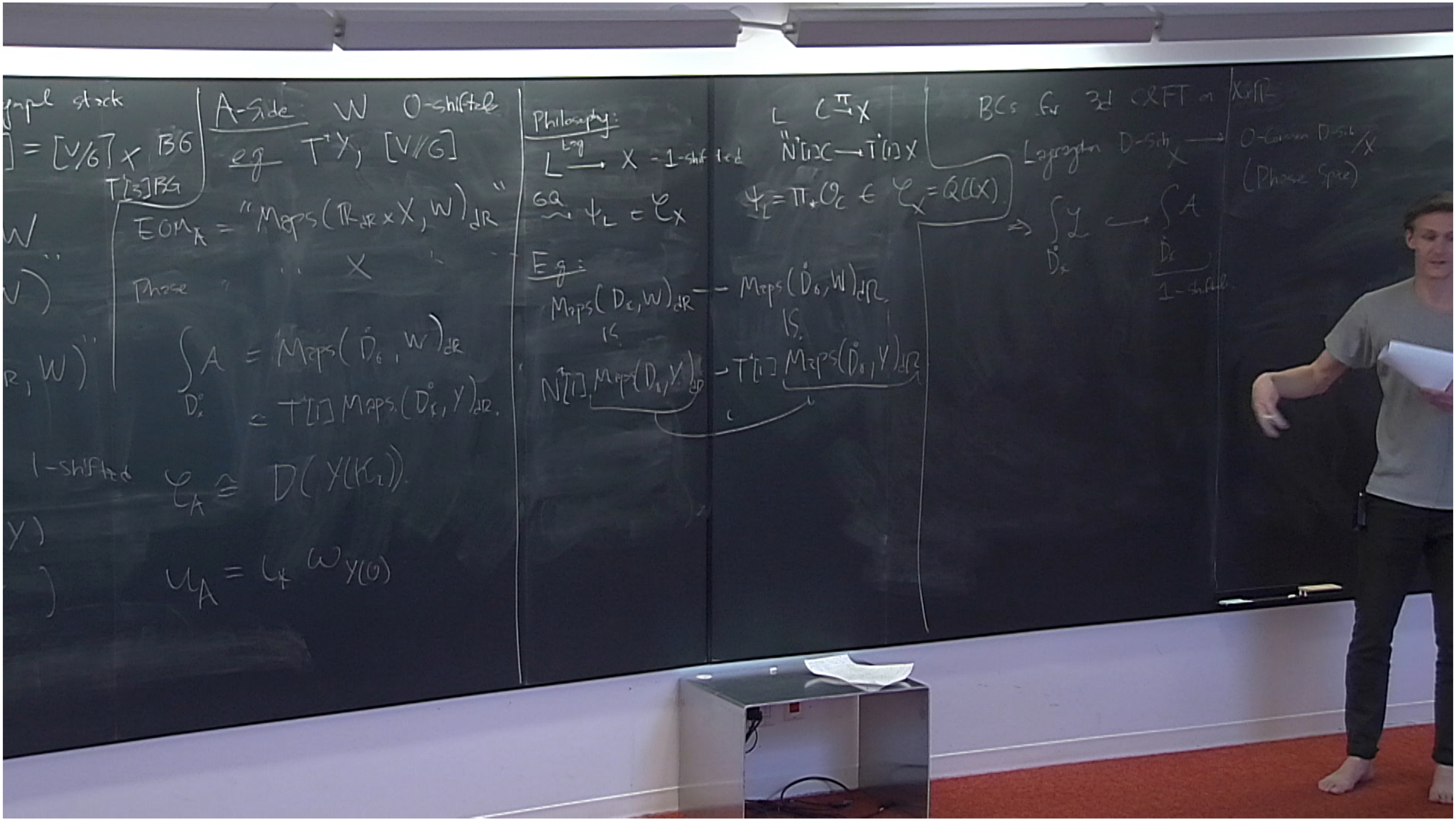
$\mathrm{EOM}_A =$ "$\mathrm{Maps}(\mathbb{R}_{dR} \times X, W)_{dR}$"

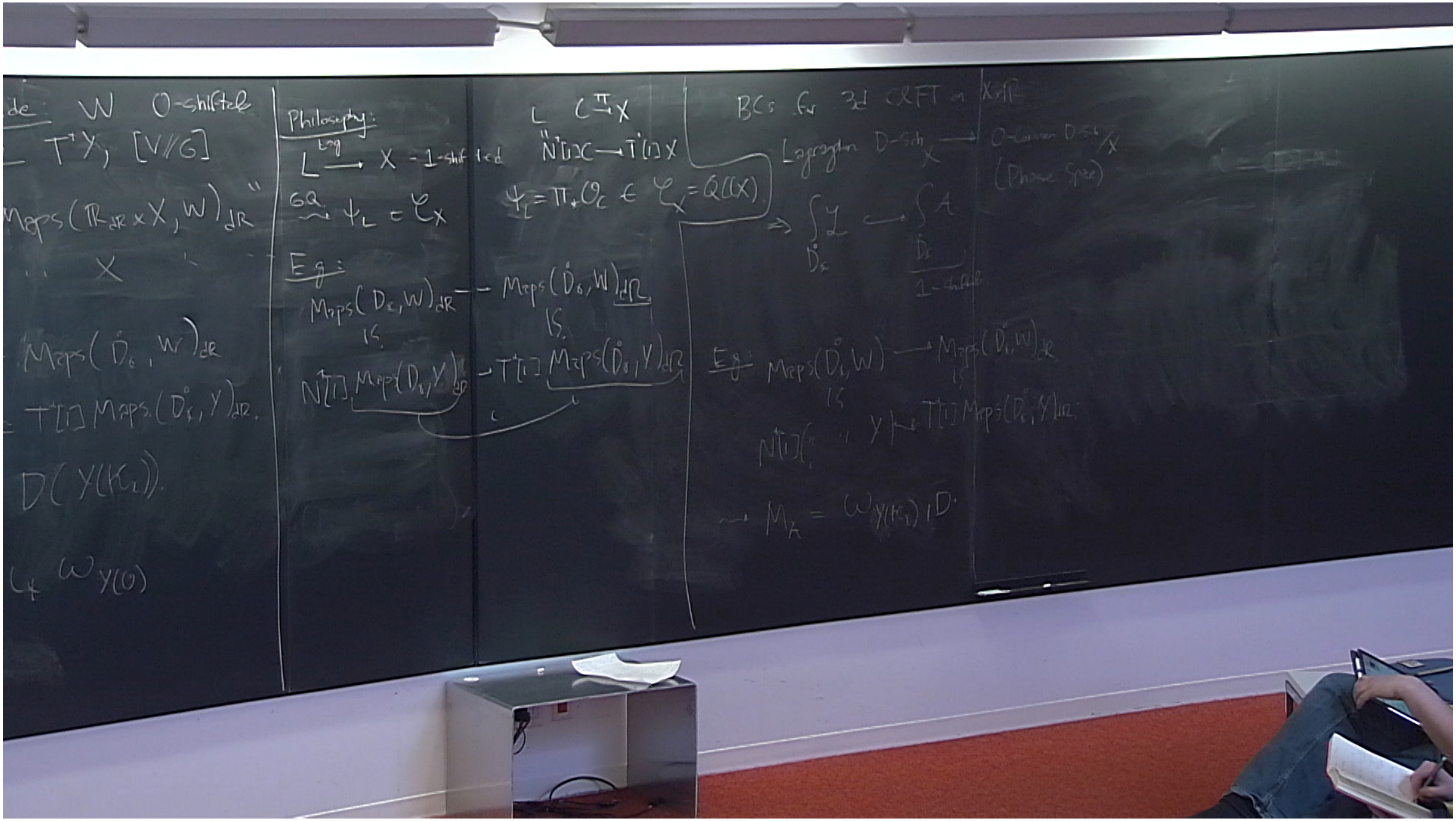
Phase " X "

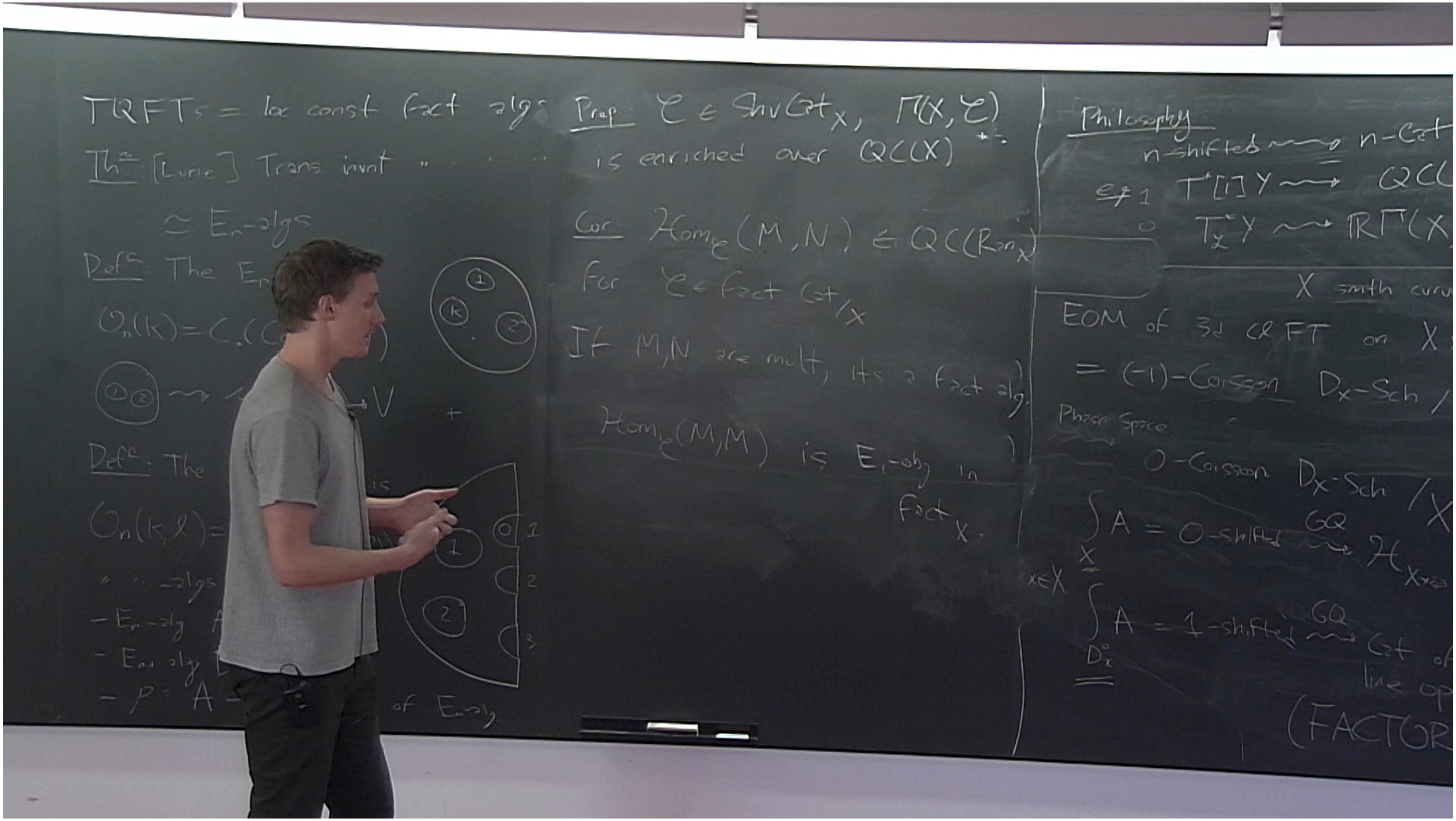
$$\int_{\mathring{D}_x} A = \mathrm{Maps}(\mathring{D}_x, W)_{dR}$$
$$\simeq T^*[1]\,\mathrm{Maps}(\mathring{D}_x, Y)_{dR}$$

$\mathcal{C}_A \cong D(Y(K_x))$

A chiral alg
$D(X)^{ch}$
$A \in D(X$
Comm $\hookrightarrow A$
Def: A L
A c


A-Side W 0-shifted
eg T*Y; [V//G]
Philosophy:
Eg:
BCs for 3d CQFT on X×R
(Phase Space)
1-shifted


W 0-shifted
T^*Y, $[V/\!/G]$
Maps$(\mathbb{R}_{dR} \times X, W)_{dR}$
X
Maps$(\mathring{D}_x, W)_{dR}$
$T^*[1]$ Maps$(\mathring{D}_x, Y)_{dR}$
$D(Y(K_x))$
$\omega_{Y(\mathcal{O})}$
Philosophy:
$L \xrightarrow{Lag} X$ -1-shifted
GQ $\rightsquigarrow \psi_L \in \mathcal{C}_X$
Eg:
Maps$(D_x, W)_{dR} \to$ Maps$(\mathring{D}_x, W)_{dR}$
$N^*[1]$ Maps$(D_x, Y)_{dR} \to T^*[1]$ Maps$(\mathring{D}_x, Y)_{dR}$
$L \overset{\pi}{\subset} X$
$N^*[1]L \to T^*[1]X$
$\psi_L = \pi_* \mathcal{O}_L \in \mathcal{C}_X = QC(X)$
BCs for 3d CRFT on $X \times \mathbb{R}$
Lagrangian D-sch $\to$ 0-Lorentz D-sch$/X$
(Phase Space)
$\Rightarrow \int_{\mathring{D}_x} \mathcal{L} \longmapsto \int_{\mathring{D}_x} A$
1-shifted
Eg: Maps$(\mathring{D}_x, W) \to$ Maps$(\mathring{D}_x, W)_{dR}$
$T^*[1]$ Maps$(\mathring{D}_x, Y)_{dR}$
$\rightsquigarrow M_A = \omega_{Y(K_x)}, D$.


TQFTs = loc const fact algs
Thm [Lurie] Trans invnt
≃ En-algs
Defn The En
Defn: The
Prop: C ∈ ShvCatX, Γ(X, C)
is enriched over QC(X)
Cor: Hom_C(M, N) ∈ QC(Ran_X)
for C ∈ Fact Cat/X
If M,N are mult, its a fact alg.
Hom_C(M,M) is E1-alg in Fact_X.
Philosophy
n-shifted ⇝ n-Cat
eg 1 T*[1]Y ⇝ QC
0 T*_X Y ⇝ RΓ(X
X smth curv
EOM of 3d Cl FT on X
= (-1)-Poisson D_X-Sch
Phase Space
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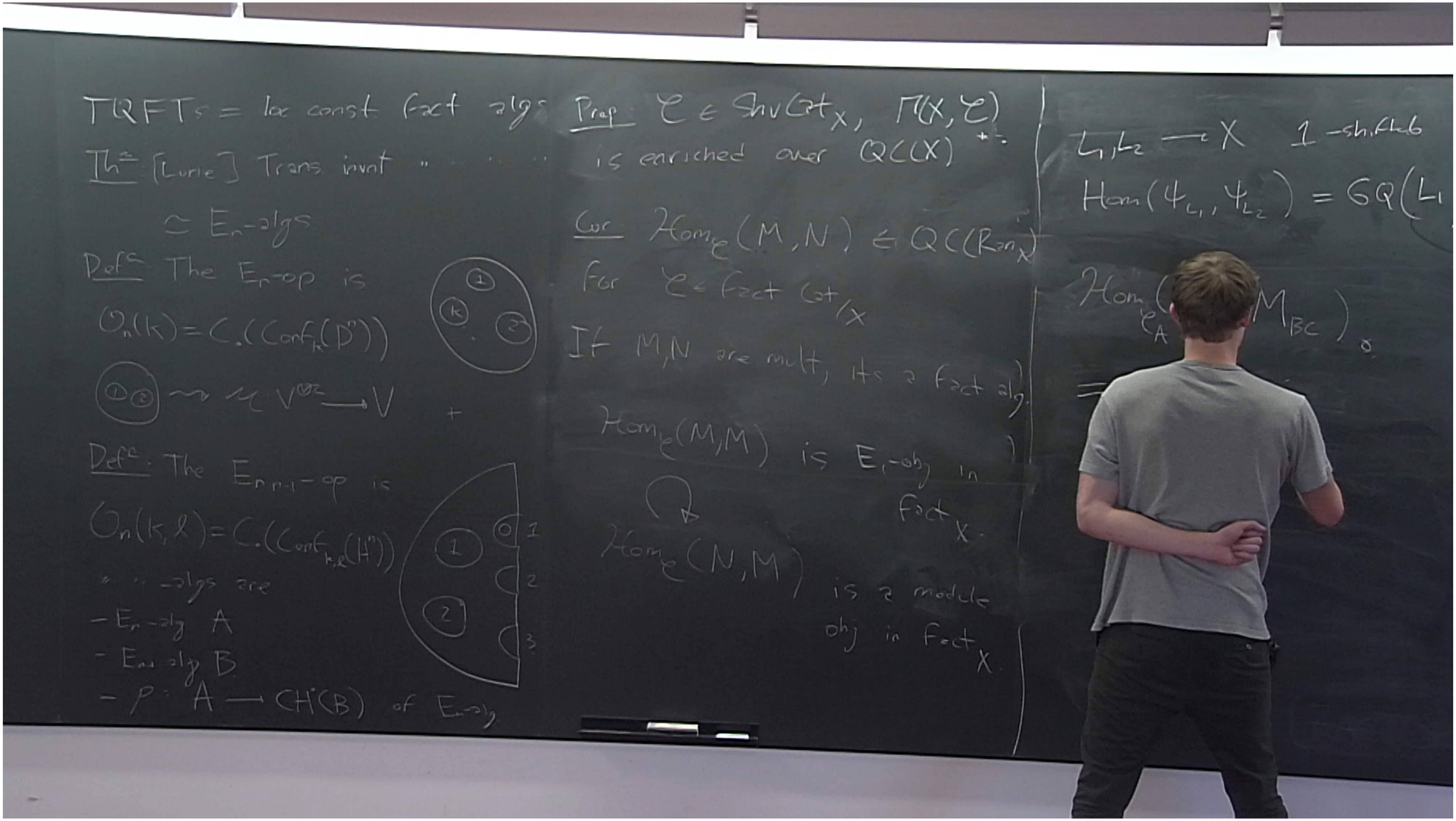
∫_X A = 0-shifted ⇝GQ H_X
∫_{D°_x} A = 1-shifted ⇝GQ Cat of line op
(FACTOR


TQFTs = loc const fact algs
Thm [Lurie] Trans invt " " "
≃ En-algs
Defn The En-op is
On(k) = C.(Confk(D^n))
Defn: The En,n-1-op is
On(k,l) = C.(Confk,l(H^n))
" " -algs are
- En-alg A
- En-1 alg B
- ρ: A → CH(B) of En-alg
Prop C ∈ ShvCatX, Γ(X, C)
is enriched over QC(X)
Cor Hom_C(M,N) ∈ QC(Ran_X)
for C ∈ Fact Cat/X
If M,N are mult, its a fact alg.
Hom_C(M,M) is E1-obj in FactX
Hom_C(N,M) is a module obj in FactX
L1, L2 → X 1-shifted
Hom(ΨL1, ΨL2) = GQ(L1
Hom_CA(MBC)

Cat_X, $\Gamma(X, \mathcal{C})$

...er $QC(X)$

$(N) \in QC(\mathrm{Ran}_X)$

Cat/X

...lt, its a fact alg.

... is E_1-obj in Fact_X

... is a module obj in Fact_X

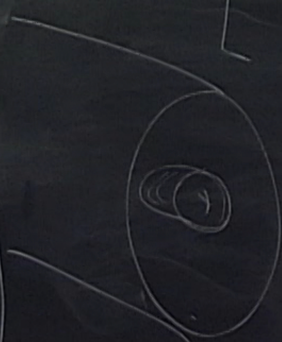
$L_1, L_2 \to X$ 1-shifted

$\mathrm{Hom}(\Psi_{L_1}, \Psi_{L_2}) = GQ(L_1 \times_X L_2)$

$\mathcal{H}om_{\mathcal{C}_A}(\mathcal{U}_A, M_{BC})$.

$= GQ\Big[$ Germs of Bulk fields along $D_x \times \{0\}$ $\times$ Fields on $\mathring{D}_x$ set BC

" " along $\mathring{D}_x$ $\Big]$



B-side: W a 2-shifted sympl stack

eg $T^*[2]Y$, BG, $[V/\!/G] = [V/G] \times_{T^*[3]BG} BG$

EOM for RW thy to W

$=$ "$\mathrm{Maps}(\mathbb{R}_{dR} \times X_{dR}, W)$"

Phase Space $=$ "$\mathrm{Maps}(X_{dR}, W)$"

$\int_{\mathring{D}_x} A = \mathrm{Maps}(\mathring{D}_{x,dR}, W)$ 1-shifted

eg $\simeq T^*[1]\mathrm{Maps}(\mathring{D}_{x,dR}, Y)$

$\xrightarrow{GQ} \mathcal{C}_B \simeq QC($ " " $)$

A-Side: ...

eg T^*Y...

$EOM_A =$ "$\mathrm{Maps}(\mathbb{R}$...

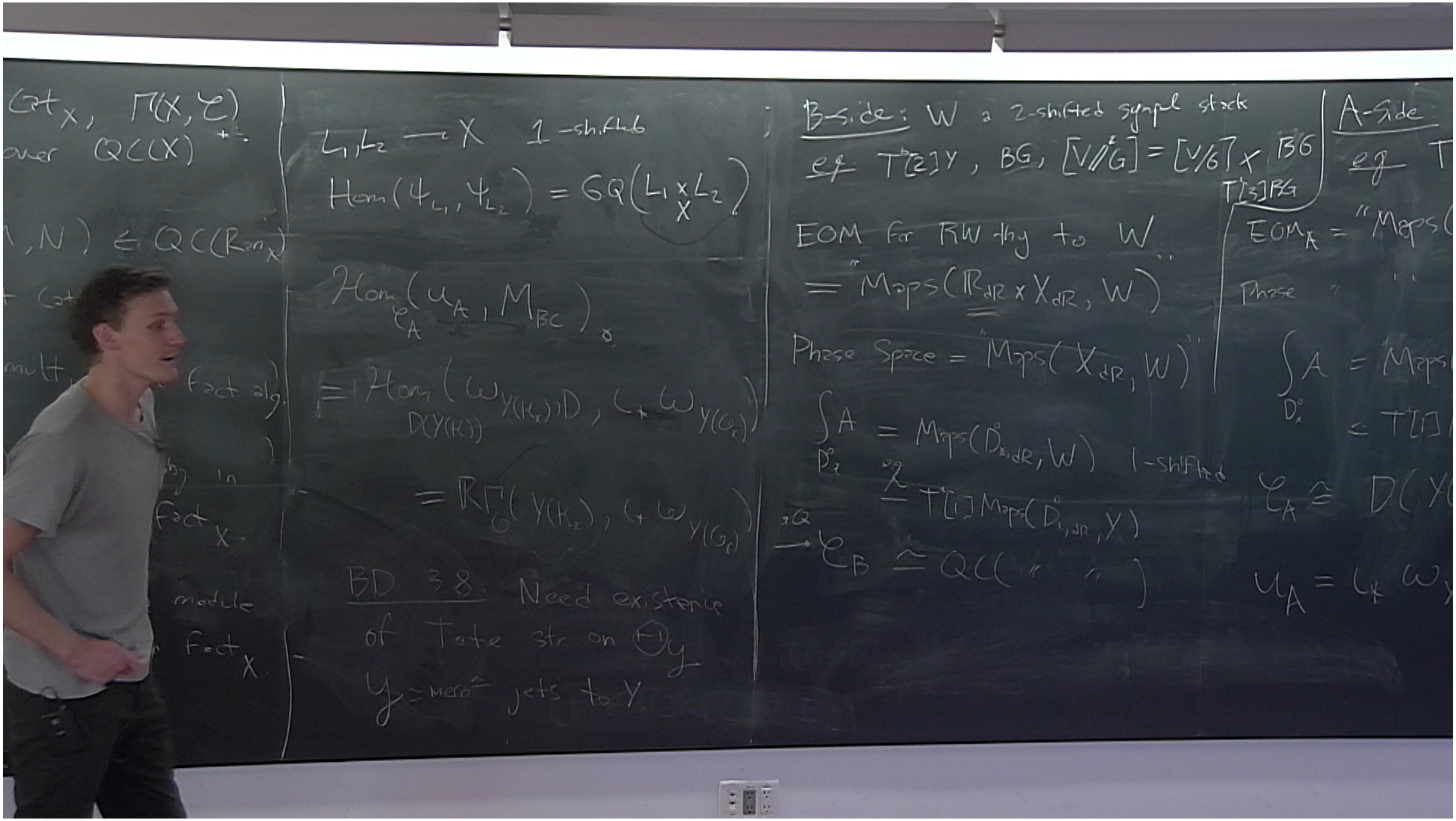
Phase ...

$\int_{\mathring{D}_x} A = \mathrm{Maps}($...

$= T^*[1] M$...

$\mathcal{C}_A \simeq D(Y(K$...

$\mathcal{U}_A = \mathcal{L}_x\, \omega_{Y(O}$...


Cat_X, Γ(X, C)
over QC(X)
L1, L2 → X 1-shifted
Hom(Ψ_L1, Ψ_L2) = GQ(L1 ×_X L2)
Hom_CA(U_A, M_BC)
= Hom_D(Y(K)) (ω_Y(K), D, ι_* ω_Y(O))
= RΓ(Y(K), ι_* ω_Y(O))
BD 3.8 Need existence of Tate str on O_Y
Y = mero jets to Y
B-side: W a 2-shifted sympl stack
eg T*[2]Y, BG, [V//G] = [V/G] ×_T*[3]BG BG
EOM for RW thy to W
= "Maps(R_dR × X_dR, W)"
Phase Space = "Maps(X_dR, W)"
∫_D°_x A = Maps(D°_x,dR, W) 1-shifted
≃ T*[1] Maps(D°_x,dR, Y)
→ C_B ≃ QC(" ")
A-side
eg T
EOM_A = "Maps(
Phase
∫_D°_x A = Maps
≃ T*[1]
C_A ≃ D(Y
U_A = ι_* ω_Y