

Title: Categorical Bernstein Operators and the Boson-Fermion correspondence.

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Abstract: Bernstein operators are vertex operators that create and annihilate Schur polynomials. These operators play a significant role in the mathematical formulation of the Boson-Fermion correspondence due to Kac and Frenkel. The role of this correspondence in mathematical physics has been widely studied as it bridges the actions of the infinite Heisenberg and Clifford algebras on Fock space. Cautis and Sussan conjectured a categorification of this correspondence within the framework of Khovanov's Heisenberg category. I will discuss how to categorify the Bernstein operators and settle the Cautis-Sussan conjecture, thus proving a categorical Boson-Fermion correspondence.

90's Frenkel \rightarrow CFT \rightarrow Boson-Fermion Correspondence.

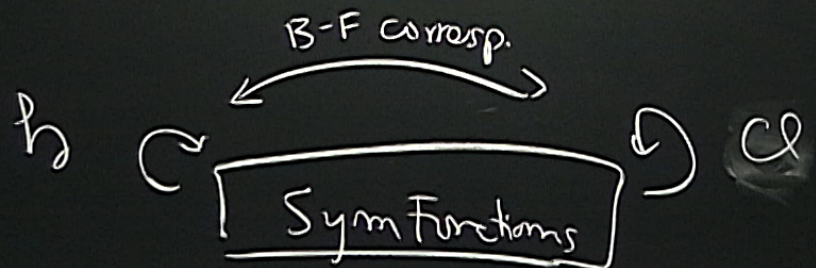
2011: Curtis-Licata \rightarrow constructed 1st ex. \rightarrow cat. vertex ops.

over a Heisenberg cat.

2013 Curtis-Sussan \rightarrow conj. a Cat-Bosm, Fem. correspondence.

2011: Cantis-Licata \rightarrow constructed $|\mathcal{S}\rangle$ ex. \rightarrow cat. vertex ops.
over a Heisenberg cat.

2013 Curtis-Sussan \rightarrow conj. a Cat-Bosm, Fen. correspondence



Sym funct

Sym functions = Sym.

gen \Rightarrow

p_n	power sum	\mathbb{Q}
h_n	homog	\mathbb{Z}
e_n	elementary	\mathbb{Z}
s_n	schur	\mathbb{Z}

Sym functions = Sym.

gen \Rightarrow p_n power sum / \mathbb{Q}

h_n homogen / \mathbb{Z}

e_n elementary / \mathbb{Z}

s_λ Schur / \mathbb{Z}

hall inner product

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}.$$

Sym functions = Sym.

gen \Rightarrow p_n power sum / $\mathbb{Q} \sim p_n^{\perp} \in \text{End}(\text{Sym})$

h_n homogen / $\mathbb{Z} \sim h_n^{\perp}$

e_n elementary / $\mathbb{Z} \sim e_n^{\perp}$

s_{λ} Schur / $\mathbb{Z} \sim s_{\lambda}^{\perp}$

Hall inner product $\langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda, \mu}$.

$f \in \text{Sym}, f^{\perp} \Rightarrow \langle f^{\perp} u, v \rangle = \langle u, f v \rangle$

Sym functions = Sym.

gen \Rightarrow p_n power sum / $\mathbb{Q} \sim p_n^t \in \text{End}(\text{Sym})$

h_n homogen / \mathbb{Z} h_n^t

e_n elementary / \mathbb{Z} e_n^t

S_λ Schur / \mathbb{Z} S_λ^t

hall inner product $\langle S_\lambda, S_\mu \rangle = \delta_{\lambda, \mu}$.

$f \in \text{Sym}, f^t \Rightarrow \langle f^t u, v \rangle = \langle u, f v \rangle$

$p_n \curvearrowright \text{Sym} \curvearrowleft p_n^\dagger$

$$[p_n, p_m^\dagger] = \hbar \delta_{n,m}$$

\Rightarrow Heisenberg alg acts on Sym

\Rightarrow Integral generators

$$p_n \curvearrowright \text{Sym} \curvearrowleft p_n^\dagger$$

$$[p_n, p_m^\dagger] = \hbar \delta_{n,m}$$

\Rightarrow Heisenberg alg acts on Sym

\Rightarrow Integral generators

$$\begin{array}{ccc} \Rightarrow & p^{(n)} & q^{(n)} \\ (n) \boxed{} & \downarrow & \downarrow \\ & h_n & h_n^\dagger \end{array}$$

$$p_n \curvearrowright \text{Sym} \curvearrowleft p_n^\dagger$$

$$[p_n, p_m^\dagger] = \hbar \delta_{n,m}$$

\Rightarrow Heisenberg alg acts on Sym

\Rightarrow Integral generators

$$\begin{array}{ccc} \Rightarrow & \left\{ p^{(n)}, q^{(n)} \right\} & \left\{ p^{(1^n)}, q^{(1^n)} \right\} \\ \downarrow & \downarrow & \downarrow \\ (n) & \hbar_n & e_n \\ & \hbar_n^\dagger & e_n^\dagger \end{array}$$

$$p_n \curvearrowright \text{Sym} \curvearrowleft p_n^\perp$$

$$[p_n, p_m^\perp] = \hbar \delta_{n,m}$$

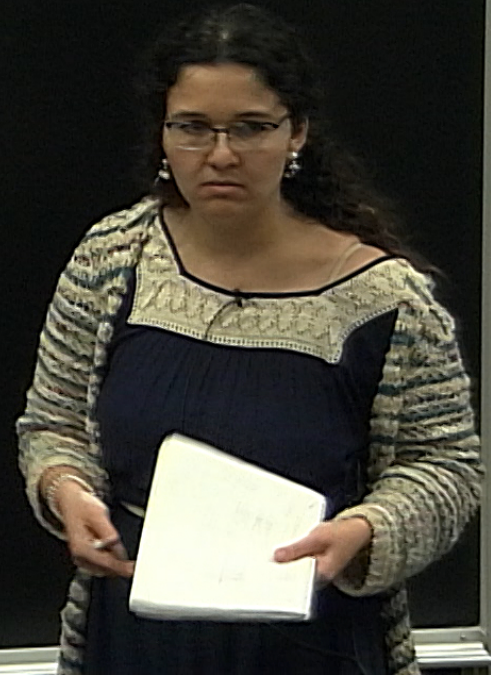
\Rightarrow Heisenberg alg acts on Sym

\Rightarrow Integral generators

$$\Rightarrow \left\{ \begin{array}{l} p^{(n)} \\ \downarrow \\ \hbar_n \end{array} \right\}, \left\{ \begin{array}{l} q^{(n)} \\ \downarrow \\ \hbar_n^\perp \end{array} \right\}$$

$$\left\{ \begin{array}{l} p^{(\perp n)} \\ \downarrow \\ e_n \end{array} \right\}, \left\{ \begin{array}{l} q^{(\perp n)} \\ \downarrow \\ e_n^\perp \end{array} \right\}$$

$$\sum_{m \in \mathbb{N}} \hbar_m z^m = \exp\left(\sum_{n \geq 0} \frac{p_n}{\hbar} z^n\right)$$



CAUTION
 We advise you to use the correct safety
 procedures when using the equipment in this
 laboratory.

Bernstein ops $a \in \mathbb{Z}$

$$B_a = \sum_{n-m=a} (-1)^m h_n e_m^\perp$$

$$B_a^* = \sum_{m-n=a} (-1)^n e_n h_m^\perp$$

h_n h_{n+1} e_n e_{n+1}

Bernstein ops $a \in \mathbb{Z}$

$$B_a = \sum_{n-m=a} (-1)^m h_n e_m^\perp$$

$$B_a^\vee = \sum_{m-n=a} (-1)^n e_n h_m^\perp$$

$$B_{a-1} B_b + B_{b-1} B_a = 0$$

$$B_{a+1}^\vee B_b^\vee + B_{b+1}^\vee B_a^\vee = 0$$

$$B_{a+1} B_{b+1}^\vee + B_b^\vee B_a = S_{a,b}$$

CAUTION

Bernstein ops $a \in \mathbb{Z}$

$$B_a = \sum_{n-m=a} (-1)^m h_n e_m^\perp$$

$$B_a^* = \sum_{m-n=a} (-1)^n e_n h_m^\perp$$

$$B_{a-1} B_b + B_{b-1} B_a = 0$$

$$B_{a+1}^* B_b^* + B_{b+1}^* B_a^* = 0$$

$$B_{a+1} B_{b+1}^* + B_b^* B_a = S_{a,b}$$

Thm (Zelevinsky)

$$(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$$

$$B_{\lambda_1} B_{\lambda_2} \dots B_{\lambda_n}(1) = S_\lambda \leftarrow \text{Schur function}$$

$$B_{\lambda_n}^* \dots B_{\lambda_1}^*(S_\lambda) = 1$$

$$\begin{array}{c} \text{Thm (Geissinger)} \\ \cong \\ K_0 \left(\bigoplus_{n \in \mathbb{N}} ([S_n] \text{-mod}) \right) \cong \text{Sym.} \end{array}$$

free char.

↓

Specht-module $S_\lambda \rightsquigarrow S_\lambda$ -Schur function

ind \rightsquigarrow algebra
 res \rightsquigarrow coalgebra



CAUTION
 For further information see the following notice.
 Notice posted in the vicinity of the board.

Khorrami's Heisenberg Cat

obj $P = \uparrow \quad Q = \downarrow$

mov $X, \curvearrowright, \cup, \cap, \cup$

nodes $\left. \begin{array}{l} \text{Diagram 1} = \uparrow\uparrow \quad \text{Diagram 2} = \text{Diagram 3} \end{array} \right\} \text{Sn-action.} \quad \uparrow = 1$

$\text{Diagram 4} = \uparrow\uparrow \begin{array}{l} \cup \\ \cap \end{array} ; \quad \text{Diagram 5} = \uparrow\downarrow \quad \uparrow = 0 \quad \downarrow = 1$

$e \in \mathbb{C}[S_n]$
 Young symmetrizer \rightsquigarrow image inside \mathcal{H}

$$e^{(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$$
 \rightsquigarrow

$$P^{(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$$

ex: $n=2$

$$\frac{1}{2} \sum_{\sigma \in S_2} \sigma = \frac{1}{2} (\uparrow\uparrow + \uparrow\downarrow)$$

anticommutator \rightsquigarrow

$$\frac{1}{2} (\uparrow\uparrow + \uparrow\downarrow) = P^{(1^n)}$$



$e(n) \in \mathbb{C}[S_n]$
 Young symmetrizer \rightsquigarrow image inside $\mathbb{C}\mathbb{P}$
 $e(n) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$ \rightsquigarrow $\begin{array}{c} \uparrow \uparrow \\ \boxed{n} \\ \uparrow \uparrow \end{array}$ \rightarrow ex: $n=2$
 $\begin{array}{c} \uparrow \uparrow \\ \boxed{2} \\ \uparrow \uparrow \end{array} = \frac{1}{2} (\uparrow\uparrow + \swarrow\searrow)$
 $p^{(n)} = \dots$
 anticommutator \rightsquigarrow $\begin{array}{c} \uparrow \uparrow \\ \boxed{\vdots} \\ \uparrow \uparrow \end{array} = p^{(1^n)}$

acts on $\mathbb{C}[S_n]$ -mod:

$\mathbb{C}[S_n]$ -mod \rightarrow $\mathbb{C}[S_{n+1}]$ -mod $\begin{array}{c} n+1 \\ \uparrow \\ n \end{array}$
 P. induction:
 Q. Restriction $\begin{array}{c} n-1 \\ \downarrow \\ n \end{array}$



antisym \rightsquigarrow $\begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \rightsquigarrow$ $\begin{matrix} P^{(n)} & \boxed{\dots} \\ \uparrow \uparrow \\ \boxed{\uparrow \downarrow} & = P^{(1^n)} \\ \uparrow \uparrow \end{matrix}$

acts on $\mathbb{C}[S_n]$ -mod:

P. induction: $\mathbb{C}[S_n]$ -mod \rightarrow $\mathbb{C}[S_{n+1}]$ -mod $\begin{matrix} n+1 \uparrow \\ n \end{matrix}$

Q. Restriction $\begin{matrix} n-1 \downarrow \\ n \end{matrix} \Rightarrow \begin{matrix} P^{(1^n)}(\mathbb{1}) = S_n \\ Q(\mathbb{1}) = 0 \end{matrix}$

Category this

$+$ \rightsquigarrow \oplus

$-$ \rightsquigarrow homological shift

alternating sum \rightsquigarrow chain complex

$$B_a = \rightarrow P^{(n+a)} \begin{matrix} (n) \\ Q \end{matrix} \rightarrow P^{(n+a-1)} \begin{matrix} (n+1) \\ Q \end{matrix} \rightarrow$$

$$B_a^{\rightarrow} = \rightarrow P^{(1+n+a)} \begin{matrix} (n) \\ Q \end{matrix} \rightarrow P^{(1+n+a+1)} \begin{matrix} (n+1) \\ Q \end{matrix} \rightarrow$$

\Uparrow

Homotopy cat of $\mathcal{H} = K(\mathcal{H})$

Category \mathcal{M}_n

$+$ \rightsquigarrow \oplus
 $-$ \rightsquigarrow homological shift
alternating sum \rightsquigarrow chain complex

$$B_a = \rightarrow P^{(n+a)} Q \rightarrow P^{(n+a-1)} Q \rightarrow \dots$$
$$B_a^{\rightarrow} = \rightarrow P^{(1+n+a)} Q \rightarrow P^{(1+n+a+1)} Q \rightarrow \dots$$

\Uparrow
Homotopy cat of $\mathcal{H} = K(\mathcal{H})$

Category \mathcal{M}_R

$$+ \rightsquigarrow \oplus$$

$- \rightsquigarrow$ homological shift

alternating sum \rightsquigarrow chain complex

$$B_a = \rightarrow P^{(n+a)} Q \rightarrow P^{(n+a-1)} Q \rightarrow \dots$$

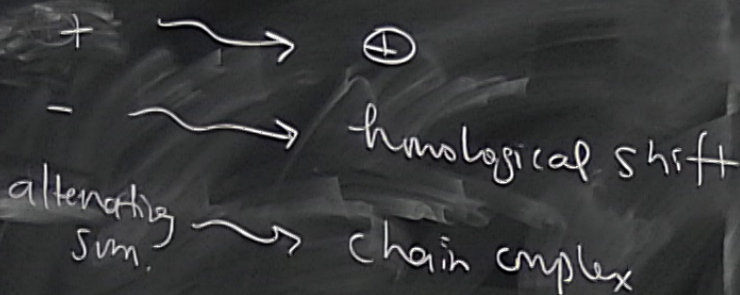
$$B_a^\rightarrow = \rightarrow P^{(1+n+a)} Q \rightarrow P^{(1+n+a-1)} Q \rightarrow \dots$$

\Uparrow
Homotopy cat of $\mathcal{M} = K(\mathcal{M})$

Thm (G) $\lambda \perp n$

$$B_{\lambda} \otimes \dots \otimes B_{\lambda_n}(\perp) = P_{(\perp)}^{(a)}$$

Categorical Thms



Thm: Cat's Bernstein ops satisfy categorical analogues of the Clifford-like relation

$$\begin{aligned}
 B_a &= \rightarrow P \ Q \rightarrow P \ Q \rightarrow \dots \\
 B_a^+ &= \rightarrow P^{(1+n)a} \ Q^{(n)} \rightarrow P^{(1+n+1)a} \ Q^{(n+1)} \rightarrow \dots \\
 &\uparrow \\
 &\text{Homotopy cont of } \mathcal{H} = K(\mathcal{H})
 \end{aligned}$$

Thm (G) $\lambda \vdash n$

$$B_{\lambda,1} \otimes \dots \otimes B_{\lambda,n}(\mathbb{1}) \cong P_{(\mathbb{1})}^{(2)}$$

$$V = \frac{11}{2} \wedge \frac{7}{2} \wedge \frac{3}{2} \wedge -\frac{1}{2} \wedge -\frac{5}{2} \wedge -\frac{7}{2} \wedge -\frac{9}{2} \wedge \dots$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$\underbrace{\frac{9}{2} \quad \frac{5}{2} \quad \frac{1}{2}}_3 \quad - \quad \underbrace{-\frac{3}{2}}_1 = 2 \text{ change.}$$

Span of vectors = Fermionic Fock, $\wedge^{\infty/2}$

\Rightarrow

$$\langle f^{\pm} u, v \rangle = \langle u, f v \rangle$$

$$V = \frac{11}{2} \wedge \frac{7}{2} \wedge \frac{3}{2} \wedge -\frac{1}{2} \wedge -\frac{5}{2} \wedge -\frac{7}{2} \wedge -\frac{9}{2} \wedge \dots$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$\underbrace{\frac{9}{2} \quad \frac{5}{2} \quad \frac{1}{2}}_3 \quad - \quad \underbrace{-\frac{3}{2}}_1 = 2 \text{ change.}$$

Span of vectors = Fermionic Fock, $\Lambda^{\infty/2}$

$$\Rightarrow \mathcal{F} = \bigoplus_{q \in \mathbb{Z}} \mathcal{F}_q$$

\uparrow change.

$$\langle f^{-1}u, v \rangle = \langle u, fv \rangle$$

$$\psi_j(z_1, z_2, \dots) = \begin{cases} (-1)^s z_1^s \dots z_{j-1/2}^s & \text{if } s \neq j - \frac{1}{2} \\ 0 & \text{if } s = j - \frac{1}{2} \end{cases}$$

$$\psi_j^*(z_1, z_2, \dots) = \begin{cases} 0 & \text{if } s \neq j - \frac{1}{2} \end{cases}$$

$$\underline{j \in \mathbb{Z}}$$

$$\Psi_j(r_1 \wedge r_2 \wedge \dots) = \begin{cases} (-1)^s r_1 \wedge \dots \wedge (j - \frac{1}{2}) \wedge \dots & \text{if } s \neq j - \frac{1}{2} \\ 0 & \text{if } s = j - \frac{1}{2} \end{cases}$$

$$\Psi_j^*(r_1 \wedge \dots) = \begin{cases} 0 & \text{if } s \neq j - \frac{1}{2} \\ (-1)^s r_1 \wedge \dots \wedge \widehat{(j - \frac{1}{2})} \wedge \dots \end{cases}$$

$j \in \mathbb{Z}$

ψ $\psi = j - 1/2$

$$\psi_j^* (\dots) = \begin{cases} 0 & \text{if } \psi \neq j - 1/2 \\ (F_1)^s \psi \dots \wedge (j - 1/2) \wedge \dots \end{cases}$$

$\Rightarrow \psi_i, \psi_i^*$ induce an action of $U_m \mathbb{F}$.

$$\psi_i \psi_j + \psi_j \psi_i = 0$$

$j \in \mathbb{Z}$

$(\dots) + (s = j - 1/2)$

$$\psi_i^* (\dots) = \begin{cases} 0 & \text{if } i \neq j - 1/2 \\ (F1)^s \dots \wedge (j - 1/2) \wedge \dots \end{cases}$$

$\Rightarrow \psi_i, \psi_i^*$ induce an action of $U_m \mathcal{F}$.

$$\psi_i \psi_j + \psi_j \psi_i = 0$$

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}$$

$$\begin{array}{ccc}
 & \xrightarrow{BF} & \\
 h \curvearrowright & & \curvearrowright \mathcal{C} \\
 \boxed{\oplus \text{Sym}} & \simeq & \boxed{\mathcal{F}} \\
 & & \curvearrowright \mathcal{C}
 \end{array}$$

Thm: (Kac) The action of \mathcal{C} on Sym .

is realized through

$$f \in \text{Sym} \quad \psi_i(t^c f) = t^{c+1} B_{i-c-1}(f)$$

$$t^c f \in t^c \text{Sym} \quad \psi_i(t^c f) = t^{c-1} B_{i-c}(f)$$

$$\text{Sym} \cong \bigoplus \mathbb{C}[S_n\text{-mod}]$$

$$\bigoplus \mathbb{C}\text{Sym} \cong$$

$$\begin{bmatrix} \vdots \\ M \\ \vdots \end{bmatrix}$$

$$M \subset \bigoplus \mathbb{C}[S_n\text{-mod}]$$

$$\Psi_i = \begin{pmatrix} \vdots & 0 & & & \\ & B_{i+1} & 0 & & \\ & & B_i & 0 & \\ & & & B_{i-1} & 0 \\ & & & & B_{i-1} & \ddots \end{pmatrix}$$

$$\Psi_i^*$$

u p p o

a 1 1 0

Thm. (9) $\psi_i \psi_j \simeq \begin{cases} \psi_j \psi_i [-1] & i > j \\ \psi_j \psi_i [+1] & i < j \\ 0 & i = j \end{cases}$

$$\psi_i \psi_j^* \simeq \begin{cases} \psi_j^* \psi_i [1] & i > j \\ \psi_j^* \psi_i [-1] & i < j \end{cases}$$

$$\psi_i \psi_i^* \Rightarrow Id \Rightarrow \psi_i^* \psi_i$$

distinguished