Title: Categorical Bernstein Operators and the Boson-Fermion correspondence.

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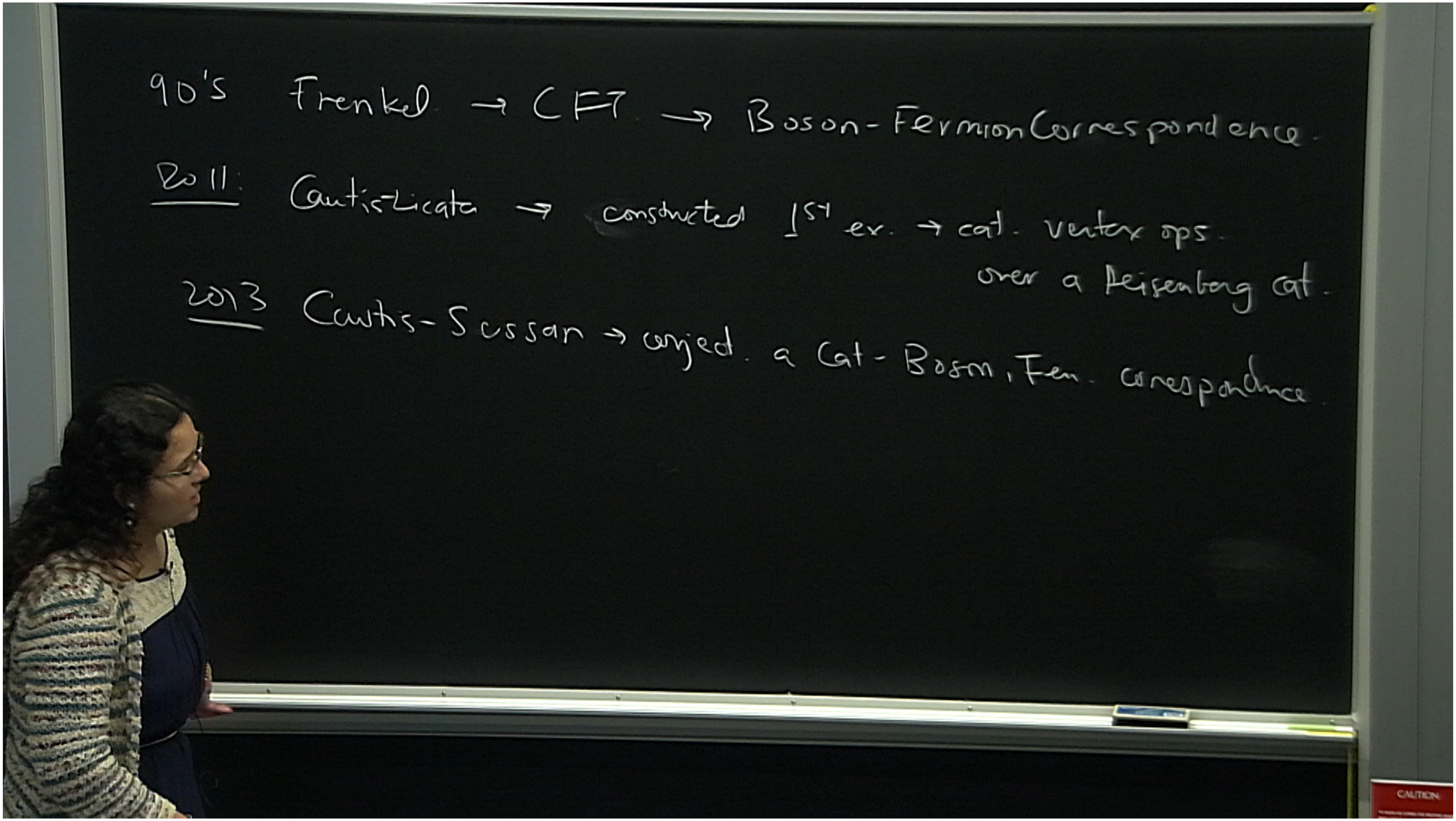
Abstract: <p>Bernstein operators are vertex operators that create and annihilate Schur polynomials. These operators play a significant role in the mathematical formulation of the Boson-Fermion correspondence due to Kac and Frenkel. The role of this correspondence in mathematical physics has been widely studied as it bridges the actions of the infinite Heisenberg and Clifford algebras on Fock space. Cautis and Sussan conjectured a categorification of this correspondence within the framework of Khovanov's Heisenberg category. I will discuss how to categorify the Bernstein operators and settle the Cautis-Sussan conjecture, thus proving a categorical Boson-Fermion correspondence.</p>


90's Frenkel → CFT → Boson-Fermion Correspondence.
2011: Cautis-Licata → constructed 1st ex. → cat. vertex ops.
over a Heisenberg cat.
2013 Cautis-Sussan → conjed. a Cat-Boson, Fer. correspondence.

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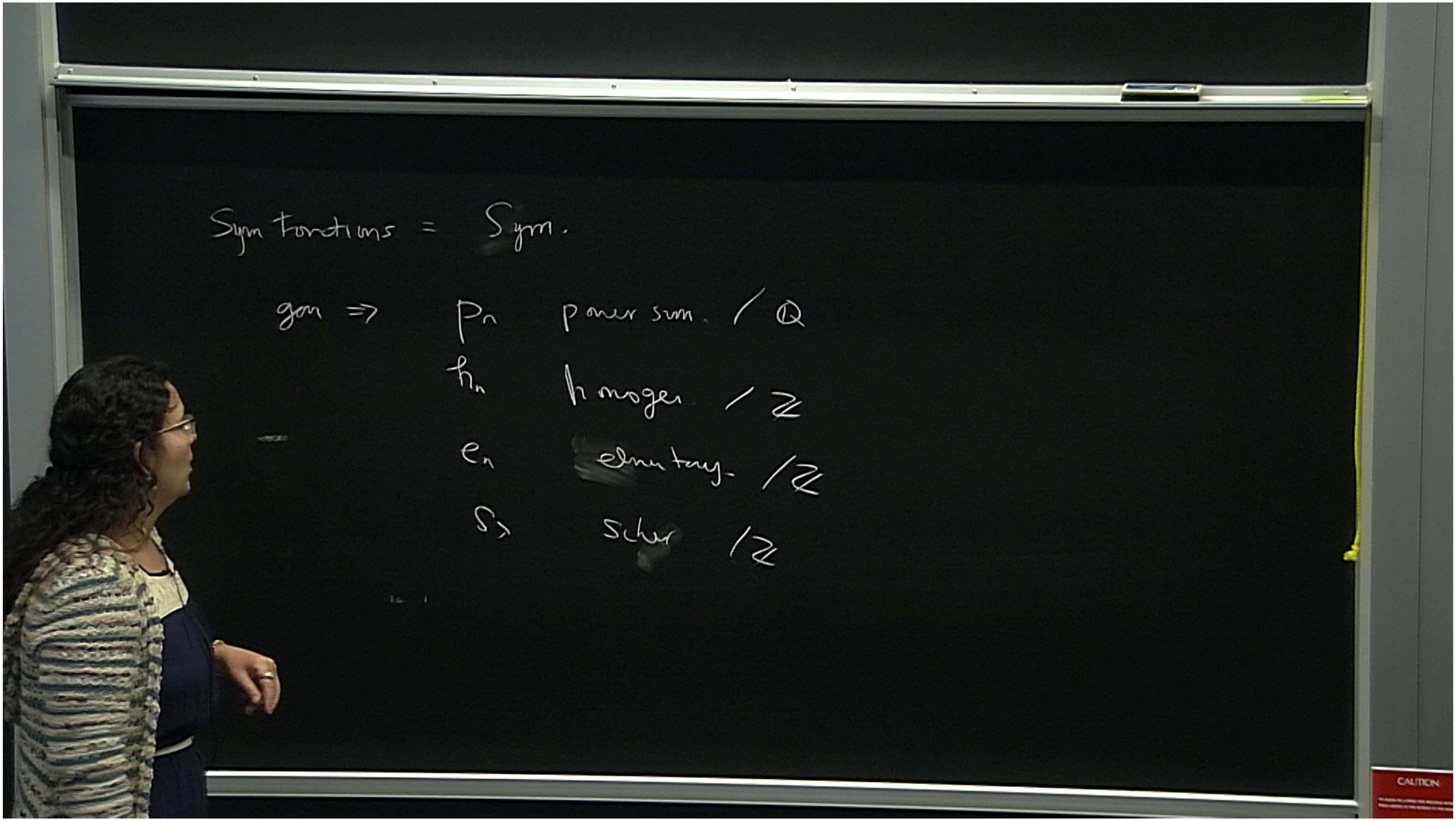
B-F corresp.

$$\mathfrak{h} \circlearrowright \boxed{\text{Sym Functions}} \circlearrowleft Cl.$$

Sym functi


Sym functions = Sym.
gen ⇒ p_n power sum. / $\mathbb{Q}$
h_n homogen. / $\mathbb{Z}$
e_n elementary. / $\mathbb{Z}$
s_λ schur / $\mathbb{Z}$
CAUTION

Sym functions = Sym.

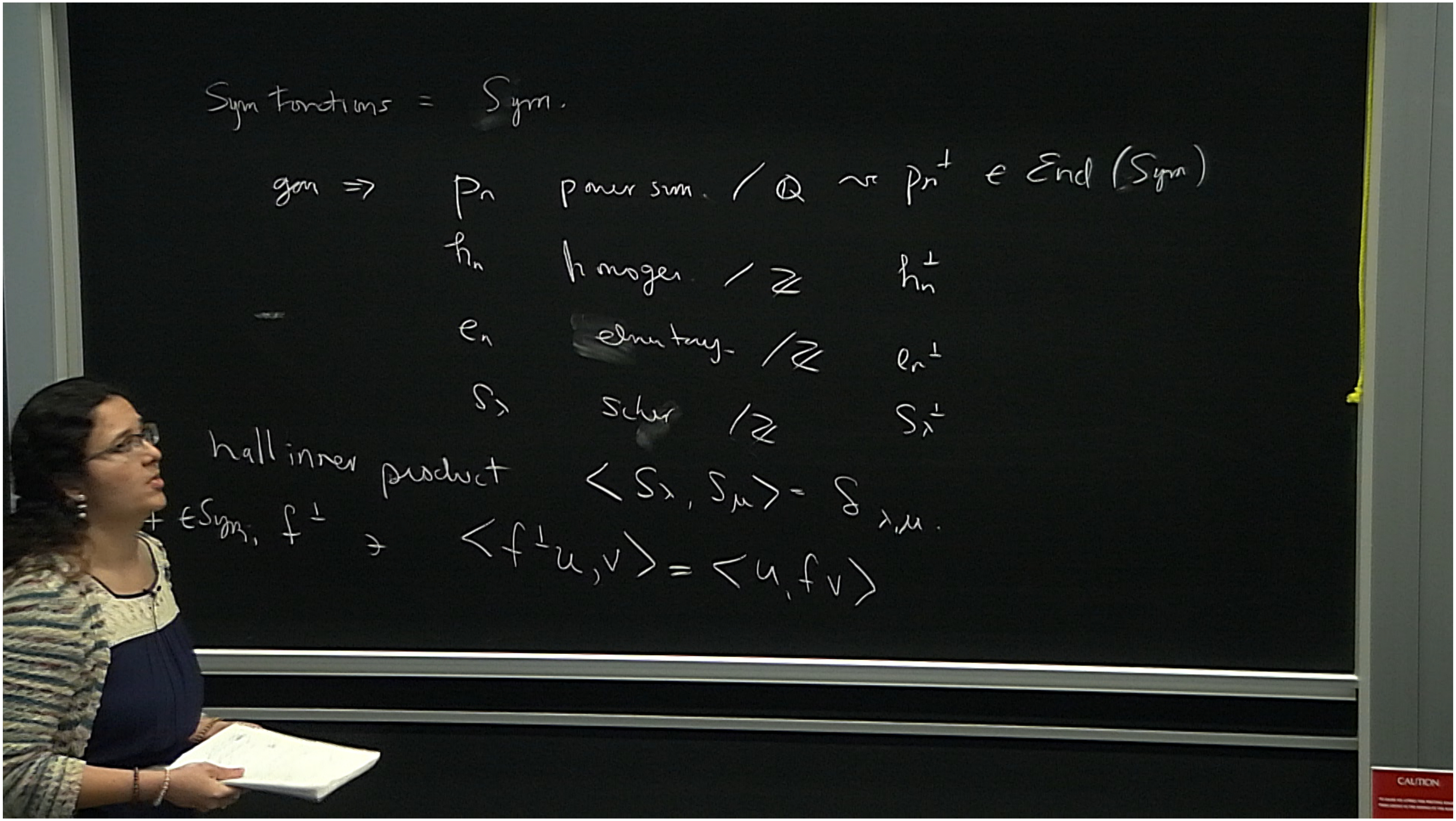
gen $\Rightarrow$ p_n power sum / $\mathbb{Q}$

h_n homogen. / $\mathbb{Z}$

e_n elementary. / $\mathbb{Z}$

s_λ schur / $\mathbb{Z}$

hall inner product $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.


Sym functions = Sym.
gen ⇒ p_n power sum / $\mathbb{Q}$ ⇝ $p_n^\perp \in End(Sym)$
h_n homogen. / $\mathbb{Z}$ $h_n^\perp$
e_n elementary / $\mathbb{Z}$ $e_n^\perp$
s_λ Schur / $\mathbb{Z}$ $s_\lambda^\perp$
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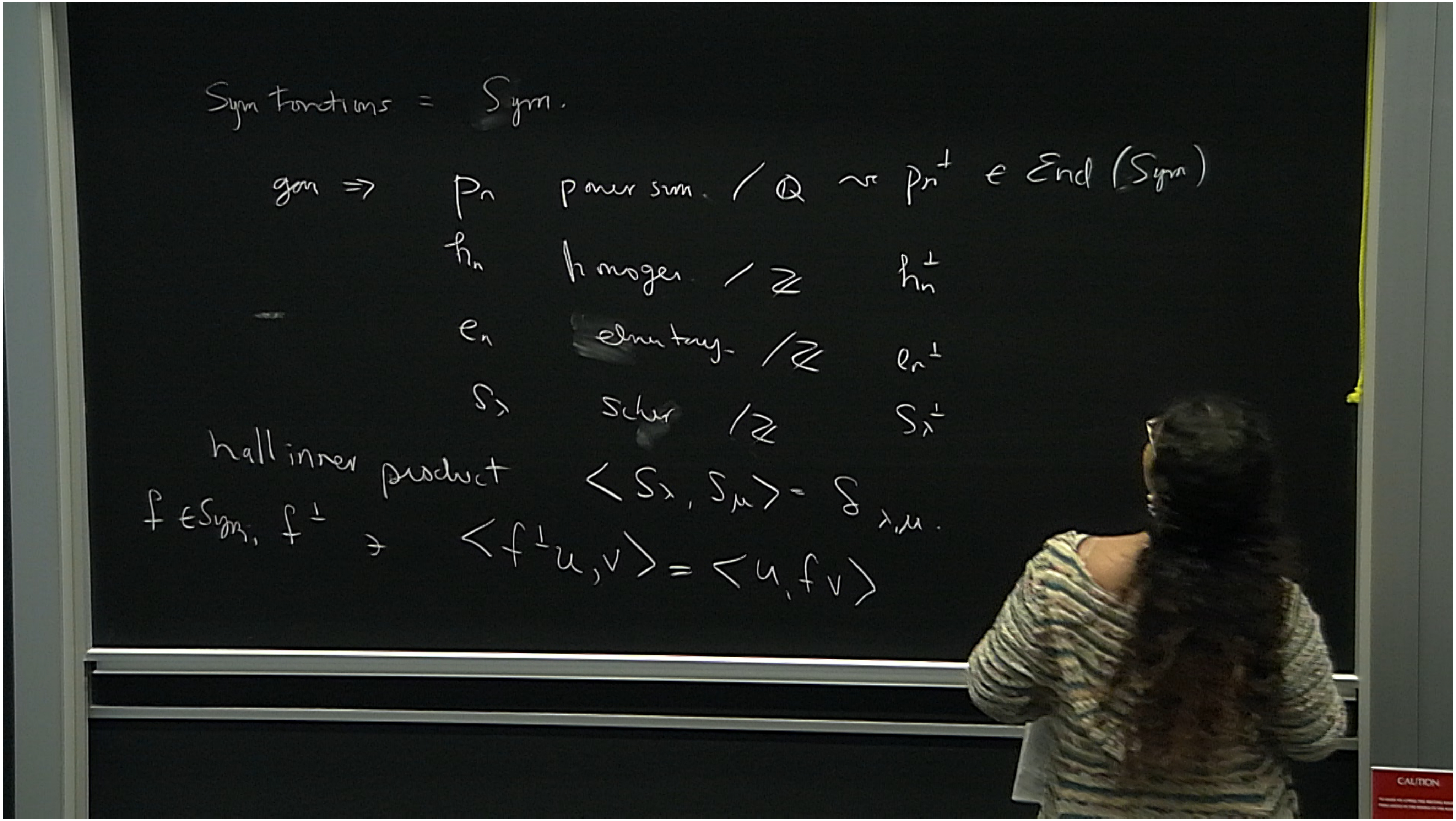
$f \in Sym$, $f^\perp$ ∋ $\langle f^\perp u, v\rangle = \langle u, fv\rangle$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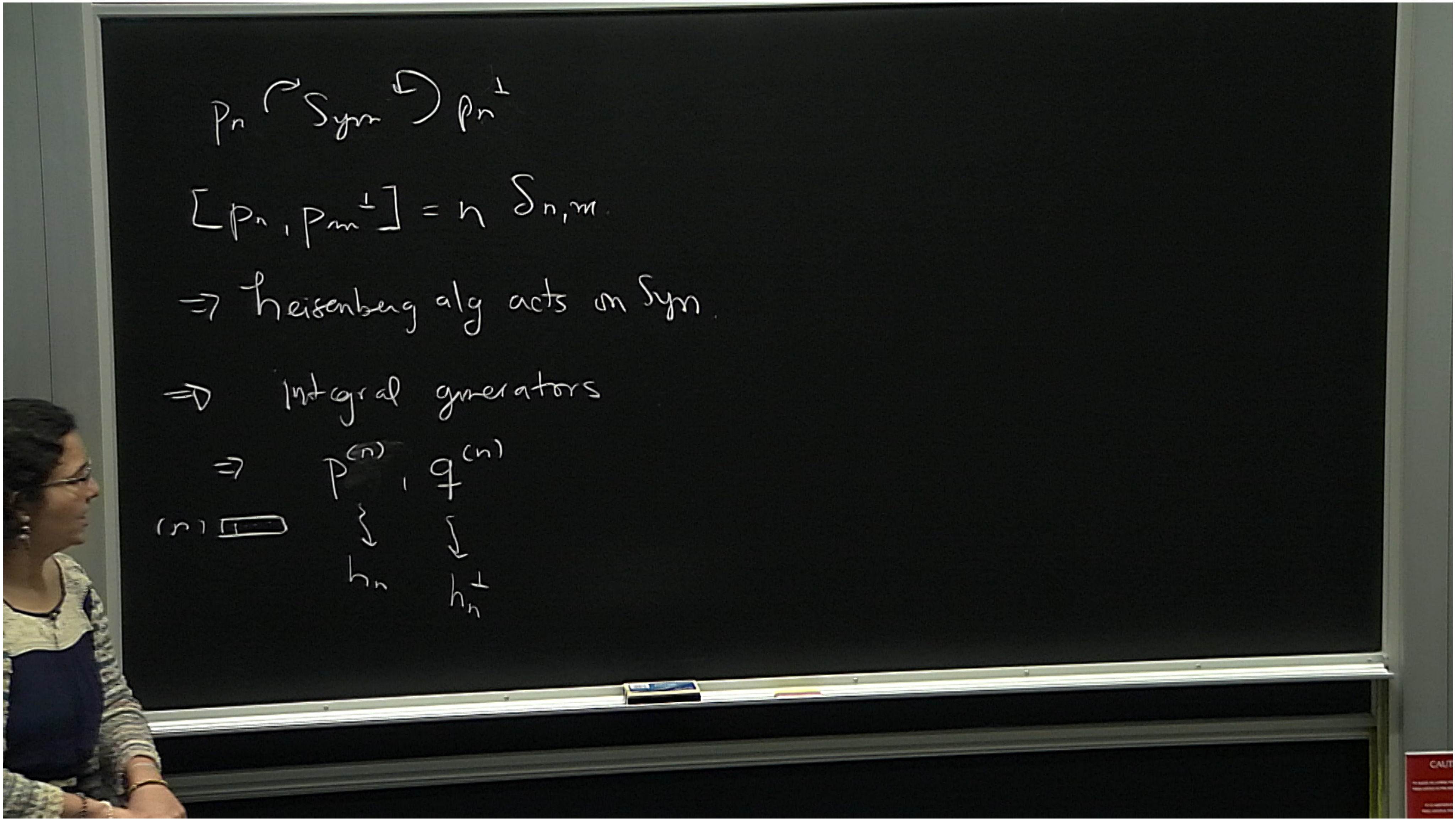
$f \in Sym$, $f^\perp \ni \langle f^\perp u, v \rangle = \langle u, f v \rangle$

$p_n \curvearrowright \mathrm{Sym} \curvearrowleft p_n^{\perp}$

$[p_n, p_m^{\perp}] = n\,\delta_{n,m}$

$\Rightarrow$ heisenberg alg acts on Sym

$\Rightarrow$ integral generators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$\Rightarrow$ integral generators
$\Rightarrow$ $p^{(n)}, q^{(n)}$
(n)
h_n $h_n^{\perp}$

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$[p_n, p_m^{\perp}] = n\,\delta_{n,m}$

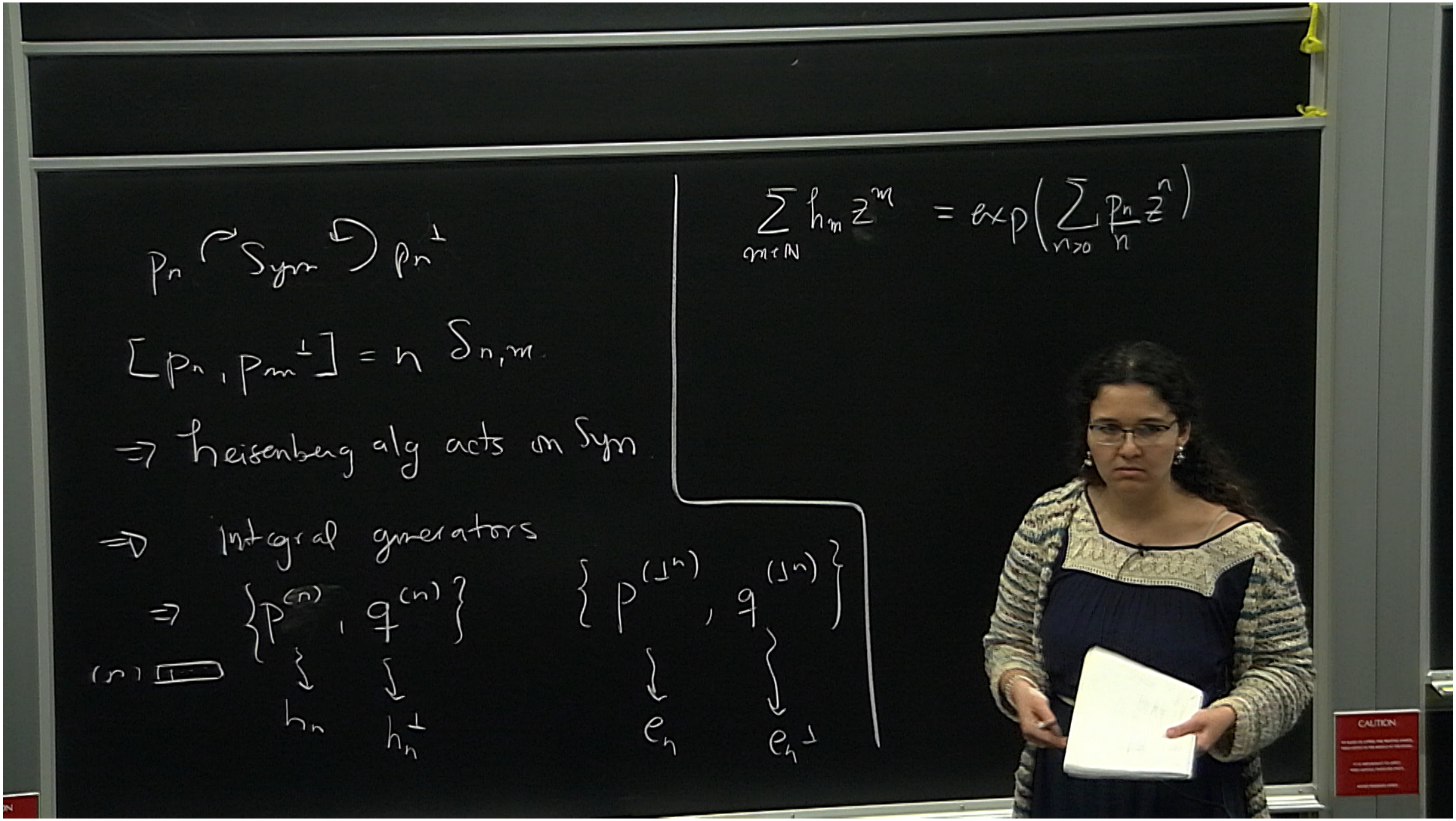
$\Rightarrow$ heisenberg alg acts on Sym

$\Rightarrow$ integral generators

$$\Rightarrow \{p^{(n)}, q^{(n)}\} \qquad \{p^{(1^n)}, q^{(1^n)}\}$$

(n)

$$p^{(n)} \rightsquigarrow h_n, \quad q^{(n)} \rightsquigarrow h_n^{\perp}, \qquad p^{(1^n)} \rightsquigarrow e_n, \quad q^{(1^n)} \rightsquigarrow e_n^{\perp}$$



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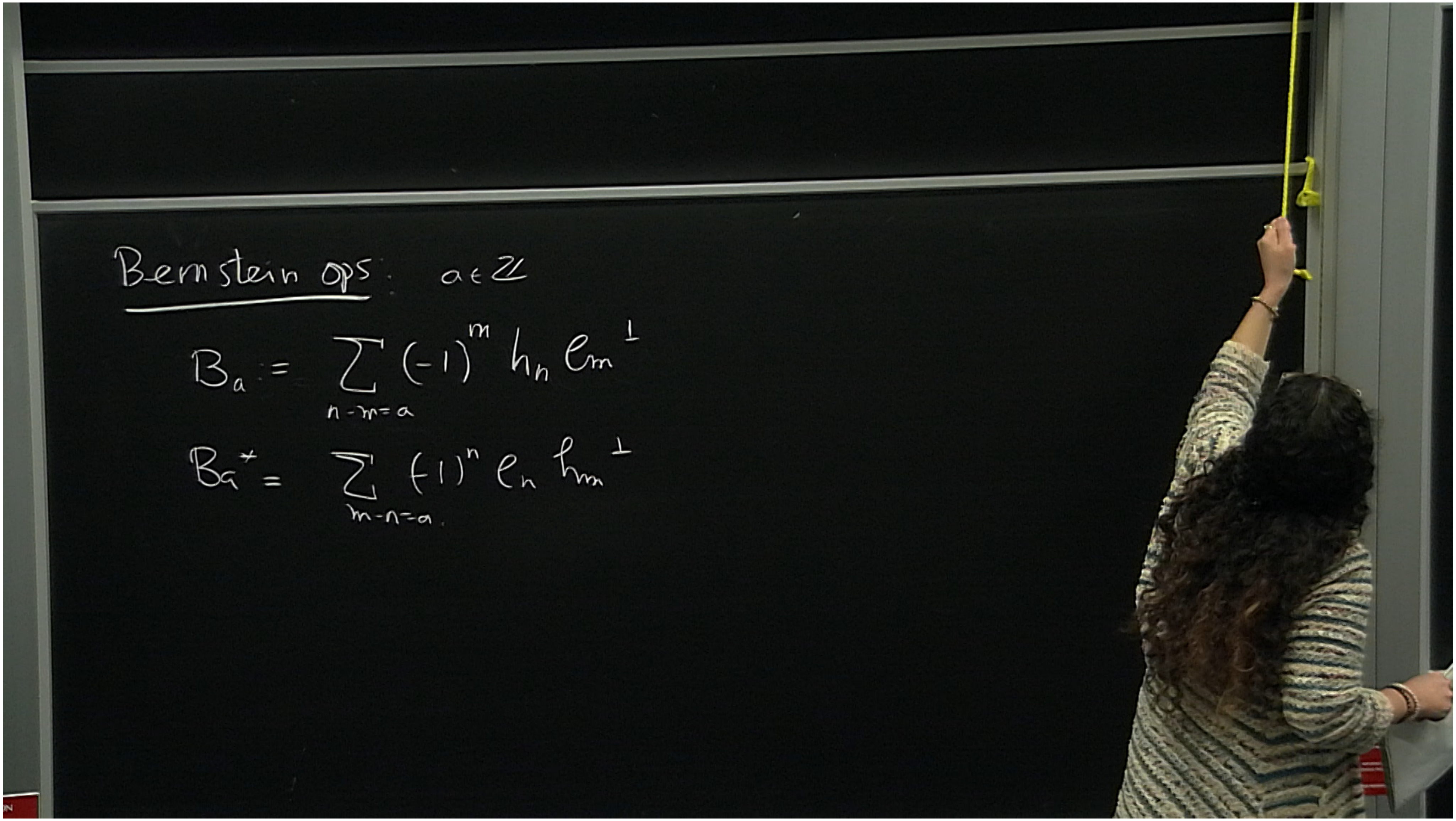
$\Rightarrow \{p^{(n)}, q^{(n)}\}$
$h_n \quad h_n^{\perp}$
$\{p^{(1^n)}, q^{(1^n)}\}$
$e_n \quad e_n^{\perp}$
$\sum_{m\in\mathbb{N}} h_m z^m = \exp\left(\sum_{n>0} \frac{p_n}{n} z^n\right)$


Bernstein ops: $a \in \mathbb{Z}$
$B_a := \sum_{n-m=a} (-1)^m h_n e_m^\perp$
$B_a^* = \sum_{m-n=a} (-1)^n e_n h_m^\perp$

$h_n \quad h_n^{\perp} \quad e_n \quad e_n^{\perp}$

Bernstein ops: $a \in \mathbb{Z}$

$$B_a := \sum_{n-m=a} (-1)^m h_n e_m^{\perp}$$

$$B_a^* = \sum_{m-n=a} (-1)^n e_n h_m^{\perp}$$

$$B_{a-1} B_b + B_{b-1} B_a = 0$$

$$B_{a+1}^* B_b^* + B_{b+1}^* B_a^* = 0$$

$$B_{a+1} B_{b+1}^* + B_b^* B_a = \delta_{a,b}$$

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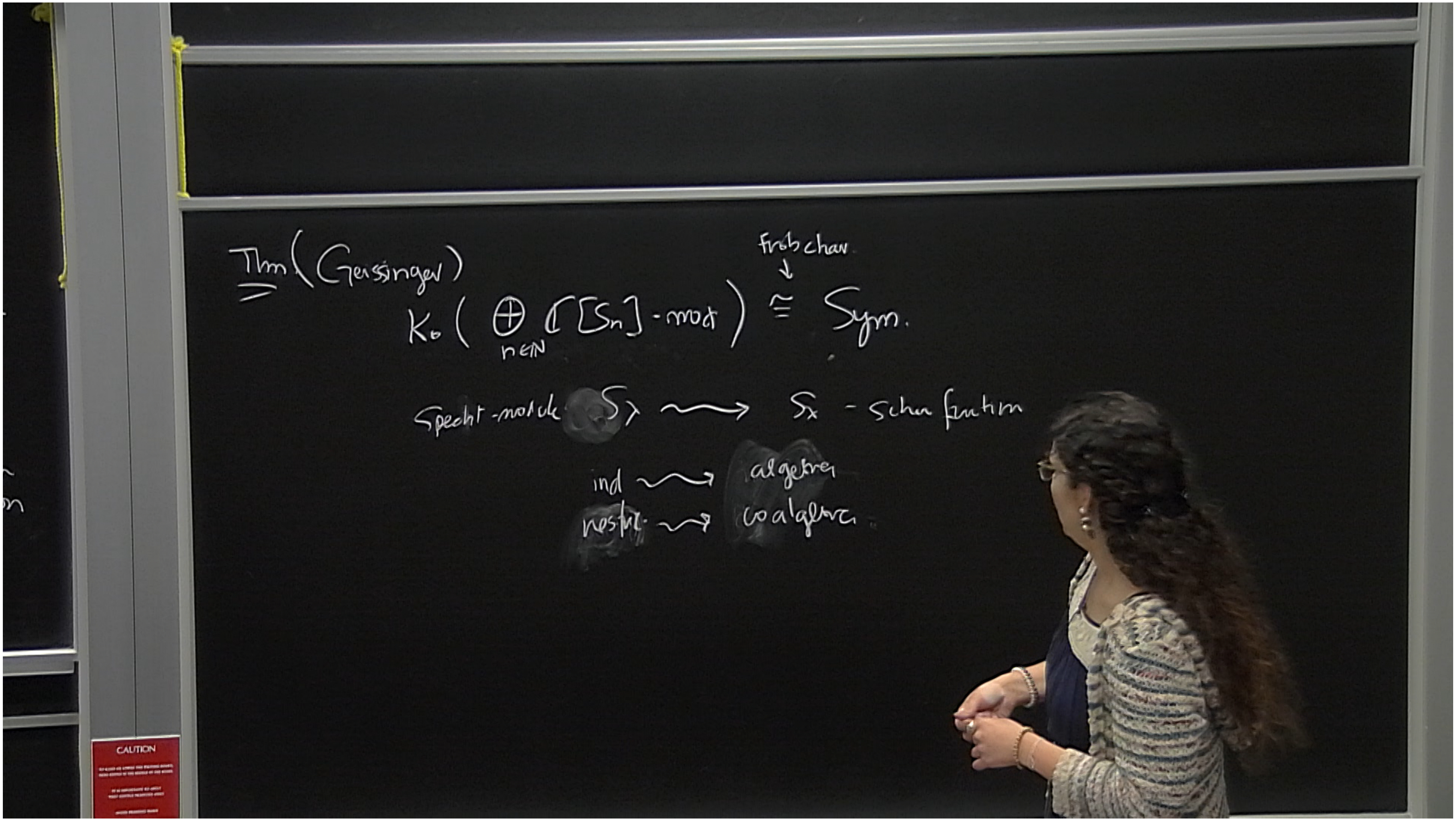
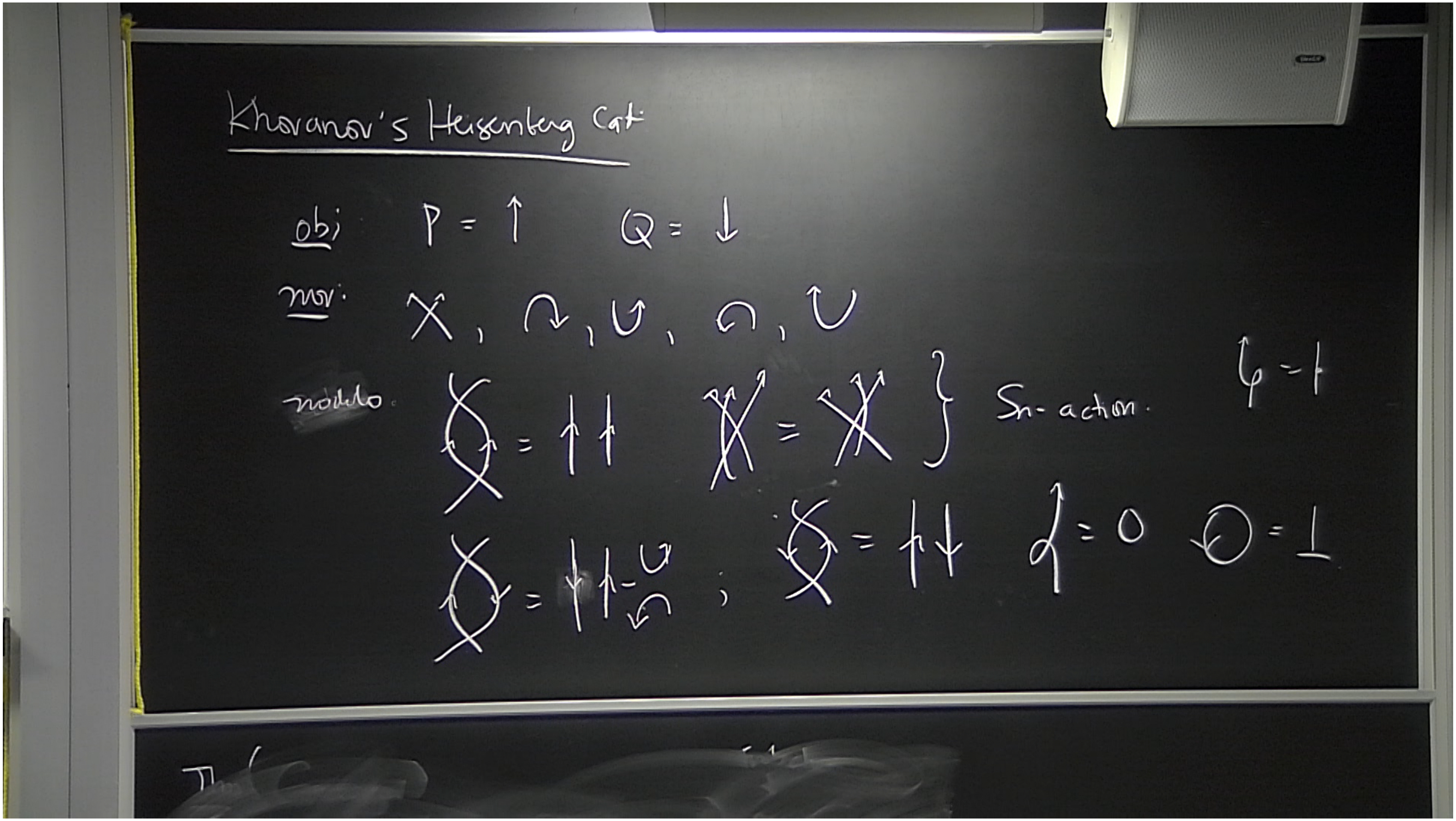
Thm (Zelevinsky)

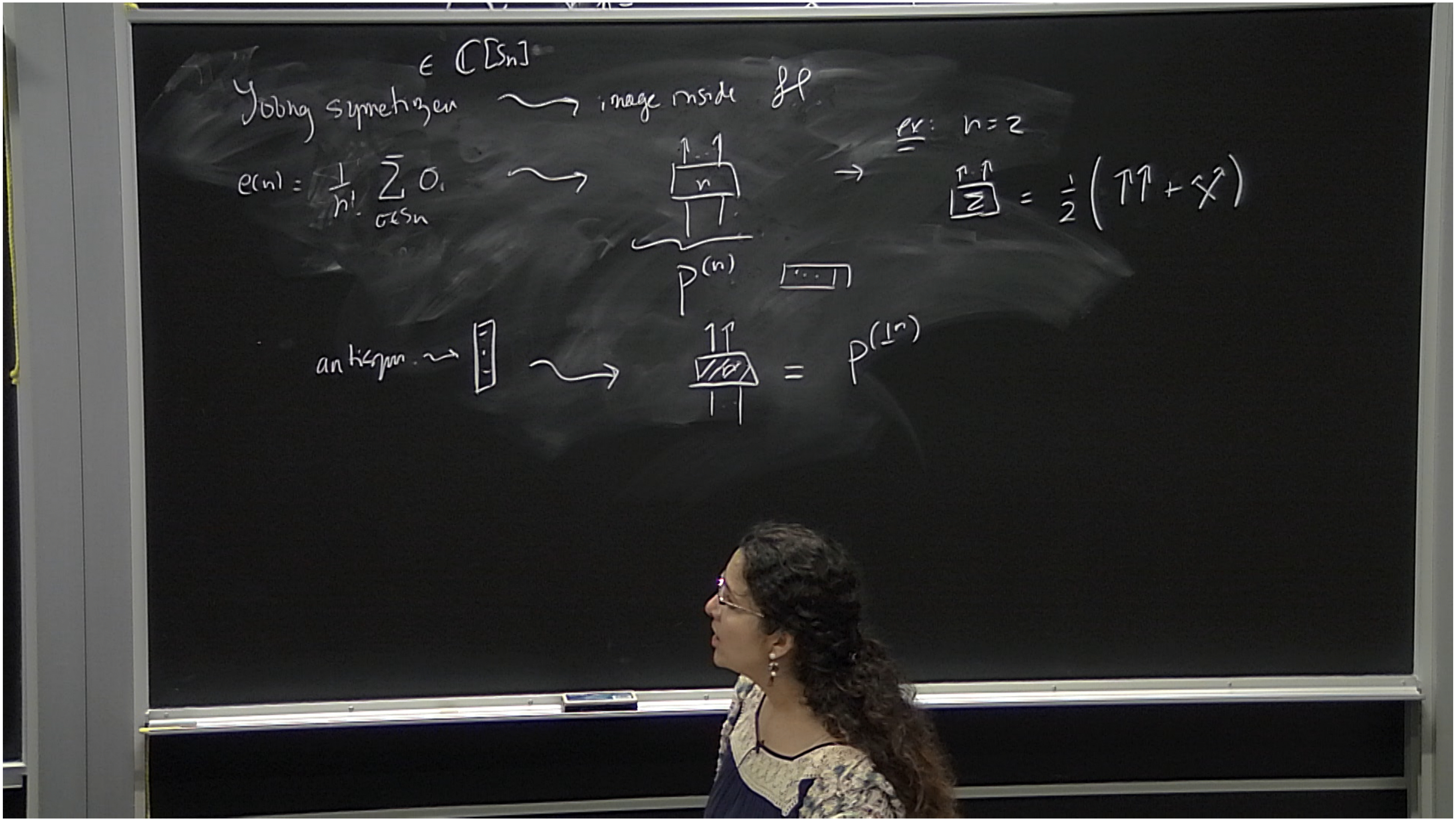
$(\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n)$

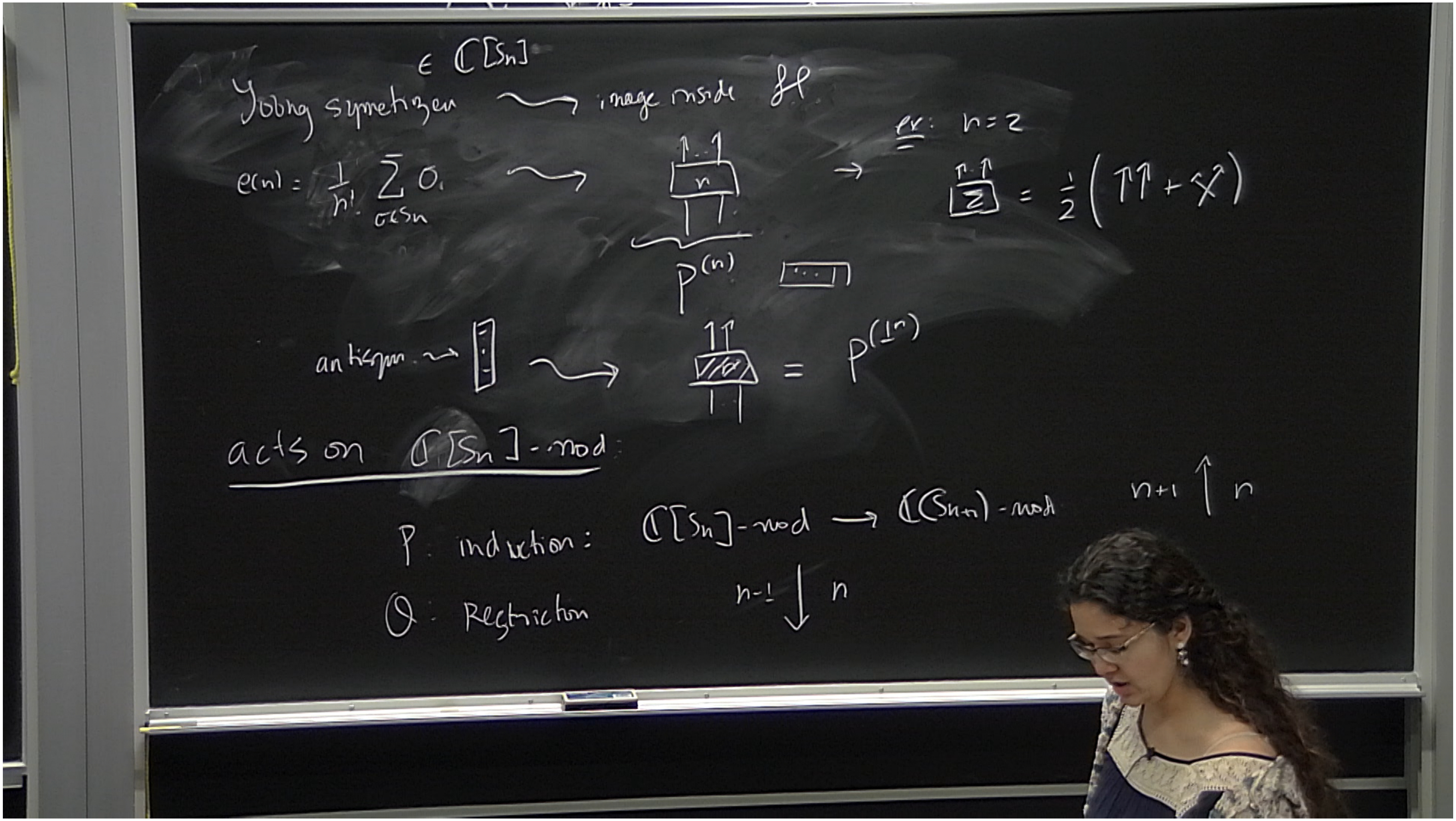
$$B_{\lambda_1} B_{\lambda_2} \ldots B_{\lambda_n}(1) = S_\lambda \leftarrow \text{Schur function}$$

$$B_{\lambda_n}^* \ldots B_{\lambda_1}^*(S_\lambda) = 1$$


Thm (Geissinger)
Frob char
$K_0\left(\bigoplus_{n\in\mathbb{N}} \mathbb{C}[S_n]\text{-mod}\right) \cong Sym.$
Specht-module $S_\lambda \rightsquigarrow s_\lambda$ - schur function
ind $\rightsquigarrow$ algebra
restr. $\rightsquigarrow$ coalgebra
CAUTION


Khovanov's Heisenberg Cat:
obj: P = ↑ Q = ↓
mor:
Sn-action.


∈ ℂ[Sn]
Young symmetrizer ⟶ image inside ℋ
ex: n=2
e(n) = 1/n! Σ σ
σ∈Sn
n
2
= 1/2 (↑↑ + ✗)
P(n)
antisym.
= P(1n)


∈ ℂ[S_n]
Young symmetizer ⟶ image inside ℋ
ex: n=2
e(n) = 1/n! Σ_{σ∈S_n} σ
= 1/2 (
P^(n)
antisym.
= P^(1^n)
acts on ℂ[S_n]-mod:
P: induction: ℂ[S_n]-mod ⟶ ℂ[S_{n+1}]-mod
n+1 ↑ n
Q: Restriction
n-1 ↓ n

$P^{(n)}$

antisym. $\rightsquigarrow$ $\longrightarrow$ $= P^{(1^n)}$

acts on $\mathbb{C}[S_n]$-mod:

P: induction: $\mathbb{C}[S_n]\text{-mod} \longrightarrow \mathbb{C}[S_{n+1}]\text{-mod}$ $\quad {}_{n+1}\uparrow_{n}$

Q: Restriction $\quad {}_{n-1}\downarrow_{n}$

$\Rightarrow \quad P^{(\lambda)}(\mathbb{1}) = S_\lambda$

$Q(\mathbb{1}) = 0$

Categorify this

$+ \rightsquigarrow \oplus$

$- \rightsquigarrow$ homological shift

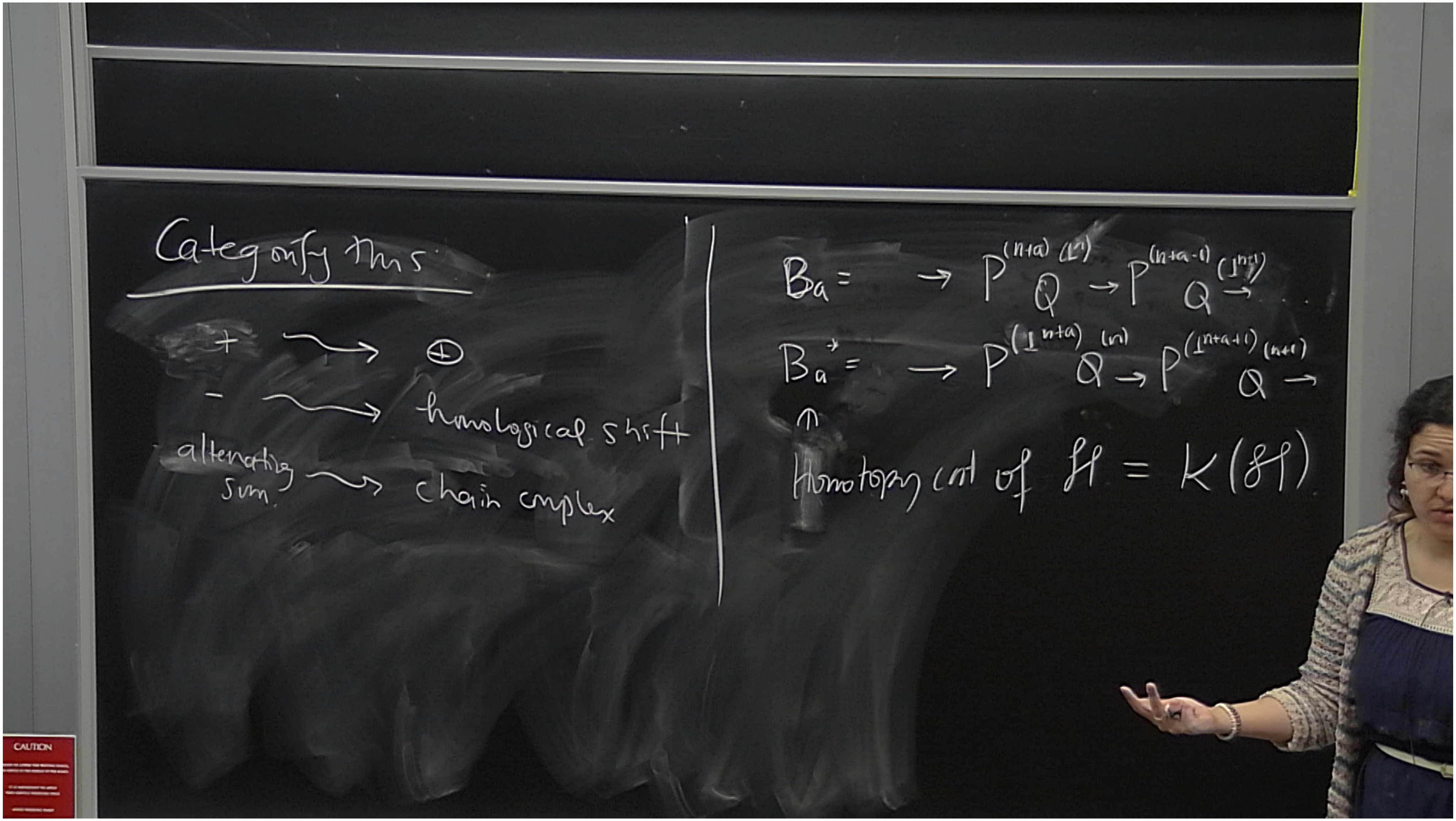
alternating sum $\rightsquigarrow$ chain complex

$$B_a = \to P^{(n+a)} Q^{(1^n)} \to P^{(n+a-1)} Q^{(1^{n-1})} \to$$

$$B_a^* = \to P^{(1^{n+a})} Q^{(n)} \to P^{(1^{n+a+1})} Q^{(n+1)} \to$$

$\Uparrow$

Homotopy cat of $\mathcal{SB} = K(\mathcal{SB})$


Categorify this
+ ⟶ ⊕
− ⟶ homological shift
alternating sum ⟶ chain complex
$B_a = \to P^{(n+a)} Q^{(1^n)} \to P^{(n+a-1)} Q^{(1^{n+1})} \to \cdots$
$B_a^* = \to P^{(1^{n+a})} Q^{(n)} \to P^{(1^{n+a+1})} Q^{(n+1)} \to$
Homotopy cat of $\mathcal{H} = K(\mathcal{H})$

Categorify this

$+ \rightsquigarrow \oplus$

$- \rightsquigarrow$ homological shift

alternating sum $\rightsquigarrow$ chain complex

$$B_a = \; \to P^{(n+a)} Q^{(1^n)} \to P^{(n+a-1)} Q^{(1^{n-1})} \to \cdots$$

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$\Uparrow$

Homotopy cat of $\mathcal{H} = K(\mathcal{H})$

Thm (G) $\lambda \vdash n$.

$$B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}(\mathbb{1}) = P^{(\lambda)}(\mathbb{1})$$

Categorify this

$+ \rightsquigarrow \oplus$

$- \rightsquigarrow$ homological shift

alternating sum $\rightsquigarrow$ chain complex

Thm: Cat's Bernstein ops satisfy categorical analogues of the Clifford-like relation

$B_a = \cdots \to P \; Q \to P^{(1^m)} \; Q \to \cdots$

$B_a^* = \; \to P^{(1^{n+a})} Q^{(n)} \to P^{(1^{n+a+1})} Q^{(n+1)} \to$

Homotopy cat of $\mathcal{H} = K(\mathcal{H})$

Thm (G). $\lambda \vdash n$.

$$B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}(\mathbb{1}) \cong P^{(\lambda)}(\mathbb{1})$$

$$V = \frac{11}{2} \wedge \frac{7}{2} \wedge \frac{3}{2} \wedge -\frac{1}{2} \wedge -\frac{5}{2} \wedge -\frac{7}{2} \wedge -\frac{9}{2} \wedge \cdots$$

$$\underbrace{9/2 \quad 5/2 \quad \tfrac{1}{2}}_{3} \qquad \underbrace{-3/2}_{1}$$

$3 - 1 = 2$ change.

Span of vectors = Fermionic Fock, $\Lambda^{\infty/2}$

$\Rightarrow$

$\langle f^{-1}u, v\rangle = \langle u, fv\rangle$

$$V = \frac{11}{2} \wedge \frac{7}{2} \wedge \frac{3}{2} \wedge -\frac{1}{2} \wedge -\frac{5}{2} \wedge -\frac{7}{2} \wedge -\frac{9}{2} \wedge \cdots$$

$$\underbrace{\tfrac{9}{2} \quad \tfrac{5}{2} \quad \tfrac{1}{2}}_{3} \; - \; \underbrace{-\tfrac{3}{2}}_{1} = 2 \text{ charge.}$$

Span of vectors = Fermionic Fock, $\Lambda^{\infty/2}$

$$\Rightarrow \quad \mathcal{F} = \bigoplus_{q \in \mathbb{Z}} \mathcal{F}_q$$

q: charge.

$$\langle f^{-1}u, v\rangle = \langle u, fv\rangle$$

$$\psi_j(i_1 \wedge i_2 \wedge \dots) = \begin{cases} (-1)^s\, i_1 \wedge \dots \; i(j-\frac{1}{2}) \wedge \dots & \text{if } i_s \neq j-\frac{1}{2} \\ 0 & \text{if } i_s = j-\frac{1}{2} \end{cases}$$

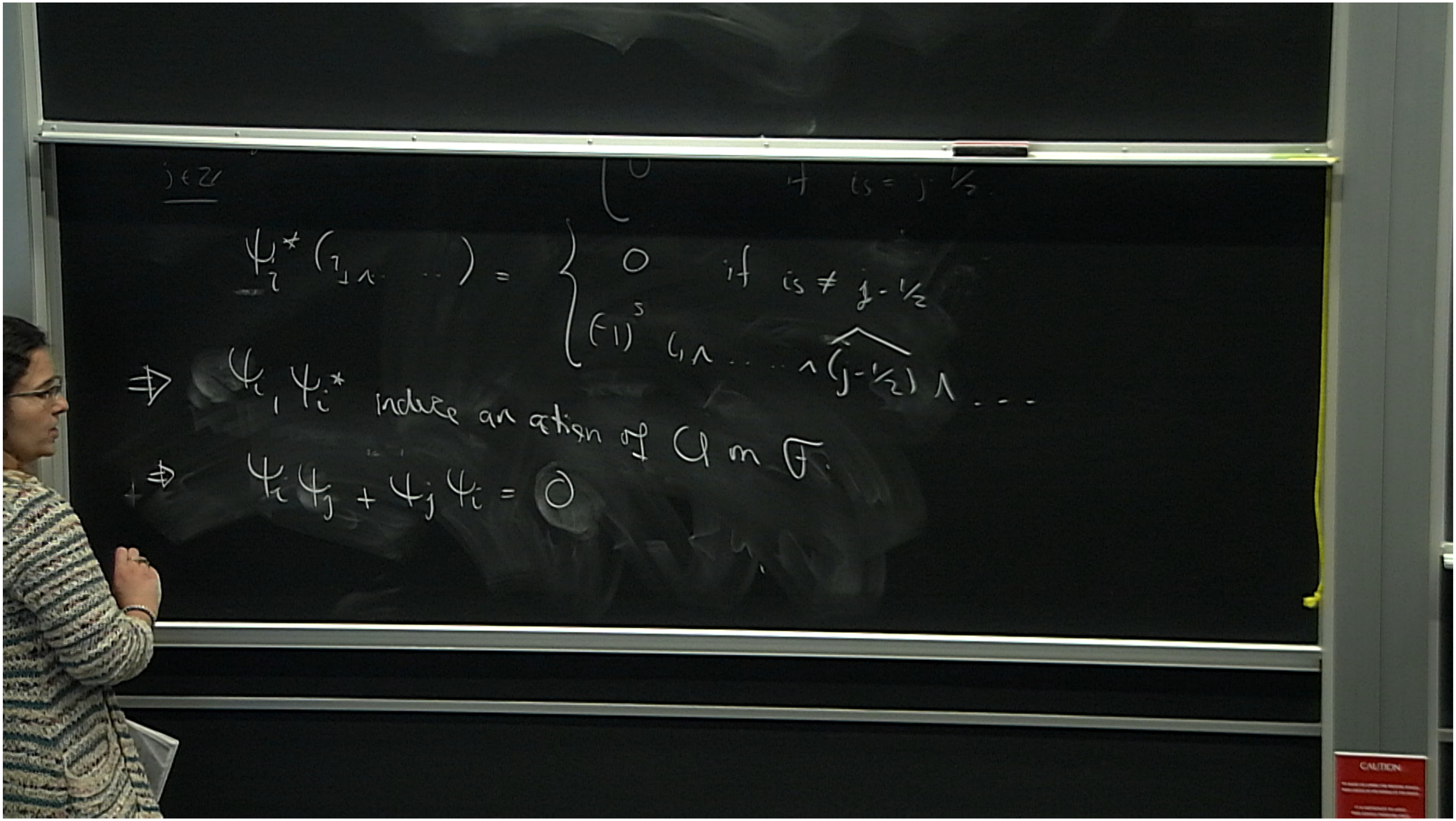
$j \in \mathbb{Z}'$

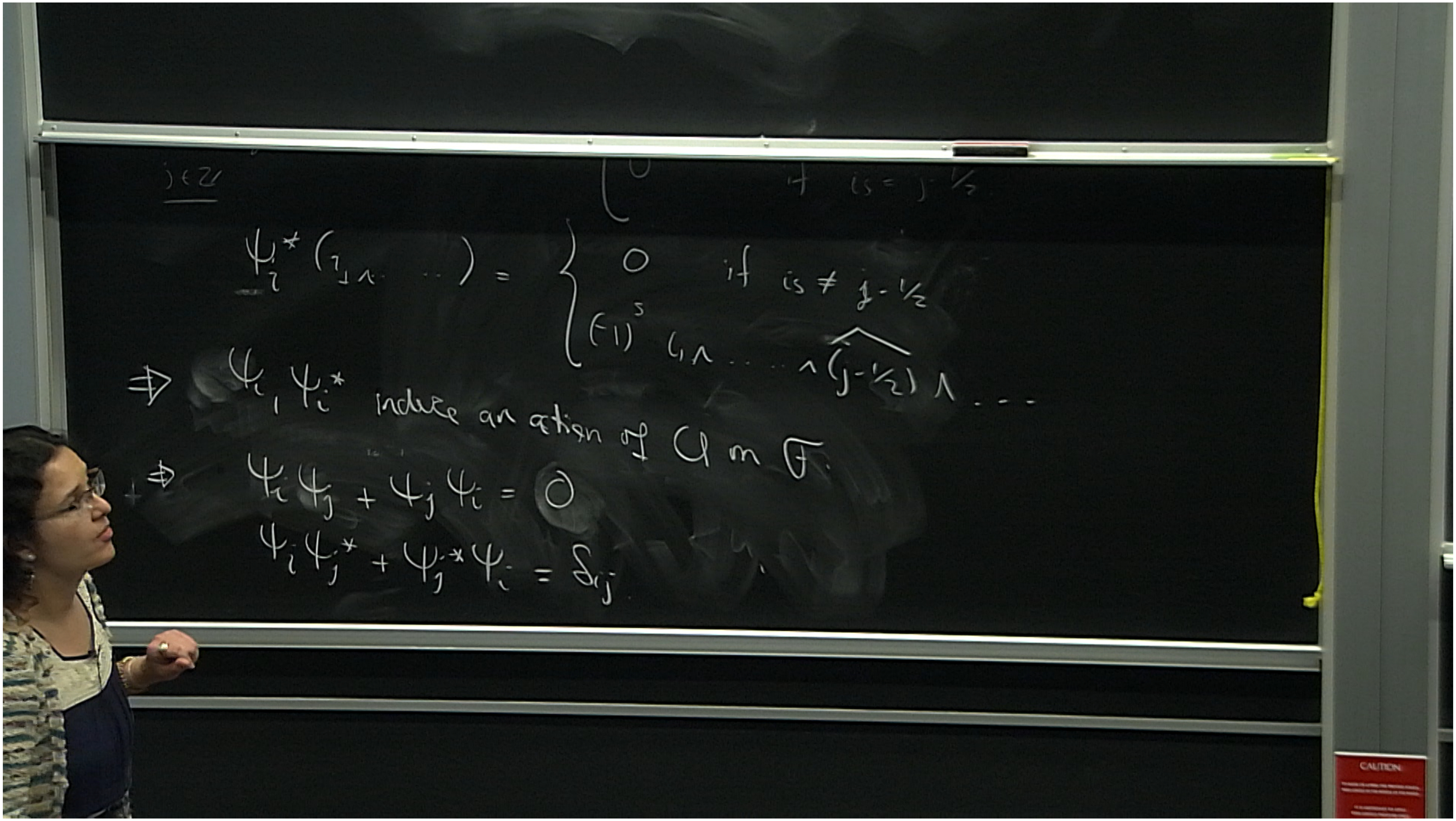
$$\psi_j^*(i_1 \wedge \dots) = \begin{cases} 0 & \text{if } i_s \neq j-\frac{1}{2} \\ & \end{cases}$$

$$\psi_j(i_1 \wedge i_2 \wedge \ldots) = \begin{cases} (-1)^s\, i_1 \wedge \ldots \wedge (j-\tfrac{1}{2}) \wedge \ldots & \text{if } i_s \neq j-\tfrac{1}{2} \\ 0 & \text{if } i_s = j-\tfrac{1}{2} \end{cases}$$

$\underline{j \in \mathbb{Z}}$

$$\psi_j^*(i_1 \wedge \ldots) = \begin{cases} 0 & \text{if } i_s \neq j-\tfrac{1}{2} \\ (-1)^s\, i_1 \wedge \ldots \wedge \widehat{(j-\tfrac{1}{2})} \wedge \ldots \end{cases}$$


$\psi_i^*(i_1 \wedge \ldots) = \begin{cases} 0 & \text{if } i_s \neq j - 1/2 \\ (-1)^s \, i_1 \wedge \ldots \wedge \widehat{(j-1/2)} \wedge \ldots \end{cases}$
$\Rightarrow$ ψ_i, ψ_i^* induce an action of Cl on $\mathcal{F}$:
$\Rightarrow$ $\psi_i \psi_j + \psi_j \psi_i = 0$


j ∈ Z
if i_s = j - 1/2
ψ_i^* (i_1 ∧ ...) = 0 if i_s ≠ j - 1/2
(-1)^s i_1 ∧ ... ∧ (j - 1/2)^ ∧ ...
⇒ ψ_i , ψ_i^* induce an action of Cl on F.
⇒ ψ_i ψ_j + ψ_j ψ_i = 0
ψ_i ψ_j^* + ψ_j^* ψ_i = δ_ij
CAUTION

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}.$$

$$V = \frac{11}{2} \wedge \frac{7}{2} \wedge \frac{3}{2} \wedge -\frac{1}{2} \wedge -\frac{5}{2} \wedge -\frac{7}{2} \wedge \frac{-9}{2} \wedge \cdots$$

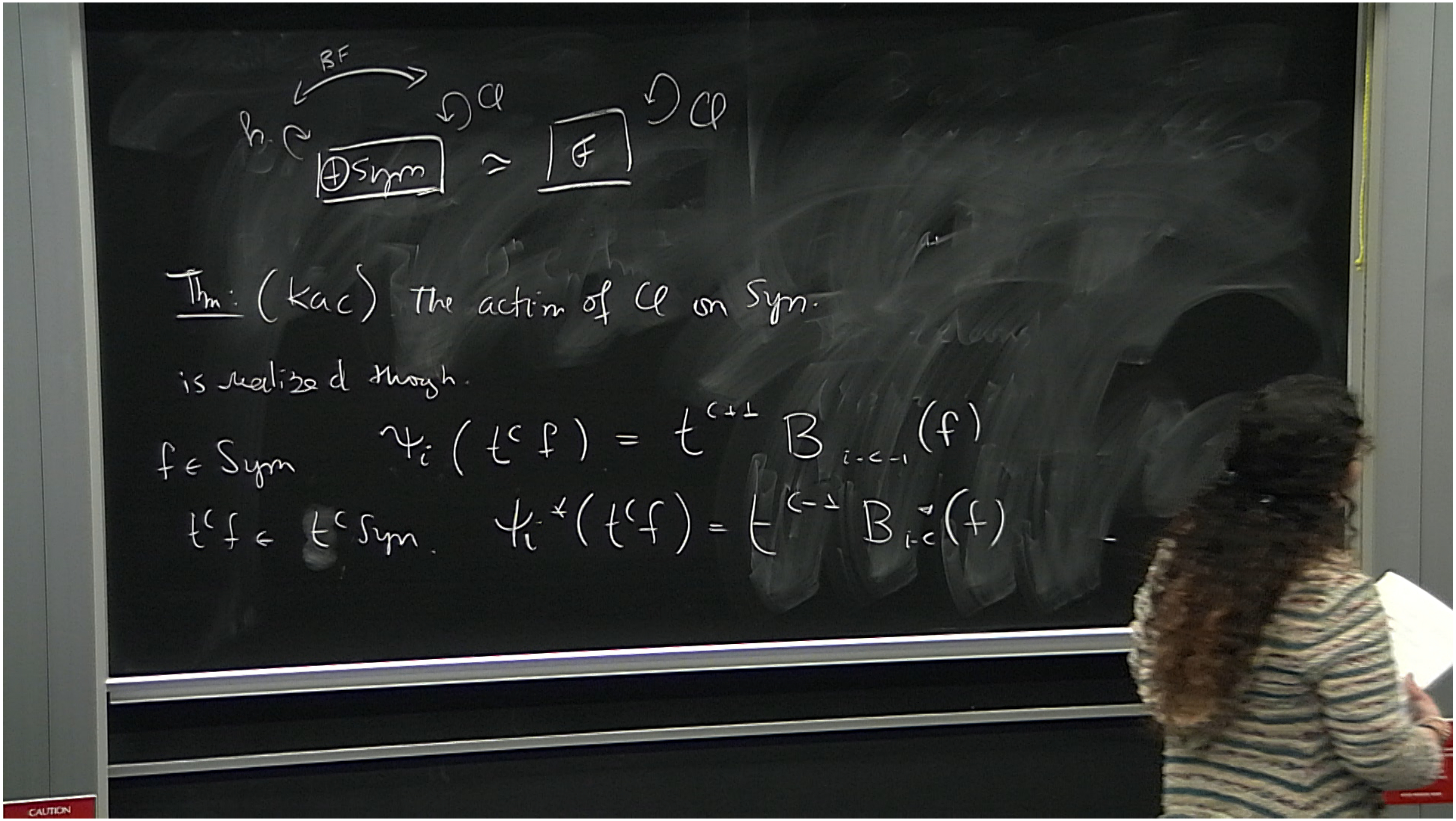
$$\underbrace{\uparrow_{9/2} \quad \uparrow_{5/2} \quad \uparrow_{\frac{1}{2}}}_{3} \qquad - \qquad \underbrace{\uparrow_{-3/2}}_{1} \;=\; 2 \text{ charge.}$$

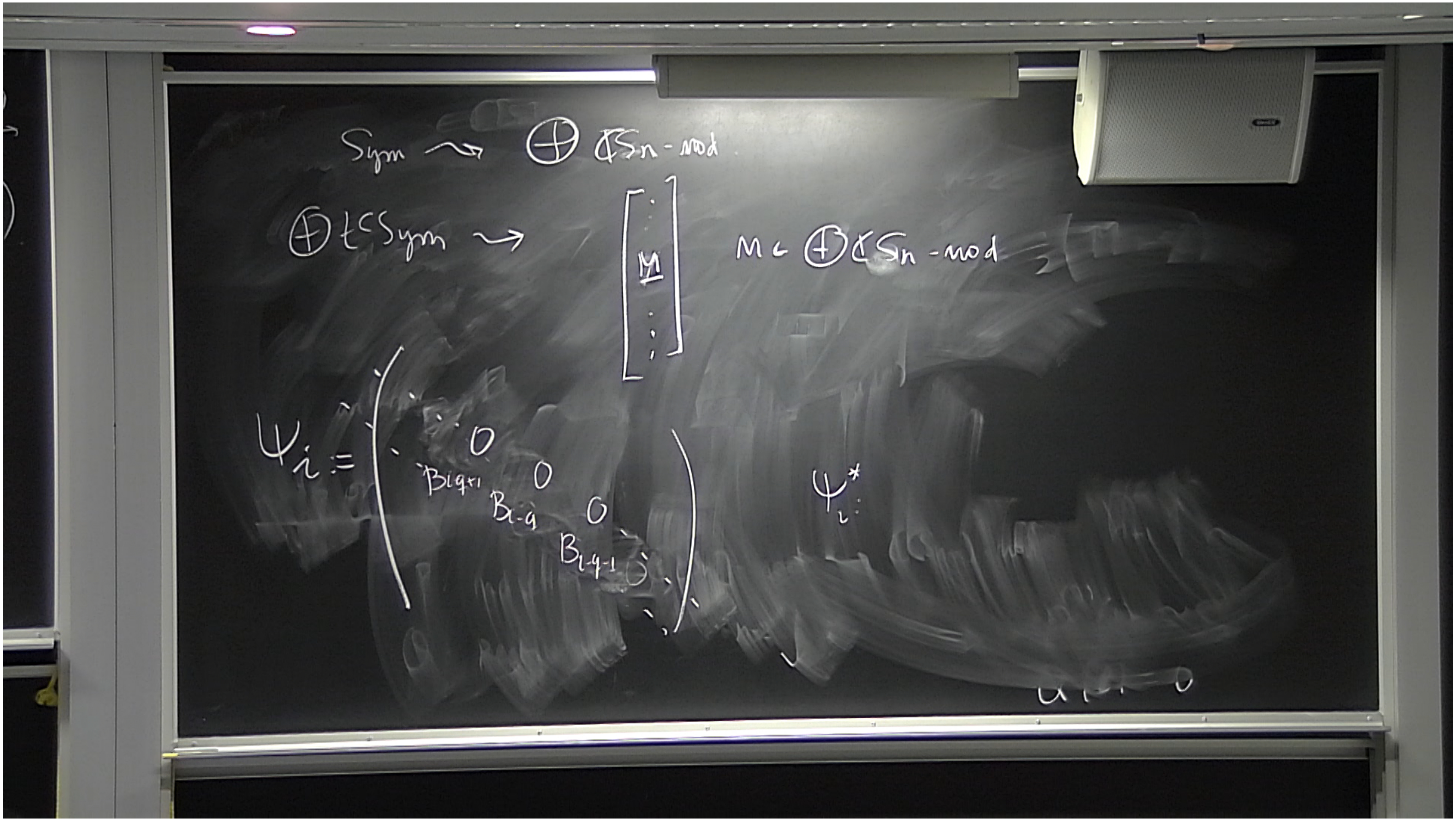
Span of vectors = Fermionic Fock, $\Lambda^{\infty/2}$

$$\Rightarrow \quad \mathcal{F} = \bigoplus_{q \in \mathbb{Z}} \mathcal{F}_q \quad \overset{\text{BF}}{\simeq} \quad \bigoplus_{q \in \mathbb{Z}} \mathrm{Sym}$$

q charge.

$\mathcal{F}_q \simeq \mathrm{Sym}$


BF
$h \circlearrowright \boxed{\oplus \mathrm{Sym}} \simeq \boxed{F} \circlearrowleft Cl$
Thm: (Kac) The action of Cl on Sym.
is realized through
$f \in \mathrm{Sym}$ $\psi_i(t^c f) = t^{c+1} B_{i-c-1}(f)$
$t^c f \in t^c \mathrm{Sym}$. $\psi_i^*(t^c f) = t^{c-1} B^*_{i-c}(f)$
CAUTION


Sym ⇝ ⊕ ℂS_n - mod
⊕ t^c Sym ⇝
M
M ∈ ⊕ ℂS_n - mod
Ψ_i =
0
B_{i,q+1} 0
B_{i,q} 0
B_{i,q-1} 0
Ψ_i^*

Thm: (G)

$$\psi_i \psi_j \simeq \begin{cases} \psi_j \psi_i [-1] & i>j \\ \psi_j \psi_i [+1] & i<j \\ 0 & i=j \end{cases}$$

$$\psi_i \psi_j^* \simeq \begin{cases} \psi_j^* \psi_i [1] & i>j \\ \psi_j^* \psi_i [-1] & i<j \end{cases}$$

$$\psi_i \psi_i^* \rightleftarrows \mathrm{Id} \rightleftarrows \psi_i^* \psi_i$$

distinguished