Title: Untangling entanglement and chaos

Date: Jan 09, 2019 04:00 PM

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Abstract: <p>How does classical chaos affect the generation of quantum entanglement? What signatures of chaos exist at the quantum level and how can they be quantified? These questions have puzzled physicists for a couple of decades now. We answer these questions in spin systems by analytically establishing a connection between entanglement generation and a measure of delocalization of a quantum state in such systems. While delocalization is a generic feature of quantum chaotic systems, it is more nuanced in regular systems. We explore when the quantum dynamics mimics a localized classical trajectory, and find criteria to quantify Bohr's correspondence principle in periodically driven spin systems. These criteria are typically violated in a deep quantum regime due to delocalized evolution. Using our criteria, we establish that entanglement is a signature of chaos only in a semiclassical regime. Our work provides a new approach to analyzing quantum chaos and designing systems that can efficiently generate entanglement. This work has been done in collaboration with Prof. Shohini Ghose.</p>

<p>References: arXiv:1806.10545 (2018) and PRE 97, 052209 (2018).</p>

Untangling entanglement and chaos

Meenu Kumari
IQC, University of Waterloo, Canada

in collaboration with
Dr. Shohini Ghose

[arXiv:1806.10545, PRE 97, 052209 (2018)]

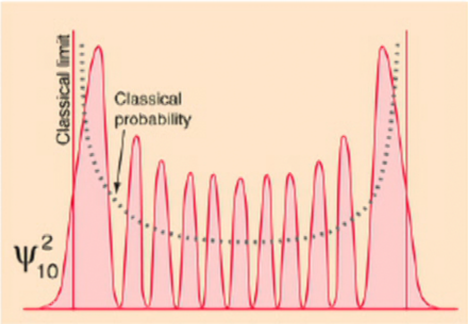
What would classical chaos, which lurks everywhere in our world, do to quantum mechanics, the theory describing the atomic and subatomic worlds?

- Martin Gutzwiller

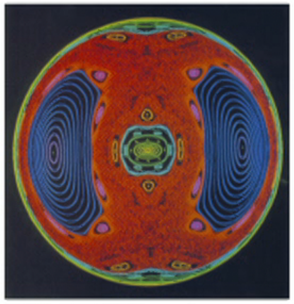
Motivation: Quantum-classical correspondence dichotomy

Semiclassical regime

Harmonic Oscillator[1]



H-atom in strong B [2]



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1 http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/hosc6.html

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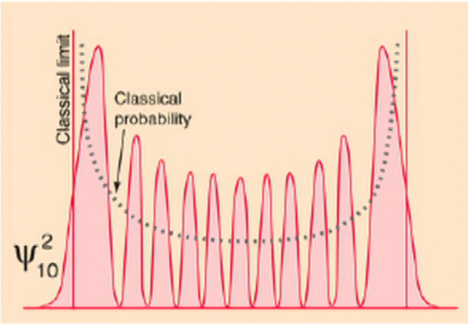

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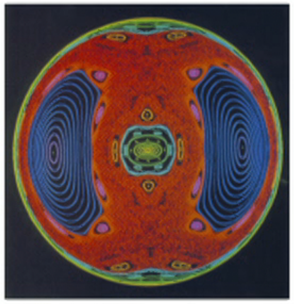
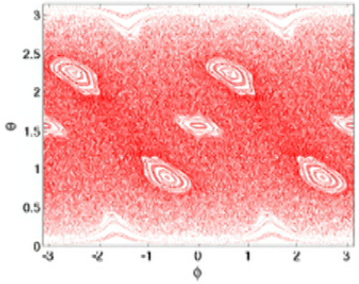
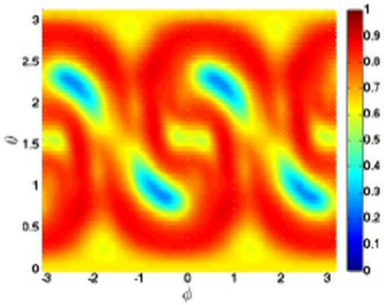


Deep quantum regime

Classical phase space



Entanglement



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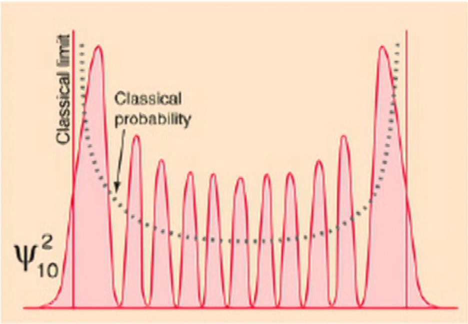

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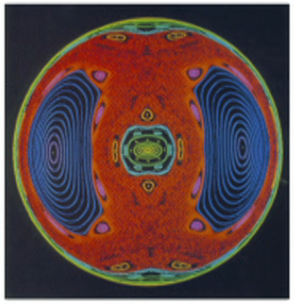

$\overset{?}{\Leftrightarrow}$

Deep quantum regime

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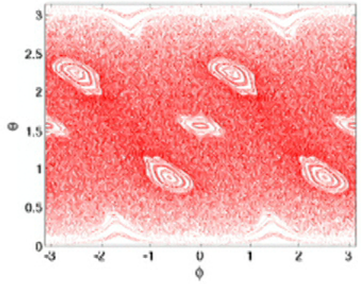
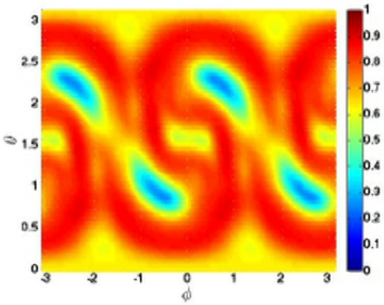


Entanglement



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Motivation: Entanglement vs classical dynamics?

High entanglement: Signature of chaos?		
Classical dynamics ↓	Deep quantum regime	Semiclassical regime
Chaotic	High	High
Regular	?	Low

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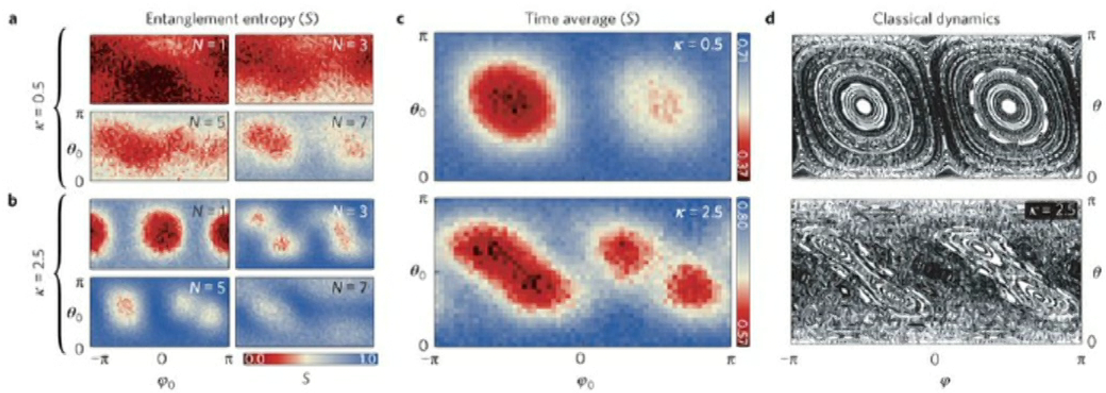
Experimental investigations:

- **Nature Physics 12, 1037 (2016):**
  In a 3-qubit kicked top,
  classically chaotic dynamics - high entanglement
  regular dynamics - low entanglement.



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Experimental investigations:

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  In a 3-qubit kicked top,
  classically chaotic dynamics - high entanglement
  regular dynamics - low entanglement.

Theoretical and numerical investigations:

- **Phys. Rev. E 95, 062222 (2017)**:
  In a 2-qubit kicked top,
  classically chaotic dynamics - medium level entanglement
  regular dynamics - either high or low entanglement.

Background

- Classical chaos
- Quantum-classical correspondence
- Chaotic model: Quantum Kicked Top (QKT)
- Entanglement in isolated spin systems?

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Classical Chaos

- Unpredictability due to sensitivity to initial conditions

[1] https://www.youtube.com/watch?v=˙i3WqnejOQk.

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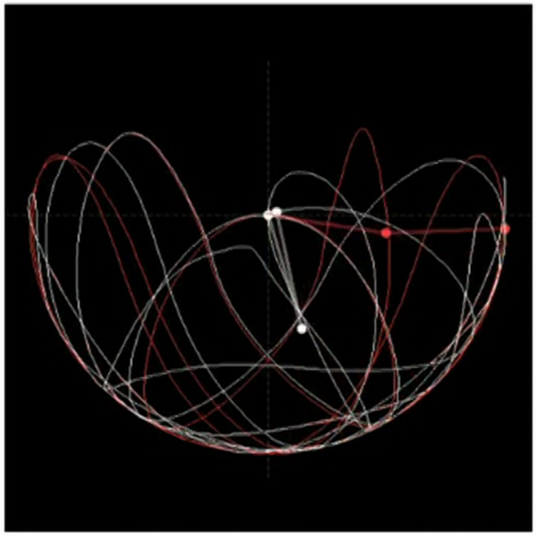

Figure: Double pendulum[1]

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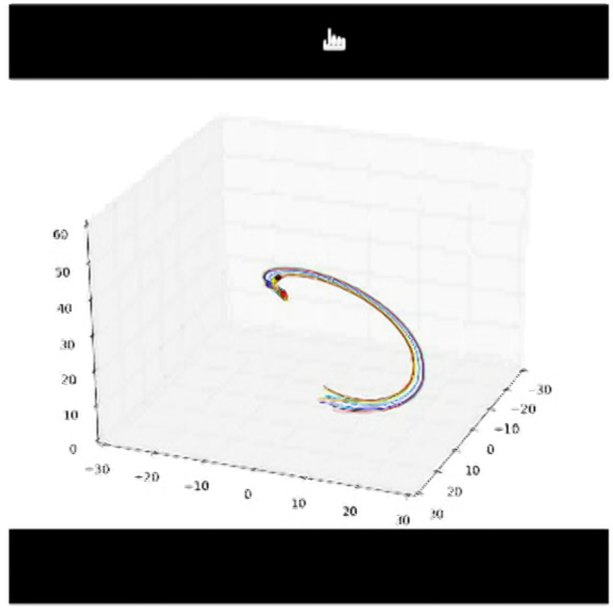
Classical Chaos

Lorenz oscillator : Classic example of chaotic behaviour

- Non-linear deterministic dynamical systems.

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$



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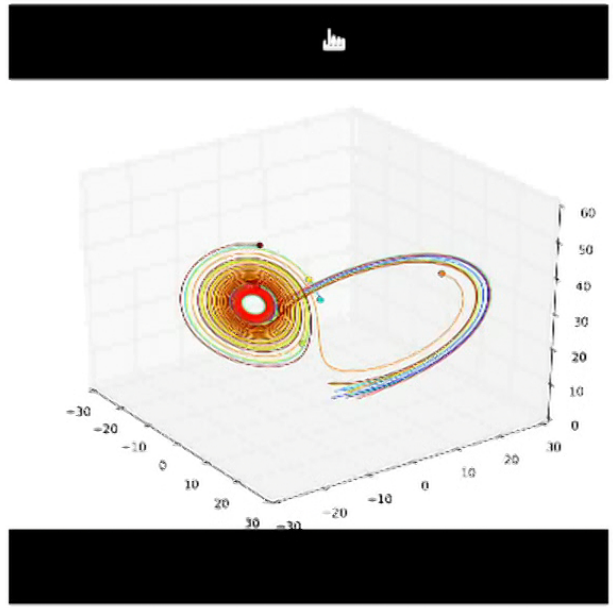

Classical chaos and Ehrenfest's correspondence principle

$$\langle X(t) \rangle \approx X_c(t)\ , \qquad t \leq t_{\mathrm{Eh}}$$

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- $t_{\mathrm{Eh}}^{regular} \approx \left(\frac{A_0}{\hbar}\right)^{\alpha}$
- $t_{\mathrm{Eh}}^{chaotic} \approx \frac{1}{\lambda} \ln\left(\frac{A_0}{\hbar}\right)$

Chaotic Model: Kicked Top

$$H = pJ_y \sum_{n=0}^{\infty} \delta(t - n\tau) + \frac{\kappa}{2j\tau} J_z^2$$

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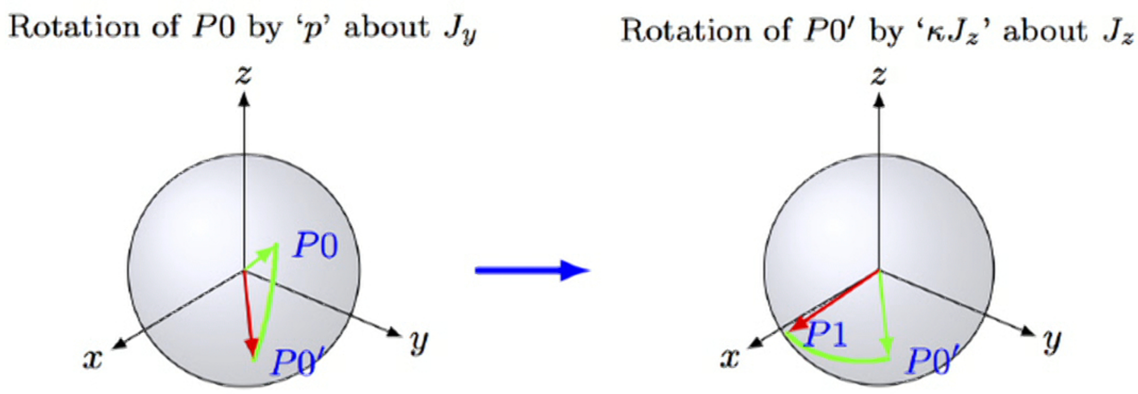

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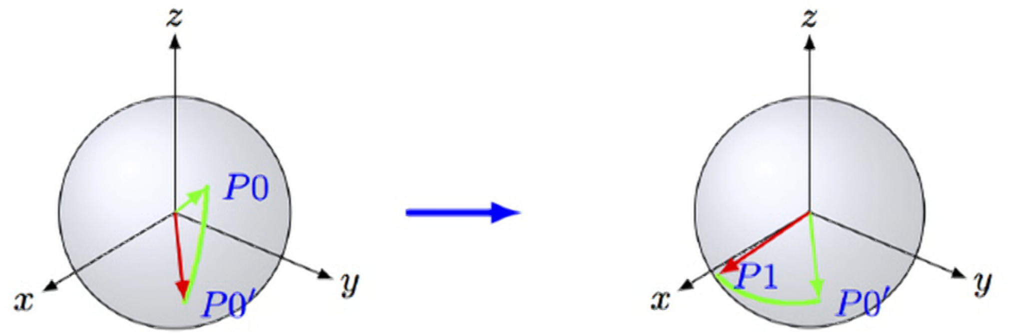
Rotation of $P0$ by 'p' about J_y — Rotation of $P0'$ by 'κJ_z' about J_z



Floquet time evolution operator:

$$U = \exp\left(-i\frac{\kappa}{2j\tau} J_z^2\right) \exp\left(-ipJ_y\right)$$

Chaotic Model: Kicked Top

$$H = \frac{\kappa}{2j\tau} J_z^2 + pJ_y \sum_{n=0}^{\infty} \delta(t - n\tau)$$

Classical equations of motion

$$J'_x = J_z \cos \kappa J_x + J_y \sin \kappa J_x$$

$$J'_y = -J_z \sin \kappa J_x + J_y \cos \kappa J_x$$

$$J'_z = -J_x$$

$J'_x = J_x(n\tau + 1), J_x = J_x(n\tau)$

$\frac{1}{j^2}\left(J_x^2 + J_y^2 + J_z^2\right) = 1.$

$\therefore$ Polar co-ordinates

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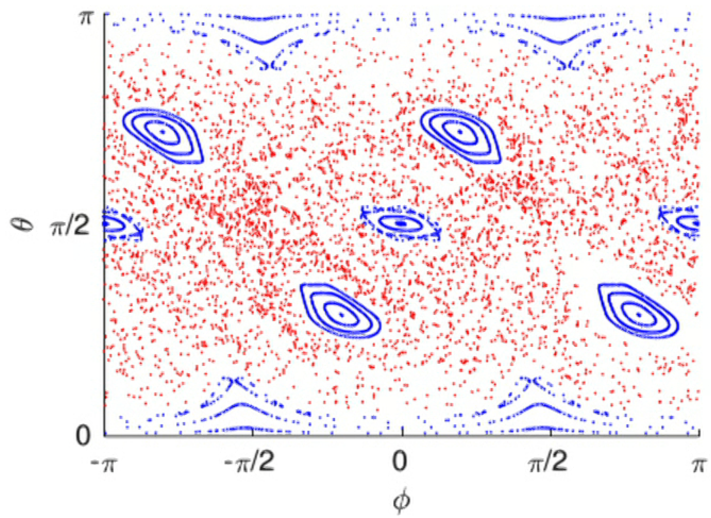

Figure: Classical stroboscopic phase space. $\kappa = 3.0$

Why kicked Top?

- Simple model, "Hydrogen atom of quantum chaos"
- Finite-dimensional system
- Experimentally realized[2]

[2] **Chaudhary et. al, Nature 461, 768-771, October 2009, and C. Neill et. al, Nature Physics 12, 1037 (2016)..**

Entanglement in isolated spin systems?

$$[H, J^2] = 0$$
$$\Rightarrow j(j+1) = \text{constant}$$

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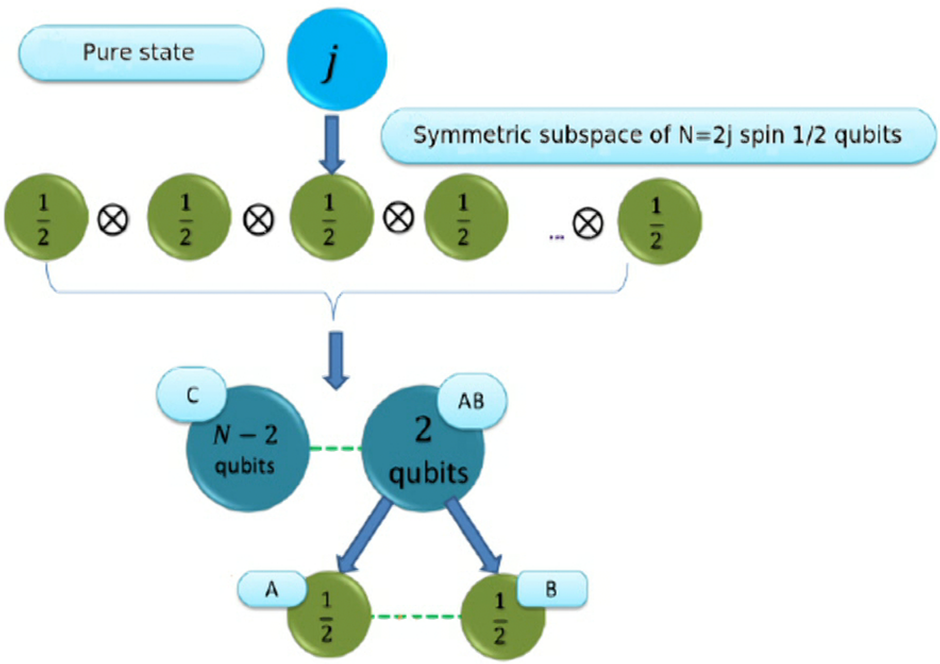

Entanglement(AB:C) $= S(\rho^{AB}) = S(\rho^{C})$

Our study: Entanglement vs classical dynamics

High entanglement: Signature of chaos?		
Classical dynamics ↓	Deep quantum regime	Semiclassical regime
Chaotic Regular	High ?	High Low

1. Propose framework to find upper bound on entanglement in spin systems.
2. When does quantum states remain close to classical-like states upon evolution?

Proposed framework: To find a bound on entanglement

Fannes-type inequality for Von Neumann entropy (S)

$$|S(\rho) - S(\sigma)| \leq D\log_2(d - 1) + h(D)$$

D = trace distance between ρ and σ; h(D) = binary entropy function.

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Choice of σ such that $S(\sigma) = 0$

$\Downarrow$

$$\text{Entanglement}\left(|\psi^{AB}\rangle\right) = S(\rho^A) \leq T\log_2(d-1) + h(T)$$

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$$\Downarrow$$

$$\begin{aligned} \text{Entanglement}\left(|\psi^{AB}\rangle\right) &= S(\rho^A) \\ &\leq T\log_2(d-1) + h(T) \end{aligned}$$

Task:

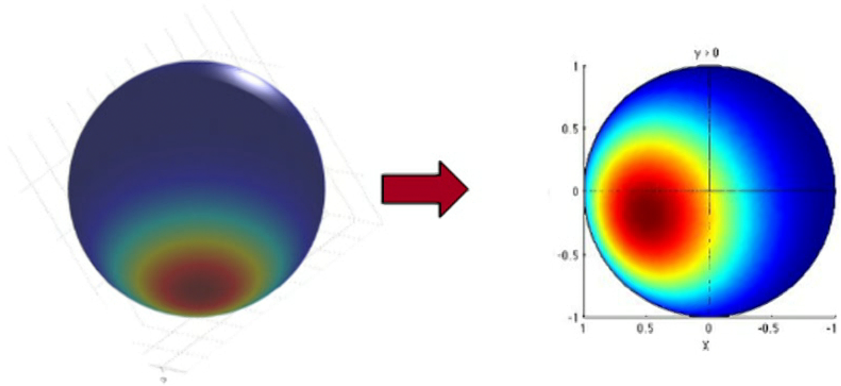
- Find σ such that $S(\rho) \lesssim D\log_2(d-1) + h(D)$

Candidate for σ: Spin Coherent States (SCS)

Spin coherent state (SCS)

$$|j,\theta,\phi\rangle = R(\theta,\phi)|j,j\rangle$$

where $R(\theta,\phi) = \exp\left(i\theta(J_x \sin\phi - J_y \cos\phi)\right)$



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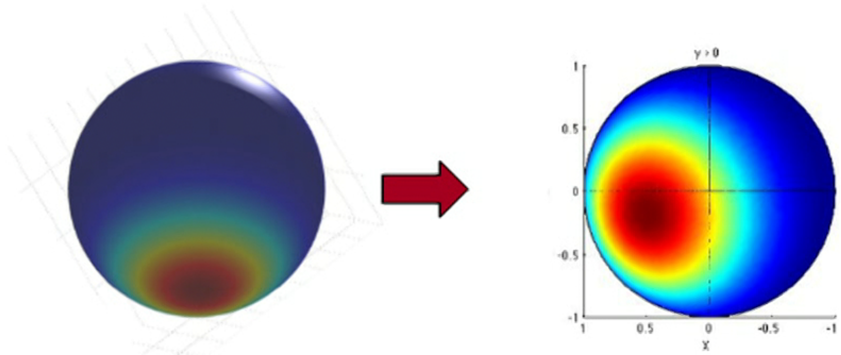


$$|j, \theta, \phi\rangle = |\theta, \phi\rangle \otimes |\theta, \phi\rangle \otimes \ldots\ldots \otimes |\theta, \phi\rangle \text{ (2j times).}$$

where $|\theta, \phi\rangle = \sin\left(\frac{\theta}{2}\right)|0\rangle + \exp\left(\mathrm{i}\phi\right)\cos\left(\frac{\theta}{2}\right)|1\rangle$.

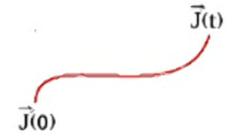
SCS: Minimum uncertainty states $\Rightarrow$ Quantum analog of classical states.

Study of entanglement generation: finding σ

$$|\psi(0)\rangle \xrightarrow[\text{for time t}]{\text{evolve with H}} |\psi(t)\rangle$$

Choices of σ: Reduced states of SCS centered at

1. $(\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle)$ with respect to $|\psi(t)\rangle$,
2. the point on the classical trajectory at time 't'



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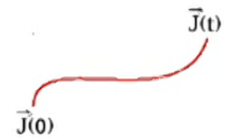

Choose SCS with with minimum trace distance from $|\psi(t)\rangle$ to calculate the upper bound on entanglement:

$$\begin{aligned} \text{Entanglement}\left(|\psi^{AB}(t)\rangle\right) &= S(\rho^A(t)) \\ &\leq D_{\text{re}}\log_2(d-1) + h(D_{\text{re}}) \end{aligned}$$

D_{re} = trace distance between reduced states.

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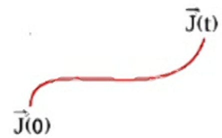

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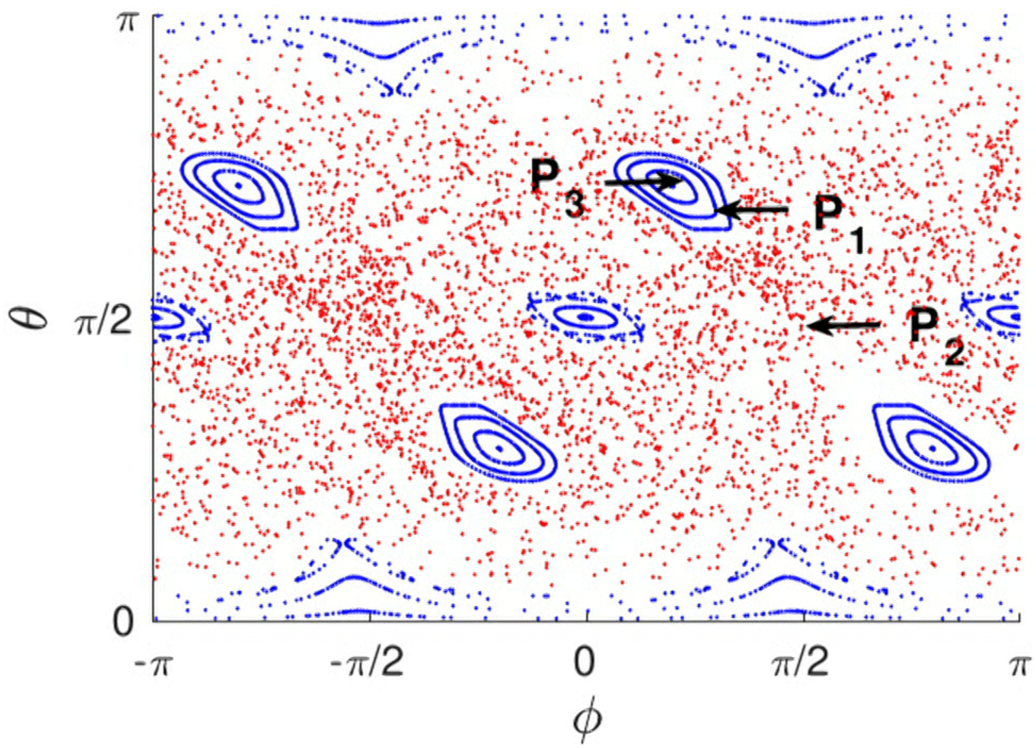
D_{re} = trace distance between reduced states.

$D_{\text{re}} \leq D$, where D = trace distance between $|\psi(t)\rangle$ and SCS.

Performance of bound on entanglement? Study in QKT

$$H = \frac{\kappa}{2j\tau} J_z^2 + p J_y \sum_{n=0}^{\infty} \delta(t - n\tau)$$



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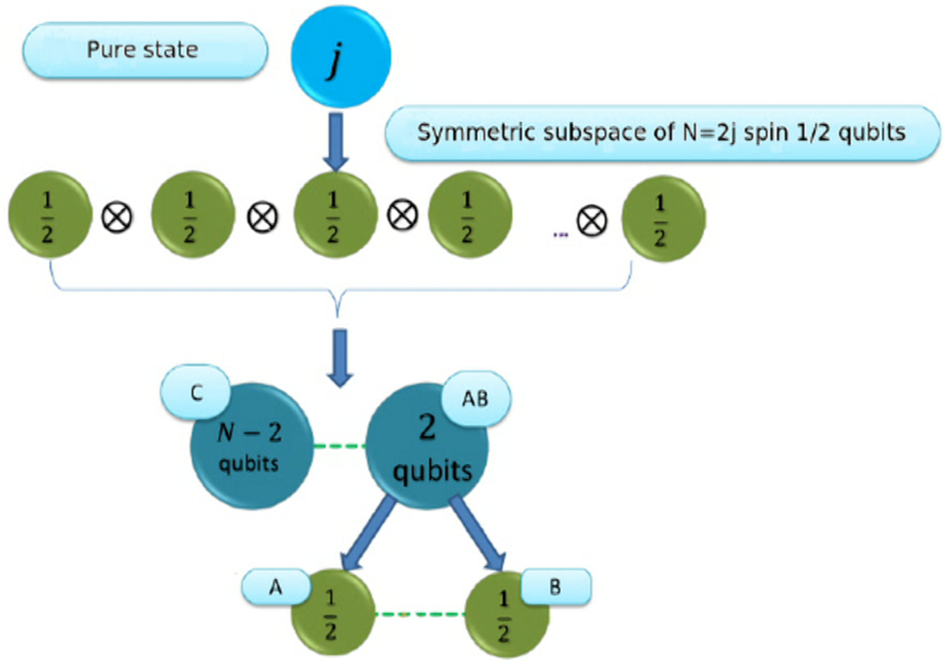
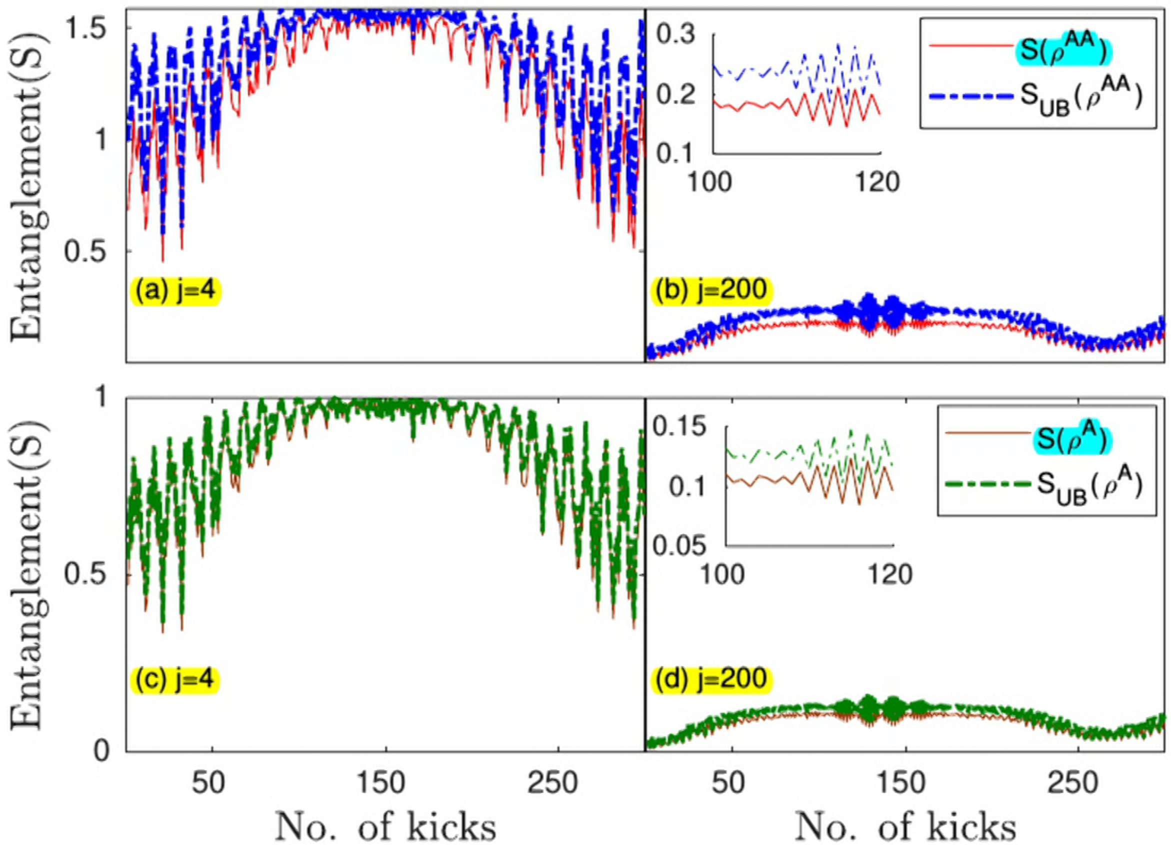


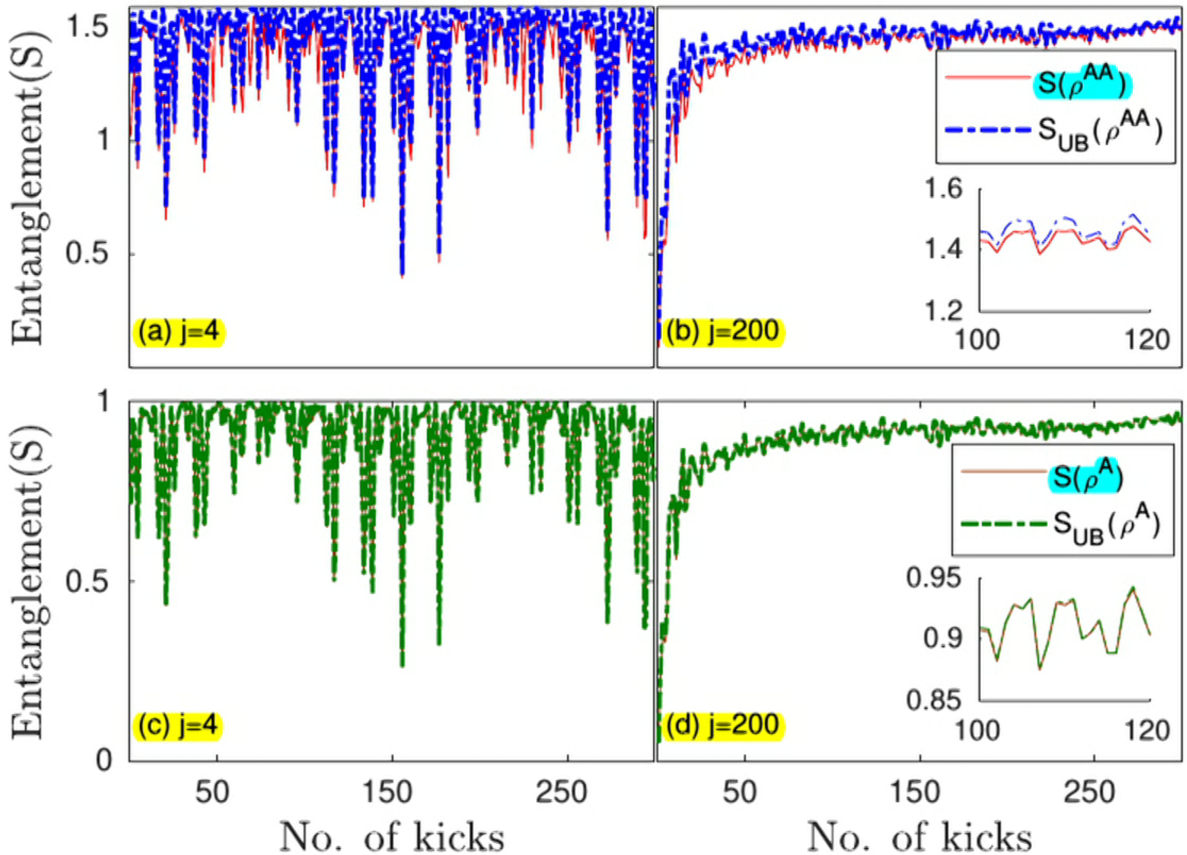
We can study entanglement in any bipartition of $2j$ **qubits**.

Performance of bound on entanglement in regular region?

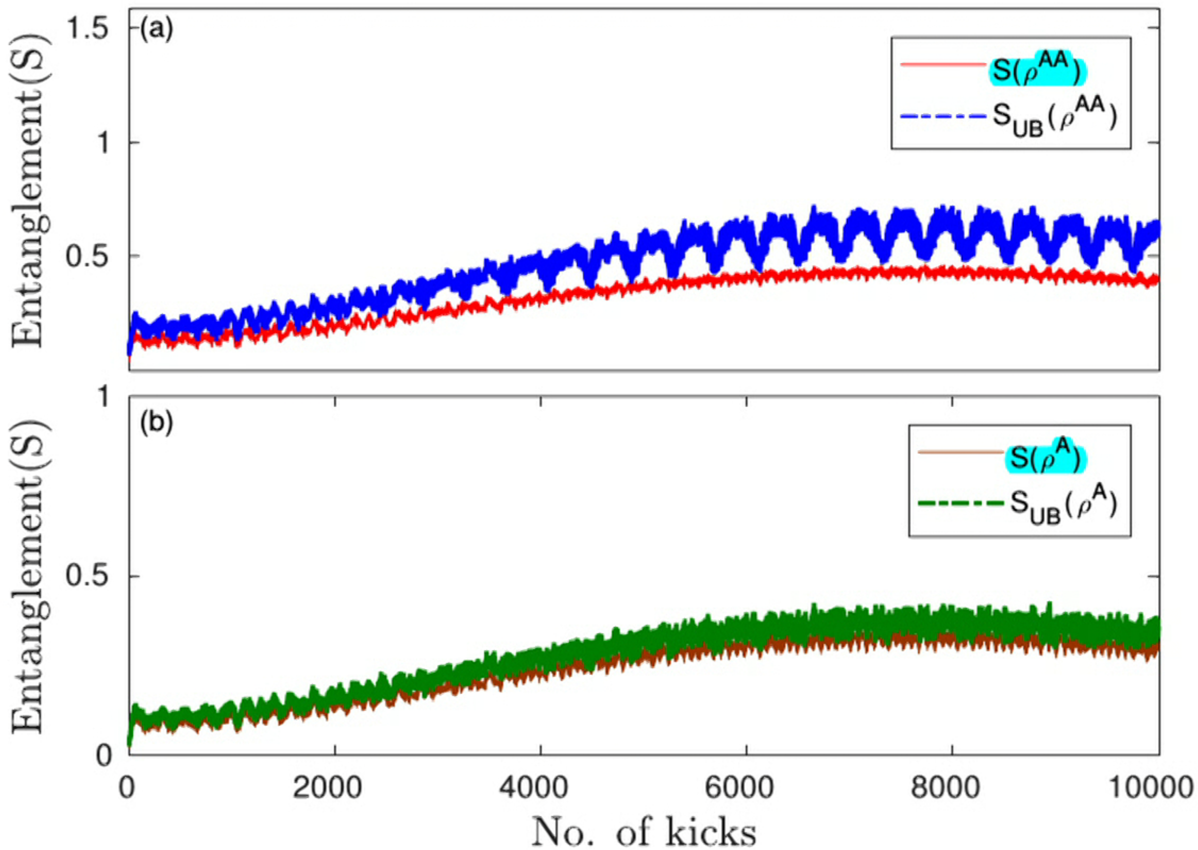
$\boldsymbol{S}(\cdot)$ = Exact entanglement; $\boldsymbol{S}_{\text{UB}}(\cdot)$ = Upper bound on entanglement in $|\psi(t)\rangle$.
$S(\rho^{A})$ = 1:(2j-1) qubit entanglement; $\boldsymbol{S}(\rho^{AA})$ = 2:(2j-2) qubit entanglement.



Performance of bound on entanglement in chaotic region?



Performance of bound on entanglement for long times?



Result: Entanglement related to delocalization

∴ With a good choice of SCS for σ,

$$\begin{aligned} \text{Entanglement}\left(|\psi^{AB}(t)\rangle\right) &= S(\rho^{A}(t)) \\ &\lesssim D_{\text{re}}\log_2(d-1) + h(D_{\text{re}}) \end{aligned}$$

Result: Entanglement related to delocalization

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$\therefore$ With a good choice of SCS for σ,

$$
\begin{aligned}
\text{Entanglement}\left(|\psi^{AB}(t)\rangle\right) &= S(\rho^A(t)) \\
&\lesssim D_{\text{re}}\log_2(d-1) + h(D_{\text{re}})
\end{aligned}
$$

Thus,

Low entanglement	$\Leftarrow$	Small trace distance to SCS	$\Rightarrow$	Localized evolution
High entanglement	$\Leftarrow$	Large trace distance to SCS	$\Rightarrow$	Delocalized evolution

Entanglement related to a measure of delocalization.

[Kumari et. al, arXiv:1806.10545]

High entanglement: Signature of chaos?		
Classical dynamics ↓	Deep quantum regime	Semiclassical regime
Chaotic	High	High
Regular	**?**	Low

Our goal: To understand ?.

High entanglement: Signature of chaos?		
Classical dynamics ↓	Deep quantum regime	Semiclassical regime
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Regular	**?**	Low

Our goal: To understand ?.

Steps: Figure out when does classical regular dynamics, especially in deep quantum regime, lead to

- localized evolution?
- delocalized evolution?

Finding criteria for localized evolution

Conjecture: Criteria for localized evolution near classical stable periodic orbits in periodically driven systems [Kumari et. al, PRE 97, 052209 (2018)]:

- Set of coherent states corresponding to the n points in the period-n orbit form an orthogonal set.
- Set of coherent states corresponding to multiple periodic orbits related by system's symmetries form an orthogonal set.

For spin systems, overlap between spin coherent states (SCS):

$$|\langle j,\theta,\phi|j,\theta_0,\phi_0\rangle| = \left(\cos\left[\frac{\chi(\theta\phi,\theta_0\phi_0)}{2}\right]\right)^{2j}$$

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Illustration of conjecture in quantum kicked top (QKT)

$$H = \frac{\kappa}{2j\tau} J_z^2 + pJ_y \sum_{n=0}^{\infty} \delta(t - n\tau)$$

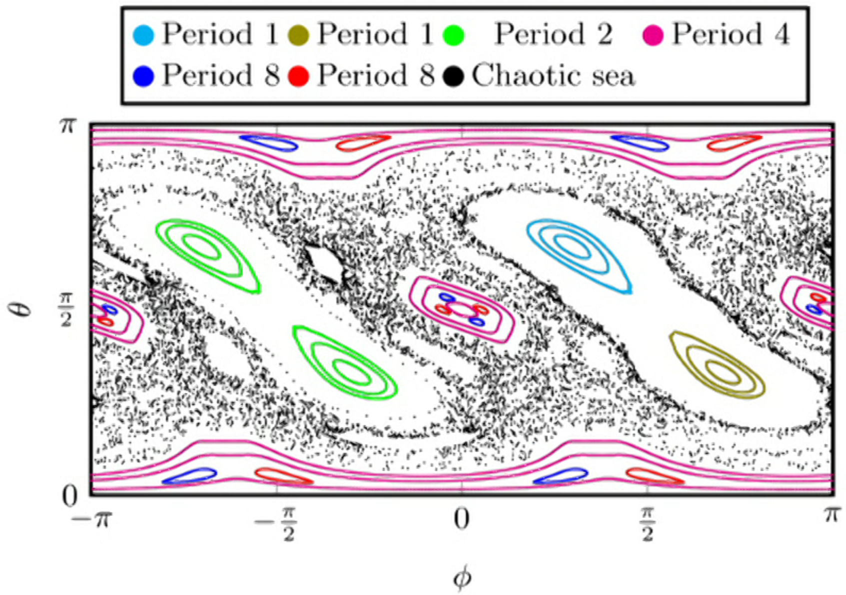

Figure: Classical stroboscopic phase space,[3] $\kappa = 2.5$

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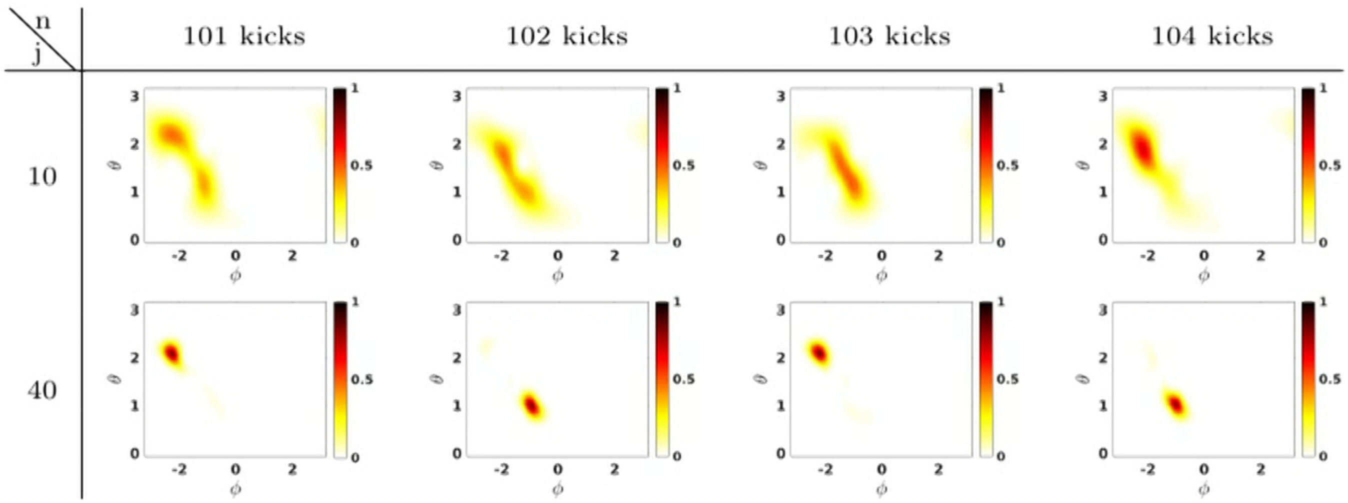
[3] **PRE 95**, **062222 (2017)**.

Illustration of conjecture in quantum kicked top (QKT)

Quantum dynamics at a classical period-2 orbit:

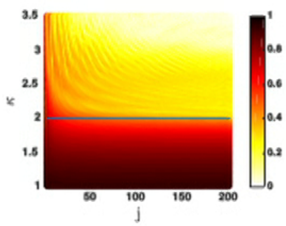
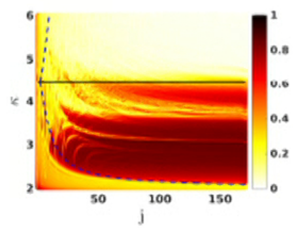
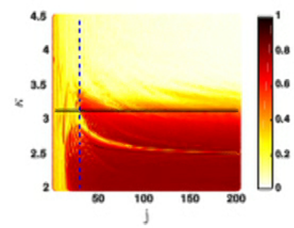
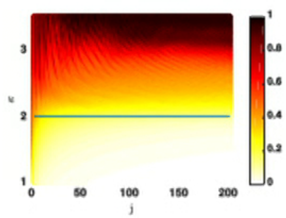
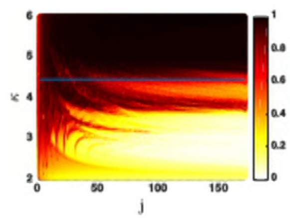
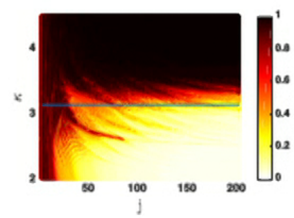
n = No. of kicks = No. of time periods



- j=10, overlap $\approx O(10^{-4})$.
- j=40, overlap $\approx O(10^{-14})$. Localized evolution.

Localized and delocalized evolution versus entanglement

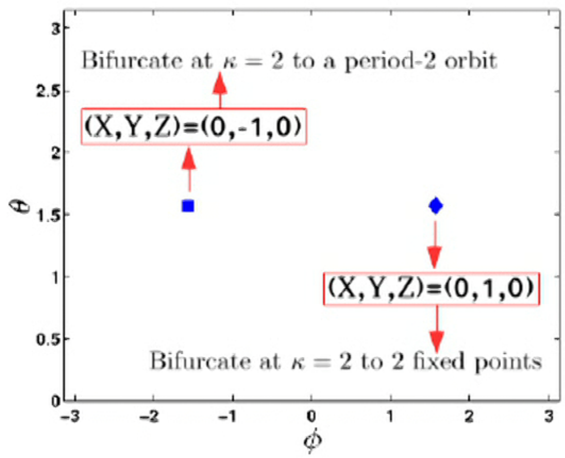
Survival probability plots: Dark red ⇒ Localized evolution
Entanglement plots: Dark red ⇒ Low entanglement

	Fixed point	Period-2 orbit	Period-4 orbit
Survival probability			
1:(2j-1) qubit entanglement			

Horizontal curve: Bifurcation parameter as function of j. (demarcates stability)
Dashed curve: κ parameter value at which criteria is satisfied.

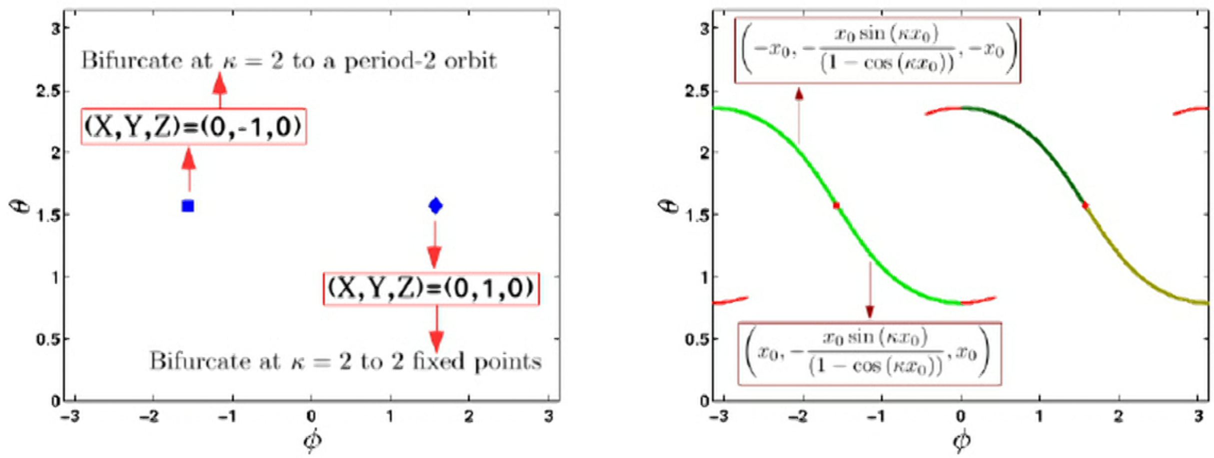
Kicked top: Fixed points and periodic orbits

$$H = \frac{\kappa}{2j\tau} J_z^2 + pJ_y \sum_{n=0}^{\infty} \delta(t - n\tau)$$



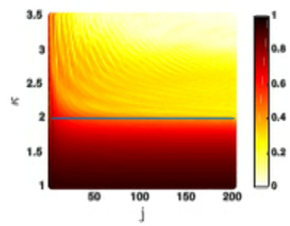
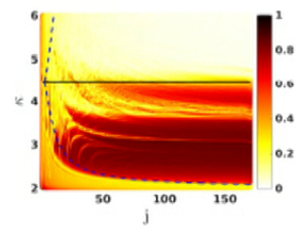
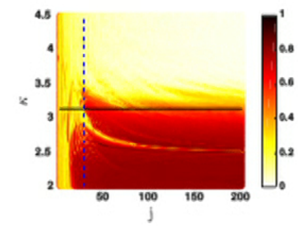
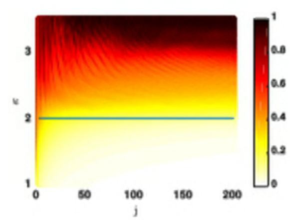
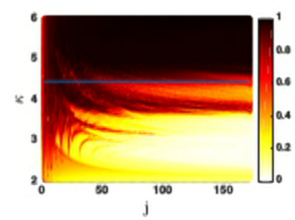
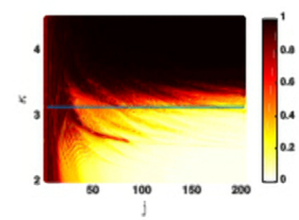
Kicked top: Fixed points and periodic orbits

$$H = \frac{\kappa}{2j\tau} J_z^2 + pJ_y \sum_{n=0}^{\infty} \delta(t - n\tau)$$

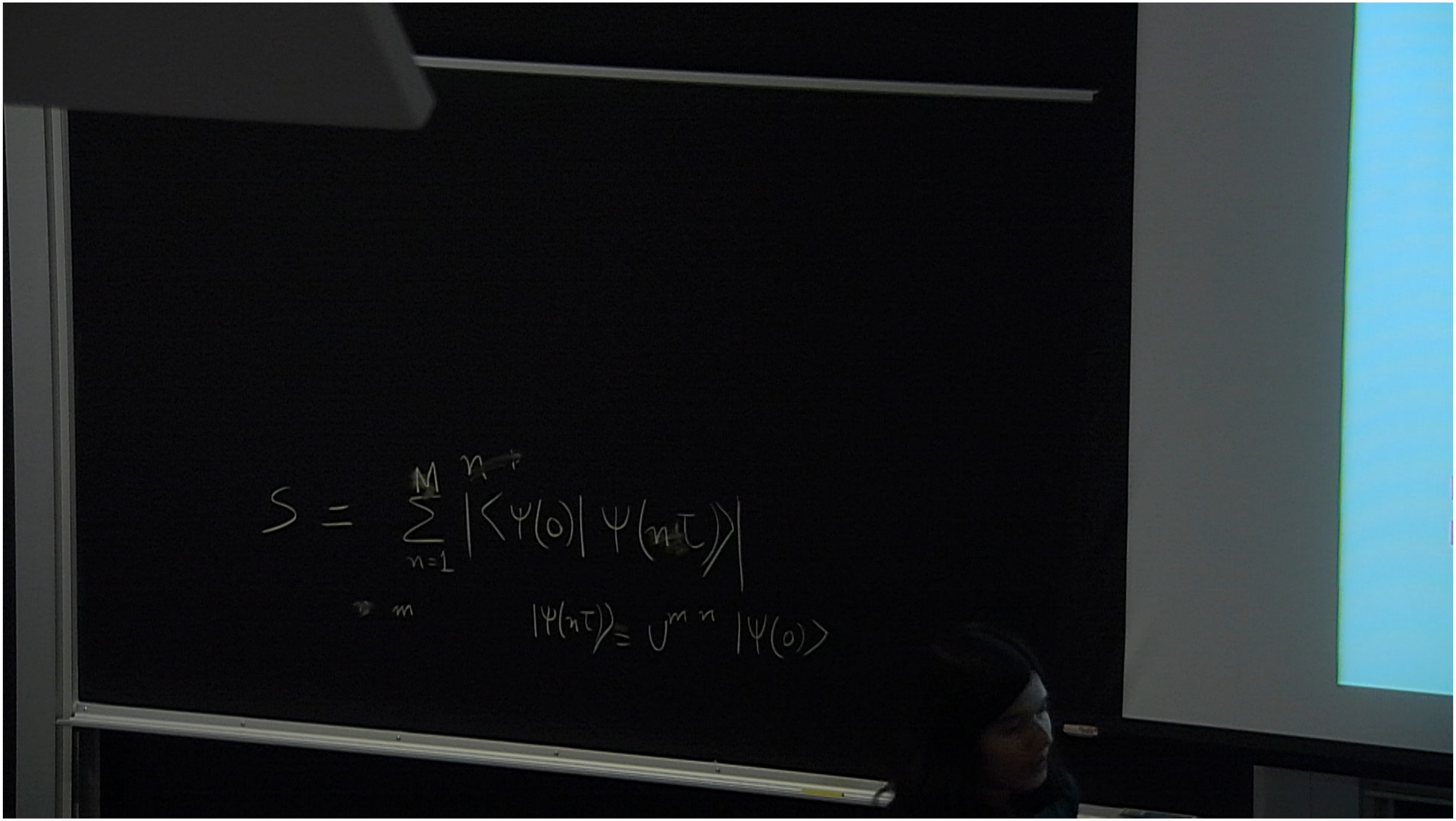


Localized and delocalized evolution versus entanglement

Survival probability plots: Dark red ⇒ Localized evolution
Entanglement plots: Dark red ⇒ Low entanglement

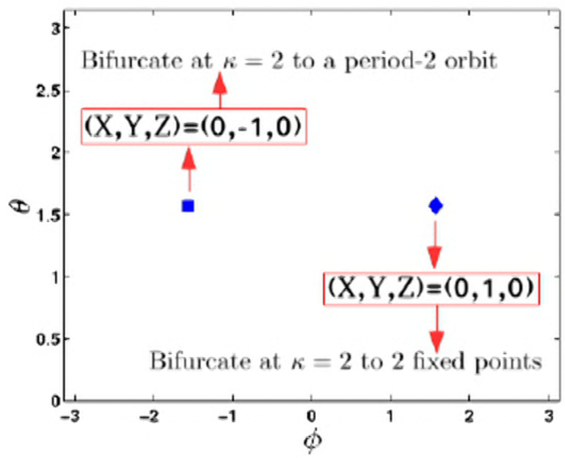
	Fixed point	Period-2 orbit	Period-4 orbit
Survival probability			
1:(2j-1) qubit entanglement			

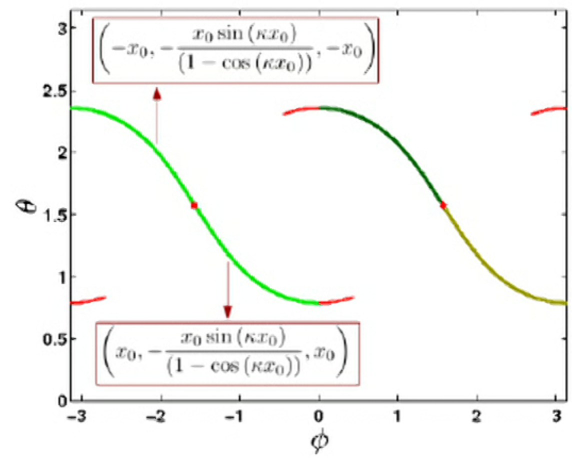
Horizontal curve: Bifurcation parameter as function of j. (demarcates stability)
Dashed curve: κ parameter value at which criteria is satisfied.


S = Σ_{n=1}^{M} |⟨ψ(0)| ψ(nτ)⟩|
m
|ψ(nτ)⟩ = U^{m n} |ψ(0)⟩

Kicked top: Fixed points and periodic orbits

$$H = \frac{\kappa}{2j\tau} J_z^2 + pJ_y \sum_{n=0}^{\infty} \delta(t - n\tau)$$





Summary

A.

Upper bound on entanglement generation

in spin systems.

B.

Quantum-classical correspondence

in periodically driven systems.

C.

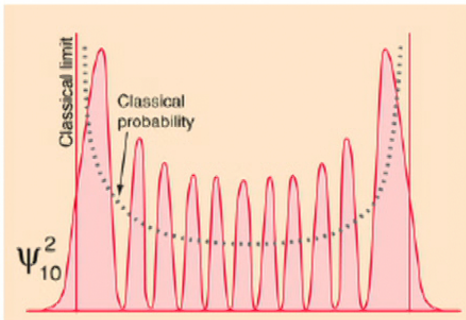
High entanglement: A signature of chaos only in semiclassical regime

in periodically driven spin systems.

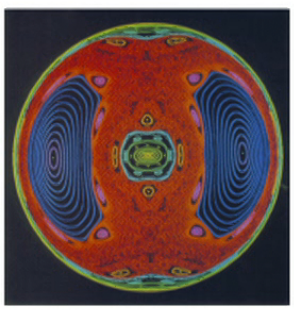
Summary

Semiclassical regime

Harmonic Oscillator[1]



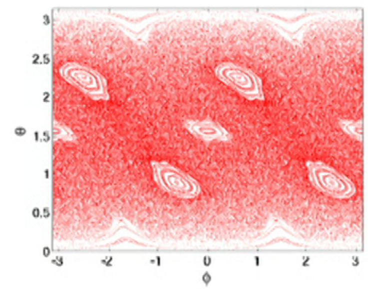
H-atom in strong B [2]



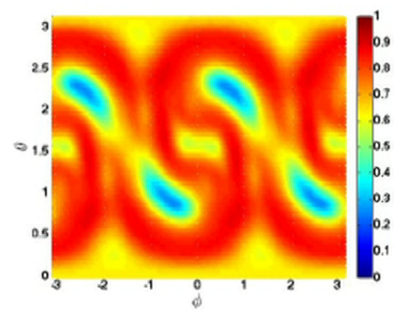
$\overset{?}{\Leftrightarrow}$

Deep quantum regime

Classical phase space



Entanglement



No more elusive!

Other projects: Study of nonlocality in kicked top

1. M. Kumari, S. Ghose and RB Mann, "Sufficient condition for nonexistence of symmetric extension of qudits using Bell inequalities", Phys. Rev. A **96**, 012128 (2017).

Theorem : Given a 2-qudit monogamous Bell inequality, any 2-qudit state that violates that inequality cannot possess any symmetric extension.

- Conjectured Bell CGLMP inequality for qutrits to be monogamous.

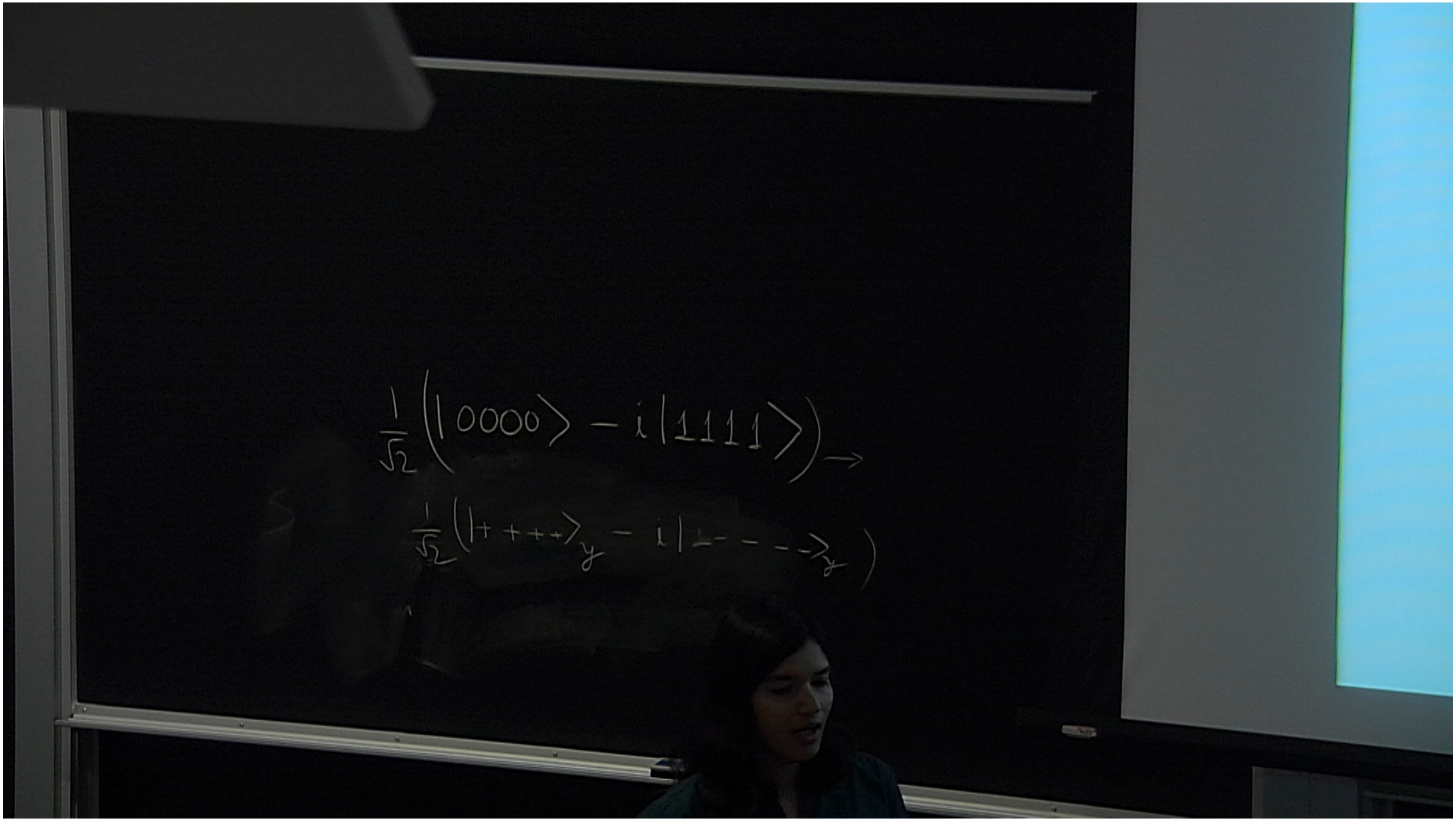
Other projects: Study of nonlocality in kicked top

2. "Kicked top as an efficient generator of nonlocal correlations", Manuscript in preparation.

Dynamical tunneling in the QKT leads to generation of states:

- $\frac{1}{\sqrt{2}}\left(\otimes^{2j}|0\rangle - i\otimes^{2j}|1\rangle\right)$ for integer j values,
- $\frac{1}{\sqrt{2}}\left(|++++\rangle_y - i|----\rangle_y\right)$

Maximal violation of multi-qubit Svetlichny's inequality by these states.

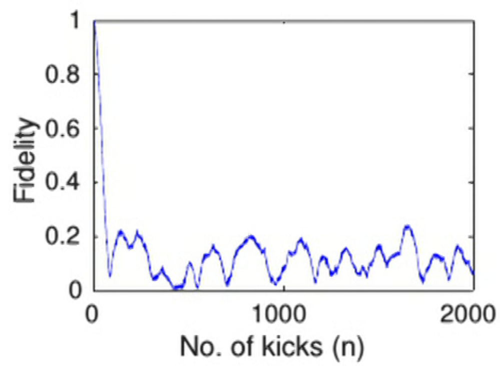


1/√2 (|0000> − i|1111>) →
1/√2 (|++++>y − i|−−−−>y)

Other projects: Stabilizing quantum dynamics

M. Kumari, E. Martin-Martinez, A. Kempf, and S. Ghose. "Stabilizing quantum dynamics through coupling to a quantized environment." arXiv preprint arXiv:1711.07906 (2017).

- Quantum dynamics sensitive to perturbations in external control parameter.
- Sensitivity measured via fidelity decay.

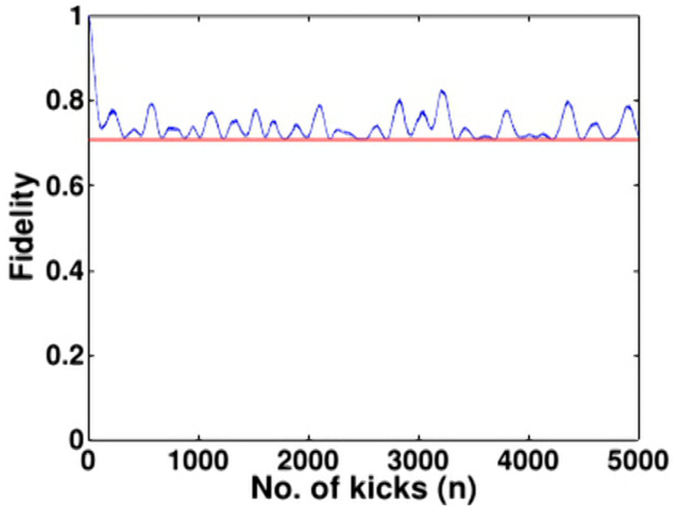


$\kappa = 3.0$, perturbation strength $\epsilon = 0.01$

Other projects: Stabilizing quantum dynamics

M. Kumari, E. Martin-Martinez, A. Kempf, and S. Ghose. "Stabilizing quantum dynamics through coupling to a quantized environment." arXiv preprint arXiv:1711.07906 (2017).

- Couple the sytem to a quantized environment.
- Derive a lower bound on the fidelity decay against perturbations in effective control parameter.



Other projects: Causal inequalities

"Are causal inequalities monogamous?", manuscript in preparation (In collaboration with Fabio Costa, University of Queensland)

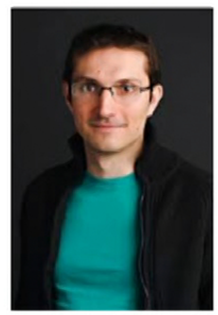
- Define symmetric extension of process matrices.
- Show that causal inequalities, GYNI and LGYNI, are weakly monogamous.

Acknowledgement

Dr. Shohini Ghose

Dr. Robert Mann

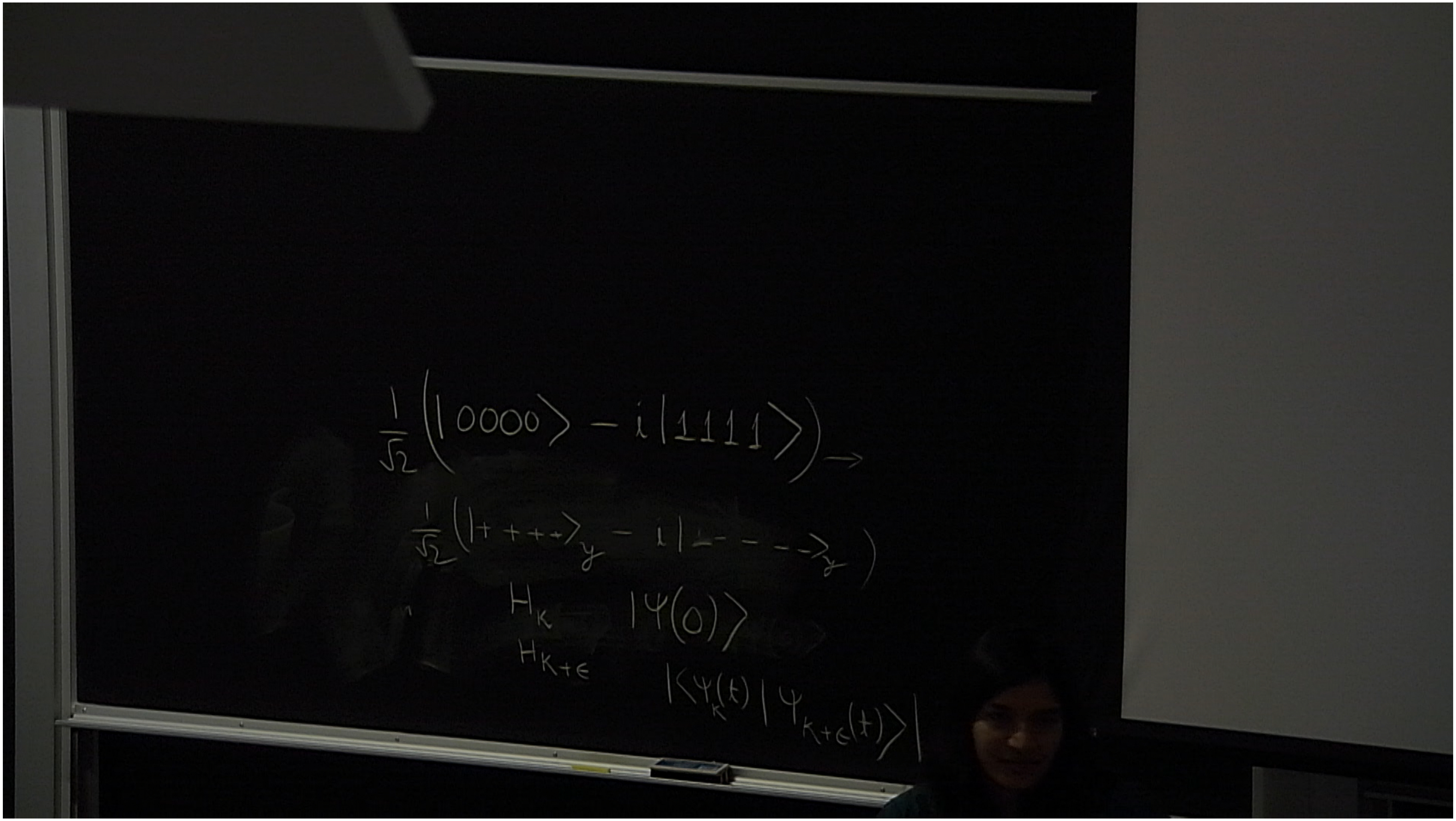
Dr. Achim Kempf



Dr. Eduardo Martin-Martinez

and Dr. Fabio Costa

Thank you


$\frac{1}{\sqrt{2}}\left(|0000\rangle - i|1111\rangle\right) \rightarrow$
$\frac{1}{\sqrt{2}}\left(|{+}{+}{+}{+}\rangle_y - i|{-}{-}{-}{-}\rangle_y\right)$
H_K $|\Psi(0)\rangle$
$H_{K+\epsilon}$ $|\langle\Psi_K(t)|\Psi_{K+\epsilon}(t)\rangle|$